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非牛頓系流在長圓套管夾層內之流動及對內外等溫熱源與等强熱源之熱傳
Some Non-Newtonian Flow through A Long Cylindrical Annulus and Heat Transfer with Constant Temperature Reservoirs and with Constant Wall rature Heat Flux

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Abstract – We may analyse the velocity distributions and the volume rate of flow of the steady fully-developed laminar flow of some non-Newtonian liquids such as Bingham Model, Ostwald-de Waele Model and Ellis Model through long cylindrical annulus. Then we may analyse the temperature distributions and the wall heat flux for heat transfer between these fluids and constant temperature reservoirs for the special case of short contact times and the temperature distributions for heat transfer between these fluids and constant wall heat flux.

摘要：賓漢型（Bingham Model）、奧斯瓦第瓦爾型（Ostward-de Waele Model）及意立司型（Ellis Model）非牛頓系液體在長圓套管夾層內流動已達穩定充分發展層流，吾人可分析其速度分佈與流量。並以等溫熱源（如低溫或高溫濕蒸氣）對其作短時間熱傳，而分析其溫度分佈與熱流率。又以等强熱源（如電熱圈或冷却圈）對其作穩定熱傳，亦可分析其溫度分佈。

一、前　言

在特殊環境使用之機械，其軸承常用非牛頓系液潤劑，卽以固質潤劑（Solid Lubricant）細粒懸浮於液油或水中構成者[4]。壓力循環潤滑軸承之潤滑劑，需加熱或冷却始可繼續應用。若其冷却器或加熱器採用圓套管夾層型，內外均有熱源，本文討論等溫熱源及等強熱源對流動於夾層內此等潤滑劑之熱傳。

二、速度分佈與體積流量

液體在長圓套管夾層內沿軸向 z 向下作穩定層流。流速 V_z，端效應可略去不計。套管總長 L，夾層最小半徑 κR，最大半徑 R。上端壓力 p_o，下端壓力 p_L。在距中心 r 處軸向剪應力 τ_{rz} 為[1]

$$\tau_{rz}=\frac{(P_o-P_L)R}{2L}\left[\frac{r}{R}-\lambda^2\left(\frac{R}{r}\right)\right] \tag{1}$$

式中 $P_o=p_o-\rho g z_o$，$P_L=p_L-\rho g z_L$，ρ 為密度，λ 為最大流速處之半徑比（在 $r=\lambda R$ 處，$V_z=V_{z\max}$）。現按不同型非牛頓系液體分別討論之。

㈠賓漢型：流速 V_z 與剪應力 τ_{rz} 之關係[1]

$$\tau_{rz} = -\mu_o \frac{dVz}{dr} + \tau_o \qquad |\tau_{rz}| > \tau_o$$

$$\frac{dVz}{dr} = 0 \qquad |\tau_{rz}| \leqslant \tau_o \tag{2}$$

式中 μ_o 與 τ_o 為賓漢参數（Bingham Parameter）可視為常數。（圖1）

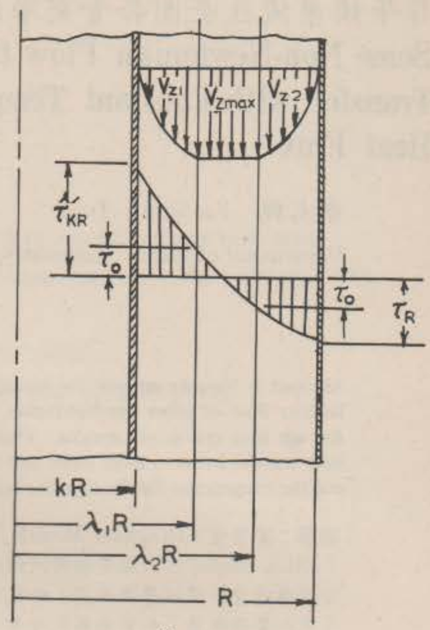

圖 1

在 $_{\kappa}R \leqslant r \leqslant \lambda_1 R$ 範圍內：$\frac{dVz}{dr} \geqslant 0$，$\tau_{rz} < 0$，

$$-\tau_o - \mu_o \frac{dVz}{dr} = \frac{(P_o - P_L)R}{2L}\left[\frac{r}{R} - \lambda^2\left(\frac{R}{r}\right)\right] \tag{3}$$

邊界條件：在 $r = {_\kappa}R$ 處，$Vz = 0$。得流速分佈為

$$Vz_1 = -\frac{(P_o - P_L)R^2}{2\mu_o L}\left[\frac{1}{2}\left(\frac{r^2}{R^2} - k^2\right) - \lambda^2 \ell n \frac{r}{kR}\right] - \frac{\tau_o R}{\mu_o}\left(\frac{r}{R} - k\right) \tag{4}$$

在 $\lambda_2 R \leqslant r \leqslant R$ 範圍內：$\frac{dVz}{dr} \leqslant 0$，$\tau_{rz} > 0$，

$$\tau_o - \mu_o \frac{dVz}{dr} = \frac{(P_o - P_L)R}{2L}\left[\frac{r}{R} - \lambda^2\left(\frac{R}{r}\right)\right] \tag{5}$$

邊界條件：在 $r = R$ 處，$Vz = 0$。得流速分佈為

$$Vz_2 = \frac{(P_o - P_L)R^2}{2\mu_o L}\left[\frac{1}{2}\left(1 - \frac{r^2}{R^2}\right) + \lambda^2 \ell n \frac{r}{R}\right] + \frac{\tau_o R}{\mu_o}\left(\frac{r}{R} - 1\right) \tag{6}$$

在 $\lambda_1 R \leqslant r \leqslant \lambda_2 R$ 範圍內：$\frac{dVz}{dr} = 0$，$Vz = Vz_{max}$，其流速為

$$\begin{aligned} Vz_{max} &= -\frac{(P_o - P_L)R^2}{2\mu_o L}\left(\frac{\lambda_2^2 - 1}{2} - \lambda^2 \ell n \lambda_2\right) - \frac{\tau_o R}{\mu_o}(1 - \lambda_2) \\ &= -\frac{(P_o - P_L)R^2}{2\mu_o L}\left(\frac{\lambda_1^2 - k^2}{2} - \lambda^2 \ell n \frac{\lambda_1}{k}\right) - \frac{\tau_o R}{\mu_o}(\lambda_1 - k) \end{aligned} \tag{7}$$

在 $r = \lambda_2 R$ 處，$\tau_o = \frac{(P_o - P_L)R}{2L}\left(\lambda_2 - \frac{\lambda^2}{\lambda_2}\right)$ (8)

在 $r = \lambda_1 R$ 處，$-\tau_o = \frac{(P_o - P_L)R}{2L}\left(\lambda_1 - \frac{\lambda^2}{\lambda_1}\right)$ (9)

由以上關係可得 λ_1，λ_2 及 λ 之值如下：

$$\lambda^2 = \lambda_1 \lambda_2 \tag{10}$$

$$\lambda_2 = \frac{2L\tau_o}{(P_o - P_L)R} + \lambda_1 \tag{11}$$

$$\left[\frac{2\tau_o L}{(P_o-P_L)R}+\lambda_1\right]\lambda_1 \ell n\left\{\frac{k}{\lambda_1}\left[\frac{2\tau_o L}{(P_o-P_L)R}+\lambda_1\right]\right\}=\frac{k^2+\lambda_1^2-1}{2}-\frac{1}{2}\left[\frac{2\tau_o L}{(P_o-P_L)R}+\lambda_1\right]^2 + (1+k)\frac{2\tau_o L}{(P_o-P_L)R} \quad (12)$$

令$\mathbb{R}=\frac{r}{R}$，其流速分佈如下：

$$V_{Z_1}=\frac{\tau_R R}{\mu_o(1-\lambda^2)}\left[\frac{k^2-\mathbb{R}^2}{2}+\lambda^2\ell n\frac{\mathbb{R}}{k}-\frac{\tau_o}{\tau_R}(1-\lambda^2)(\mathbb{R}-k)\right] \qquad k\leqslant\mathbb{R}\leqslant\lambda_1 \quad (13)$$

$$V_{Z\max}=\frac{\tau_R R}{\mu_o(1-\lambda^2)}\left[\frac{1-\lambda_2^2}{2}+\lambda^2\ell n\,\lambda_2-\frac{\tau_o}{\tau_R}(1-\lambda^2)(1-\lambda_2)\right] \qquad \lambda_1\leqslant\mathbb{R}\leqslant\lambda_2 \quad (14)$$

$$=\frac{\tau_R R}{\mu_o(1-\lambda^2)}\left[\frac{k^2-\lambda_1^2}{2}+\lambda^2\ell n\frac{\lambda_1}{k}-\frac{\tau_o}{\tau_R}(1-\lambda^2)(\lambda_1-k)\right]$$

$$V_{Z_2}=\frac{\tau_R R}{\mu_o(1-\lambda^2)}\left[\frac{1-\mathbb{R}^2}{2}+\lambda^2\ell n\mathbb{R}-\frac{\tau_o}{\tau_R}(1-\lambda^2)(1-\mathbb{R})\right] \qquad \lambda_2\leqslant\mathbb{R}\leqslant 1 \quad (15)$$

式中$\tau_R=\frac{(P_o-P_L)R}{2L}(1-\lambda^2)$為夾層外管內壁剪應力。

體積流量為

$$\dot{V}=\int_o^{2\pi}\int_{kR}^{\lambda_1 R}V_{Z_1}\,r\,dr\,d\phi+\int_o^{2\pi}\int_{\lambda_1 R}^{\lambda_2 R}V_{Z\max}\,r\,dr\,d\phi+\int_o^{2\pi}\int_{\lambda_2 R}^{R}V_{Z_2}\,r\,dr\,d\phi$$

$$=\frac{\pi(P_o-P_L)R^4}{8\mu_o L}\left[1-k^4-2\lambda^2\left(\lambda_1^2-\lambda_2^2-k^2+2\lambda_1^2\ell n\frac{\lambda_2 k}{\lambda_1}\right)-\lambda_2^4-\lambda_1^4+2k^2\lambda_1^2-2\lambda_1^2\right] - \frac{\tau_o\pi R^3}{3\mu_o}\left[1+k^3+2\lambda_1^3-\lambda_2^3+3\lambda_1^2(\lambda_2-k-1)\right] \quad (16)$$

平均流速為

$$V_m=\frac{(P_o-P_L)R^2}{8\mu_o L(1-k^2)}\left[1-k^4-2\lambda^2\left(\lambda_1^2-\lambda_2^2-k^2+2\lambda_1^2\ell n\frac{\lambda_2 k}{\lambda_1}\right)-\lambda_2^4-\lambda_1^4+2k^2\lambda_1^2-2\lambda_1^2\right] - \frac{\tau_o R}{3\mu_o(1-k^2)}\left[1+k^3+2\lambda_1^3-\lambda_2^3+3\lambda_1^2(\lambda_2-k-1)\right] \quad (17)$$

(二)奧斯瓦第瓦爾型：流速 Vz 與剪應力 τ_{rz} 之關係[1](圖2)

$$\tau_{rz}=-m\left|\frac{dV_z}{dr}\right|^{n-1}\frac{dV_z}{dr} \quad (18)$$

式中 m及 n 為實驗參數，可視為常數。則

$$-\frac{dV_z}{dr}=\left[\frac{(P_o-P_L)R}{2Lm}\left(\frac{r}{R}-\frac{\lambda^2 R}{r}\right)\right]^{1/n} \qquad \frac{1}{n}\neq 2,4,6,\cdots \quad (19)$$

在 $k\leqslant\mathbb{R}\leqslant\lambda$ 範圍內：邊界條件(1)在$\mathbb{R}=k$處，$V_z=0$；(2)在$\mathbb{R}=\lambda$處，$V_z=V_{Z\max}$。流速分佈為

$$V_{z_1} = \left[\frac{\tau_R}{m(1-\lambda^2)}\right]^{1/n} R \left[\frac{(-\lambda^2)^{1/n}\left(k^{1-\frac{1}{n}} - \mathbb{R}^{1-\frac{1}{n}}\right)}{1-\frac{1}{n}} + \frac{\frac{1}{n}(-\lambda^2)^{\frac{1}{n}-1}\left(k^{3-\frac{1}{n}} - \mathbb{R}^{3-\frac{1}{n}}\right)}{3-\frac{1}{n}} \right.$$
$$+ \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)(-\lambda^2)^{\frac{1}{n}-2}}{(2!)\left(5-\frac{1}{n}\right)}\left(k^{5-\frac{1}{n}}-\mathbb{R}^{5-\frac{1}{n}}\right) + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)(-\lambda^2)^{\frac{1}{n}-3}}{(3!)\left(7-\frac{1}{n}\right)}$$
$$\left.\left(k^{7-\frac{1}{n}}-\mathbb{R}^{7-\frac{1}{n}}\right) + \cdots \right] \qquad k \leqslant \mathbb{R} \leqslant \lambda \qquad (20)$$

$$V_{z_{max}} = \left[\frac{\tau_R}{m(1-\lambda^2)}\right]^{\frac{1}{n}} R \left[\frac{(-\lambda^2)^{\frac{1}{n}}\left(k^{1-\frac{1}{n}} - \lambda^{1-\frac{1}{n}}\right)}{1-\frac{1}{n}} + \frac{\frac{1}{n}(-\lambda^2)^{\frac{1}{n}-1}}{3-\frac{1}{n}} \right.$$
$$\left(k^{3-\frac{1}{n}} - \lambda^{3-\frac{1}{n}}\right) + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)(-\lambda^2)^{\frac{1}{n}-2}}{(2!)\left(5-\frac{1}{n}\right)}\left(k^{5-\frac{1}{n}}-\lambda^{5-\frac{1}{n}}\right)$$
$$\left. + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)(-\lambda^2)^{\frac{1}{n}-3}}{(3!)\left(7-\frac{1}{n}\right)}\left(k^{7-\frac{1}{n}}-\lambda^{7-\frac{1}{n}}\right) + \cdots \right] \qquad \mathbb{R} = \lambda \qquad (21)$$

在 $\lambda \leqslant \mathbb{R} \leqslant 1$ 範圍內：邊界條件(1)在 $\mathbb{R} = 1$ 處，$V_z = 0$；(2)在 $\mathbb{R} = \lambda$ 處，$V_z = V_{z_{max}}$。流速分佈為

$$V_{z_2} = \left[\frac{\tau_R}{m(1-\lambda^2)}\right]^{\frac{1}{n}} R \left[\frac{1-\mathbb{R}^{\frac{1}{n}+1}}{\frac{1}{n}+1} + \frac{(-\lambda^2)\left(1-\mathbb{R}^{\frac{1}{n}-1}\right)}{\left(\frac{1}{n}-1\right)n} + \frac{(-\lambda^2)^2\left(\frac{1}{n}\right)\left(\frac{1}{n}-1\right)}{(2!)\left(\frac{1}{n}-3\right)} \right.$$
$$\left(1-\mathbb{R}^{\frac{1}{n}-3}\right) + \frac{(-\lambda^2)^3\left(\frac{1}{n}\right)\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{(3!)\left(\frac{1}{n}-5\right)}\left(1-\mathbb{R}^{\frac{1}{n}-5}\right)$$
$$\left. + \frac{(-\lambda^2)^4\left(\frac{1}{n}\right)\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\left(\frac{1}{n}-3\right)}{(4!)\left(\frac{1}{n}-7\right)}\left(1-\mathbb{R}^{\frac{1}{n}-7}\right) + \cdots \right] \qquad \lambda \leqslant \mathbb{R} \leqslant 1 \qquad (22)$$

$$V_{z_{max}} = \left[\frac{\tau_R}{m(1-\lambda^2)}\right]^{\frac{1}{n}} R \left[\frac{1-\lambda^{\frac{1}{n}+1}}{\frac{1}{n}+1} + \frac{(-\lambda^2)\left(1-\lambda^{\frac{1}{n}-1}\right)}{\left(\frac{1}{n}-1\right)n} + \frac{(-\lambda^2)^2\left(\frac{1}{n}\right)\left(\frac{1}{n}-1\right)}{(2!)\left(\frac{1}{n}-3\right)} \right.$$
$$\left(1-\lambda^{\frac{1}{n}-3}\right) + \frac{(-\lambda^2)^3\left(\frac{1}{n}\right)\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)}{(3!)\left(\frac{1}{n}-5\right)}\left(1-\lambda^{\frac{1}{n}-5}\right)$$

$$+\frac{(-\lambda^2)^4(\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)(\frac{1}{n}-3)}{(4!)(\frac{1}{n}-7)}(1-\lambda^{\frac{1}{n}-7})+\cdots] \quad \mathbb{R}=\lambda \quad (23)$$

由(21)與(23)兩式相等可得 λ 值。體積流量為

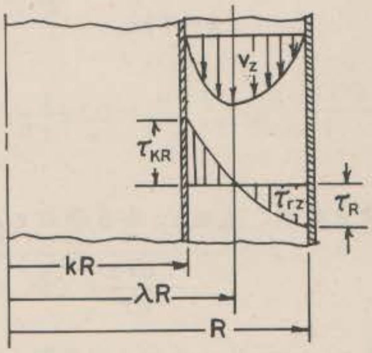

圖 2

$$\dot{V}=\int_{o}^{2\pi}\int_{kR}^{\lambda R} Vz_1\, rdrd\phi+\int_{o}^{2\pi}\int_{\lambda R}^{R} Vz_2\, rdrd\phi$$

$$=2\pi R^3\left[\frac{\tau_R}{m(1-\lambda^2)}\right]^{\frac{1}{n}}\left\{\frac{(-\lambda^2)^{\frac{1}{n}}}{1-\frac{1}{n}}\left[\frac{k^{3-\frac{1}{n}}-\lambda^{3-\frac{1}{n}}}{3-\frac{1}{n}}+\frac{k^{1-\frac{1}{n}}}{2}(\lambda^2-k^2)\right]+\frac{\frac{1}{n}(-\lambda^2)^{\frac{1}{n}-1}}{3-\frac{1}{n}}\left[\frac{k^{5-\frac{1}{n}}-\lambda^{3-\frac{1}{n}}}{5-\frac{1}{n}}+\frac{k^{3-\frac{1}{n}}}{2}(\lambda^2-k^2)\right]+\frac{\frac{1}{n}(\frac{1}{n}-1)(-\lambda^2)^{\frac{1}{n}-2}}{(2!)(5-\frac{1}{n})}\right.$$

$$\left[\frac{k^{7-\frac{1}{n}}-\lambda^{7-\frac{1}{n}}}{7-\frac{1}{n}}+\frac{k^{5-\frac{1}{n}}}{2}(\lambda^2-k^2)\right]+\frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)(-\lambda^2)^{\frac{1}{n}-3}}{(3!)(7-\frac{1}{n})}\left[\frac{k^{9-\frac{1}{n}}-\lambda^{9-\frac{1}{n}}}{9-\frac{1}{n}}\right.$$

$$\left.+\frac{k^{7-\frac{1}{n}}}{2}(\lambda^2-k^2)\right]+\cdots\Big\}+2\pi R^3\left[\frac{\tau_R}{m(1-\lambda^2)}\right]^{\frac{1}{n}}\left\{\frac{1}{\frac{1}{n}+1}\left(\frac{1-\lambda^2}{2}-\frac{1-\lambda^{\frac{1}{n}+3}}{\frac{1}{n}+3}\right)\right.$$

$$\left.+\frac{(-\lambda^2)}{(\frac{1}{n}-1)n}\left(\frac{1-\lambda^2}{2}-\frac{1-\lambda^{\frac{1}{n}+1}}{\frac{1}{n}+1}\right)+\frac{(-\lambda^2)^2(\frac{1}{n})(\frac{1}{n}-1)}{(2!)(\frac{1}{n}-3)}\left(\frac{1-\lambda^2}{2}-\frac{1-\lambda^{\frac{1}{n}}}{\frac{1}{n}-1}\right)+\cdots\right\} \quad (24)$$

平均流速為

$$Vm=\frac{2R}{(1-k^2)}\left[\frac{\tau_R}{m(1-\lambda^2)}\right]^{\frac{1}{n}}\left\{\frac{(-\lambda^2)^{\frac{1}{n}}}{1-\frac{1}{n}}\left[\frac{k^{3-\frac{1}{n}}-\lambda^{3-\frac{1}{n}}}{3-\frac{1}{n}}+\frac{k^{1-\frac{1}{n}}}{2}(\lambda^2-k^2)\right]\right.$$

$$+\frac{\frac{1}{n}(-\lambda^2)^{\frac{1}{n}-1}}{3-\frac{1}{n}}\left[\frac{k^{5-\frac{1}{n}}-\lambda^{5-\frac{1}{n}}}{5-\frac{1}{n}}+\frac{k^{3-\frac{1}{n}}}{2}(\lambda^2-k^2)\right]+\frac{\frac{1}{n}(\frac{1}{n}-1)(-\lambda^2)^{\frac{1}{n}-2}}{(2!)(5-\frac{1}{n})}$$

$$\left[\frac{k^{7-\frac{1}{n}}-\lambda^{7-\frac{1}{n}}}{7-\frac{1}{n}}+\frac{k^{5-\frac{1}{n}}}{2}(\lambda^2-k^2)\right]+\frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)(-\lambda^2)^{\frac{1}{n}-3}}{(3!)(7-\frac{1}{n})}$$

$$[\frac{k^{9-\frac{1}{n}}-\lambda^{9-\frac{1}{n}}}{9-\frac{1}{n}}+\frac{k^{7-\frac{1}{n}}}{2}(\lambda^2-k^2)]+\cdots+\frac{1}{\frac{1}{n}+1}(\frac{1-\lambda^2}{2}-\frac{1-\lambda^{\frac{1}{n}+3}}{\frac{1}{n}+3})$$

$$+\frac{(-\lambda^2)}{(\frac{1}{n}-1)n}(\frac{1-\lambda^2}{2}-\frac{1-\lambda^{\frac{1}{n}+1}}{\frac{1}{n}+1})+\frac{(-\lambda^2)^2(\frac{1}{n})(\frac{1}{n}-1)}{(2!)(\frac{1}{n}-3)}(\frac{1-\lambda^2}{2}-\frac{1-\lambda^{\frac{1}{n}-1}}{\frac{1}{n}-1})$$

$$+\cdots\} \tag{25}$$

(三)意立司型：流速 Vz 與剪應力 τ_{rz} 之關係[1]

$$-\frac{dVz}{dr}=(\varphi_0+\varphi_1|\tau_{rz}|^{\alpha-1})\tau_{rz} \tag{26}$$

式中 φ_0、φ_1 及 α 為意立司參數 (Ellis Parameters) 可視為常數。在 $k\leqslant \mathbb{R}\leqslant\lambda$ 範圍內：邊界條件(1)在 $\mathbb{R}=k$ 處，$Vz=0$；(2)在 $\mathbb{R}=\lambda$ 處，$Vz=Vz_{max}$。流速分佈為

$$Vz_1=\frac{\tau_R R}{1-\lambda^2}\{\varphi_0(\frac{k^2-\mathbb{R}^2}{2}+\lambda^2\ell n\frac{\mathbb{R}}{k})+\varphi_1(\frac{\tau_R}{1-\lambda^2})^{\alpha-1}[\frac{(-\lambda^2)^{\alpha}}{1-\alpha}(k^{1-\alpha}-\mathbb{R}^{1-\alpha})$$

$$+\frac{\alpha(-\lambda^2)^{\alpha-1}}{3-\alpha}(k^{3-\alpha}-\mathbb{R}^{3-\alpha})+\frac{\alpha(\alpha-1)(-\lambda^2)^{\alpha-2}}{(2!)(5-\alpha)}(k^{5-\alpha}-\mathbb{R}^{5-\alpha})$$

$$+\cdots]\} \qquad k\leqslant\mathbb{R}\leqslant\lambda \tag{27}$$

$$Vz_{max}=\frac{\tau_R R}{1-\lambda^2}\{\varphi_0(\frac{k^2-\lambda^2}{2}-\lambda^2\ell n\frac{k}{\lambda})+\varphi_1(\frac{\tau_R}{1-\lambda^2})^{\alpha-1}[\frac{(-\lambda^2)^{\alpha}}{1-\alpha}(k^{1-\alpha}-\lambda^{1-\alpha})$$

$$+\frac{\alpha(-\lambda^2)^{\alpha-1}}{3-\alpha}(k^{3-\alpha}-\lambda^{3-\alpha})+\frac{\alpha(\alpha-1)(-\lambda^2)^{\alpha-2}}{(2!)(5-\alpha)}(k^{5-\alpha}-\lambda^{5-\alpha})$$

$$+\cdots]\} \qquad \mathbb{R}=\lambda \tag{28}$$

在 $\lambda\leqslant\mathbb{R}\leqslant 1$ 範圍內：邊界條件(1)在 $\mathbb{R}=1$ 處，$Vz=0$；(2)在 $\mathbb{R}=\lambda$ 處，$Vz=Vz_{max}$。流速分佈為

$$Vz_2=\frac{\tau_R R}{1-\lambda^2}\{\varphi_0(\frac{1-\mathbb{R}^2}{2}+\lambda^2\ell n\mathbb{R})+\varphi_1(\frac{\tau_R}{1-\lambda^2})^{\alpha-1}[\frac{1}{\alpha+1}(1-\mathbb{R}^{\alpha+1})+\frac{\alpha(-\lambda^2)}{\alpha-1}$$

$$(1-\mathbb{R}^{\alpha-1})+\frac{\alpha(\alpha-1)(-\lambda^2)^2}{(2!)(\alpha-3)}(1-\mathbb{R}^{\alpha-3})+\frac{\alpha(\alpha-1)(\alpha-2)(-\lambda^2)^3}{(3!)(\alpha-5)}(1-\mathbb{R}^{\alpha-5})$$

$$+\cdots]\} \qquad \lambda\leqslant\mathbb{R}\leqslant 1 \tag{29}$$

$$Vz_{\max}=\frac{\tau_R R}{1-\lambda^2}\{\varphi_0(\frac{1-\lambda^2}{2}+\lambda^2\ell n\lambda)+\varphi_1(\frac{\tau_R}{1-\lambda^2})^{\alpha-1}[\frac{1}{\alpha+1}(1-\lambda^{\alpha+1})+\frac{\alpha(-\lambda^2)}{\alpha-1}$$

$$(1-\lambda^{\alpha-1})+\frac{\alpha(\alpha-1)(-\lambda^2)^2}{(2!)(\alpha-3)}(1-\lambda^{\alpha-3})+\frac{\alpha(\alpha-1)(\alpha-2)(-\lambda^2)^3}{(3!)(\alpha-5)}(1-\lambda^{\alpha-5})$$

$$+\cdots]\} \qquad \mathbb{R}=\lambda \tag{30}$$

由 (28) 與 (30) 兩式相等可得 λ 值。體積流量為

$$\dot{V}=\int_0^{2\pi}\int_{kR}^{\lambda R}Vz_1 rdrd\phi+\int_0^{2\pi}\int_{\lambda R}^{R}Vz_2 rdrd\phi$$

$$=2\pi R^3(\frac{\tau_R}{1-\lambda^2})\{\varphi_0[\frac{k^2}{4}(\lambda^2-k^2)+\frac{1-\lambda^2}{4}+\frac{k^4-1}{8}+\lambda^2(\frac{k^2-\lambda^2}{4}-\frac{1-\lambda^2}{4}-\frac{\lambda^2}{2}\ell nk)]$$

$$+\varphi_1(\frac{\tau_R}{1-\lambda^2})^{\alpha-1}[\frac{(-\lambda^2)^2}{1-\alpha}(\frac{k^{1-\alpha}\lambda^2-k^{3-\alpha}}{2}-\frac{\lambda^{3-\alpha}-k^{3-\alpha}}{3-\alpha})+\frac{\alpha(-\lambda^2)^{\alpha-1}}{3-\alpha}$$

$$(\frac{k^{3-\alpha}\lambda^2-k^{5-\alpha}}{2}-\frac{\lambda^{5-\alpha}-k^{5-\alpha}}{5-\alpha})+\frac{\alpha(\alpha-1)(-\lambda^2)^{\alpha-2}}{(2!)(5-\alpha)}(\frac{k^{5-\alpha}\lambda^2-k^{7-\alpha}}{2}$$

$$-\frac{\lambda^{7-\alpha}-k^{7-\alpha}}{7-\alpha})+\cdots+\frac{1}{\alpha+1}(\frac{1-\lambda^2}{2}-\frac{1-\lambda^{\alpha+3}}{\alpha+3})+\frac{\alpha(-\lambda^2)}{\alpha-1}(\frac{1-\lambda^2}{2}-\frac{1-\lambda^{\alpha+1}}{\alpha+1})$$

$$+\frac{\alpha(\alpha-1)(-\lambda^2)^2}{(2!)(\alpha-3)}(\frac{1-\lambda^2}{2}-\frac{1-\lambda^{\alpha-1}}{\alpha-1})+\cdots]\} \tag{31}$$

平均流速為

$$Vm=(\frac{2R}{1-k^2})(\frac{\tau_R}{1-\lambda^2})\{\varphi_0[\frac{k^2}{4}(\lambda^2-k^2)+\frac{1-\lambda^2}{4}+\frac{k^4-1}{8}+\lambda^2(\frac{k^2-\lambda^2}{4}-\frac{\lambda^2}{2}\ell nk-\frac{1-\lambda^2}{4})]$$

$$+\varphi_1(\frac{\tau_R}{1-\lambda^2})^{\alpha-1}[\frac{(-\lambda^2)^2}{1-\alpha}(\frac{k^{1-\alpha}\lambda^2-k^{3-\alpha}}{2}-\frac{\lambda^{3-\alpha}-k^{3-\alpha}}{3-\alpha})+\frac{\alpha(-\lambda^2)^{\alpha-1}}{3-\alpha}$$

$$(\frac{k^{3-\alpha}\lambda^2-k^{5-\alpha}}{2}-\frac{\lambda^{5-\alpha}-k^{5-\alpha}}{5-\alpha})+\frac{\alpha(\alpha-1)(-\lambda^2)^{\alpha-2}}{(2!)(5-\alpha)}(\frac{k^{5-\alpha}\lambda^2-k^{7-\alpha}}{2}$$

$$-\frac{\lambda^{7-\alpha}-k^{7-\alpha}}{7-\alpha})+\cdots+\frac{1}{\alpha+1}(\frac{1-\lambda^2}{2}-\frac{1-\lambda^{\alpha+3}}{\alpha+3})+\frac{\alpha(-\lambda^2)}{\alpha-1}(\frac{1-\lambda^2}{2}-\frac{1-\lambda^{\alpha+1}}{\alpha+1})$$

$$+\frac{\alpha(\alpha-1)(-\lambda^2)^2}{(2!)(\alpha-3)}(\frac{1-\lambda^2}{2}-\frac{1-\lambda^{\alpha-1}}{\alpha-1})+\cdots]\} \qquad (32)$$

三、等溫熱源對非牛頓系液體作短時間之熱傳

非牛頓系液體在長圓套管夾層內流動已達充分發展層流後，自 $z=0$ 處起繞以等溫熱源，使管壁與液體在 $z<0$ 處溫度仍為 T_0（無熱流影響）。有熱源部份即 $z\geqslant 0$ 處內管壁溫度均為T_1，外管壁溫度均為 T_2 。若軸向傳導熱流與對流熱流相較是非常小（通常如此），其流動能量微分方程式為（圖 3）[1][2][8]

$$\rho c_P V_Z\frac{\partial T}{\partial z}=\frac{k}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) \qquad (33)$$

式中 ρ 為液體密度，c_P 為比熱，k 為熱傳導係數，取在流動平均溫度（flow-average temperature）之值，均可視為常數[1][2][8]。

(一)賓漢型：以流速分佈代入(33)式解之。因接觸時間短受熱流影響部份不會很深，設僅 $kR\leqslant r\leqslant\lambda_1 R$ 及 $\lambda_2 R\leqslant r\leqslant R$ 部份受影響。

1. 在 $kR\leqslant r\leqslant\lambda_1 R$ 範圍內：將(4)式代入(33)式：

$$\rho c_P\{\frac{(P_0-P_L)R^2}{2\mu_0 L}[\frac{1}{2}(k^2-\frac{r^2}{k^2})$$

$$+\lambda^2\ell n\frac{r}{kR}]-\frac{\tau_0 R}{\mu_0}(\frac{r}{R}-k)\}\frac{\partial T}{\partial z}$$

$$=\frac{k}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) \qquad (34)$$

取自內管外壁向外座標 $w=r-kR$，而$\frac{w}{kR}$為小值，於是

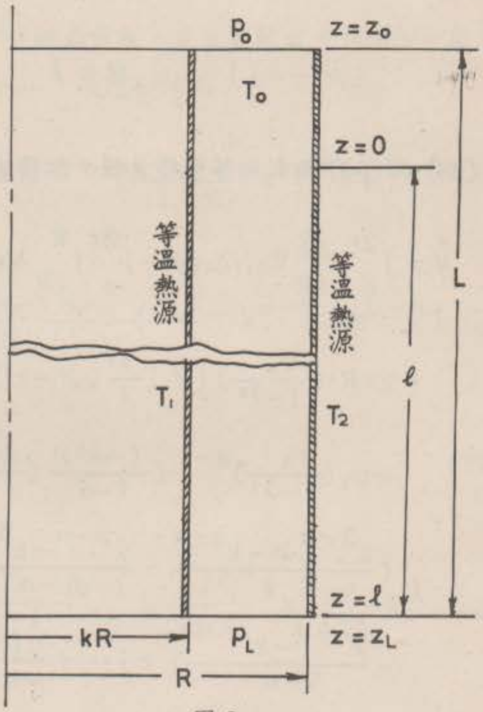

圖 3

$$k^2-\frac{r^2}{R^2}=k^2-(\frac{w}{R}+k)^2\simeq-2k\frac{w}{R}$$

$$\ell n\frac{r}{kR}=\ell n(\frac{w}{kR}+1)=\frac{w}{kR}-\frac{1}{2}(\frac{w}{kR})^2+\frac{1}{3}(\frac{w}{kR})^3-\cdots\simeq\frac{w}{kR}$$

(34) 式簡略為

$$\rho c_P\{\frac{-(P_0-P_L)R^2}{2\mu_0 L}(\frac{w}{R})(k-\frac{\lambda^2}{k})-\frac{\tau_0 R}{\mu_0}(\frac{w}{R})\}\frac{\partial T}{\partial z}=k\frac{\partial^2 T}{\partial w^2} \qquad (35)$$

因內管外壁處剪應力為 $\tau_{KR}=\dfrac{(P_o-P_L)R}{2L}(k-\dfrac{\lambda^2}{k})$，故 (35) 式化為

$$-\frac{\rho c_P}{\mu_o}(\tau_{KR}+\tau_o)w\frac{\partial T}{\partial z}=k\frac{\partial^2 T}{\partial w^2} \tag{36}$$

設下列無單位量群 (dimensionless groups) 為新變數：

$$\theta=\frac{T-T_0}{T_1-T_0}\ ,\quad Z=\frac{z}{R}\ ,\quad W=\frac{w}{R}\ ,\quad N=\frac{\rho c_P}{k}\left(\frac{-\tau_{KR}-\tau_o}{\mu_o}\right)R^2$$

則

$$N\frac{\partial\theta}{\partial Z}=\frac{1}{W}\frac{\partial^2\theta}{\partial W^2} \tag{37}$$

設有一解 $\theta=f(\eta)$，而令 $\eta=(\frac{N}{9Z})^{1/3}W$，(37) 式化為

$$\frac{d^2f}{d\eta^2}+3\eta^2\frac{df}{d\eta}=0 \tag{38}$$

邊界條件(1)在 $w=0$ 處 ($W=0$, $\eta=0$)，$T=T_1$, $f=1$; (2)在 $w=w_o$, $z=z_o'$ 處，$T=T_o$, $f=0$。(內管表面在 $z<0$ 處，$T=T_o$, $z_o'>0$) 得溫度分佈為[8]

$$\theta=\frac{T-T_0}{T_1-T_0}=1-\frac{1}{\psi(\eta_o)}\left(\eta-\frac{3\cdot1}{3\cdot4}\eta^4+\frac{3\cdot4\cdot3\cdot1}{6\cdot7\cdot3\cdot4}\eta^7-\frac{3\cdot7\cdot3\cdot4\cdot3\cdot1}{9\cdot10\cdot6\cdot7\cdot3\cdot4}\eta^{10}+\cdots\right) \tag{39}$$

式中 $\psi(\eta_o)=\left(\eta_o-\frac{3\cdot1}{3\cdot4}\eta_o^4+\frac{3\cdot4\cdot3\cdot1}{6\cdot7\cdot3\cdot4}\eta_o^7-\frac{3\cdot7\cdot3\cdot4\cdot3\cdot1}{9\cdot10\cdot6\cdot7\cdot3\cdot4}\eta_o^{10}+\cdots\right)$

而令 $W_o=\frac{w_o}{R}$，$Z_o=\frac{z_o'}{R}$，$\eta_o=(\frac{N}{9Z_o})^{1/3}W_o$。

單位面積熱流率 (heat flux) 為

$$q=-k\frac{\partial T}{\partial w}\Big|_{w=0}=-k\frac{\partial T}{\partial\eta}\frac{\partial\eta}{\partial w}\Big|_{w=0}=-k(T_1-T_0)\frac{df}{d\eta}\frac{\partial\eta}{\partial w}\Big|_{w=0}$$

$$=\frac{k(T_1-T_0)}{\psi(\eta_o)}\left(\frac{N}{9Z}\right)^{1/3}\left(\frac{1}{R}\right)=k^{2/3}(T_1-T_0)\frac{1}{\psi(\eta_o)}\left[\frac{\rho c_P}{9\mu_o z}(-\tau_{KR}-\tau_o)\right]^{1/3} \tag{40}$$

夾層內面總熱流率為

$$Q=\int_0^{\ell}q2\pi Rdz=3\pi Rk^{2/3}(T_1-T_0)\frac{1}{\psi(\eta_o)}\left[\frac{\rho c_P}{9\mu_o}(-\tau_{KR}-\tau_o)\right]^{1/3}(\ell)^{2/3} \tag{41}$$

2 在 $\lambda_2R\leqslant r\leqslant R$ 範圍內：將(6)式代入 (33) 式：

$$\rho c_P\left\{\frac{(P_o-P_L)R^2}{2\mu_oL}\left[\frac{1}{2}\left(1-\frac{r^2}{R^2}\right)+\lambda^2\ell n\frac{r}{R}\right]+\frac{\tau_oR}{\mu_o}\left(\frac{r}{R}-1\right)\right\}\frac{\partial T}{\partial z}$$

$$=\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{42}$$

取自外管內壁向內座標 $s = R - r$，而 $\frac{s}{R}$ 為小值，則

$$1 - \frac{r^2}{R^2} = 1 - (1 - \frac{s}{R})^2 \simeq 2\frac{s}{R}$$

$$\ell n\frac{r}{R} = \ell n(1 - \frac{s}{R}) = -\frac{s}{R} - \frac{1}{2}(-\frac{s}{R})^2 + \frac{1}{3}(-\frac{s}{R})^3 - \cdots \simeq -\frac{s}{R}$$

(42) 式簡略為

$$\rho c_P\{\frac{(P_0-P_L)R^2}{2\mu_0 L}(\frac{s}{R} - \lambda^2\frac{s}{R}) - \frac{\tau_0 R}{\mu_0}(\frac{s}{R})\}\frac{\partial T}{\partial z} = k\frac{\partial^2 T}{\partial s^2}$$

即

$$\frac{\rho c_P}{\mu_0}(\tau_R - \tau_0)\, s\frac{\partial T}{\partial z} = k\frac{\partial^2 T}{\partial s^2} \tag{43}$$

設下列無單位量群為新變數：

$$\theta = \frac{T - T_0}{T_2 - T_0}, \quad Z = \frac{z}{R}, \quad S = \frac{s}{R}, \quad N = \frac{\rho c_P}{k}(\frac{\tau_R - \tau_0}{\mu_0})R^2$$

則

$$N\frac{\partial\theta}{\partial Z} = \frac{1}{S}\frac{\partial^2\theta}{\partial S^2} \tag{44}$$

令 $\eta = (\frac{N}{9Z})^{1/3} S$，(44) 式亦化為

$$\frac{d^2 f}{d\eta^2} + 3\eta^2\frac{df}{d\eta} = 0$$

邊界條件(1)在 $s = 0$ ($S = 0$，$\eta = 0$) 處，$T = T_2$，$f = 1$；(2)在 $s = s_0$，$z = z_0'$ 處，$T = T_0$，$f = 0$。(外管內面在 $z < 0$ 處 $T = T_0$，$z_0' > 0$) 得溫度分佈與 (39) 式相同，但 $\eta_0 = (\frac{N}{9Z_0})^{1/3} S_0$，而令 $S_0 = \frac{s_0}{R}$， $Z_0 = \frac{z_0'}{R}$。單位面積熱流率為

$$q = k^{2/3}(T_2 - T_0)\frac{1}{\psi(\eta_0)}[\frac{\rho c_P}{9\mu_0 z}(\tau_R - \tau_0)]^{1/3} \tag{45}$$

夾層外面總熱流率為

$$Q = 3\pi R k^{2/3}(T_2 - T_0)\frac{1}{\psi(\eta_0)}[\frac{\rho c_P}{9\mu_0}(\tau_R - \tau_0)]^{1/3}\ell^{2/3} \tag{46}$$

(二)奧斯瓦第瓦爾型：以流速分佈代入 (33) 式解之。

1. 在 $k \leqslant \mathbb{R} \leqslant 1$ 範圍內：將 (20) 式代入 (33) 式：

$$\frac{\rho c_P(P_0-P_L)^{\frac{1}{n}} R^{\frac{1}{n}} R}{(2Lm)^{1/n}}[\frac{(-\lambda^2)^{\frac{1}{n}}}{1-\frac{1}{n}}(k^{1-\frac{1}{n}} - \mathbb{R}^{1-\frac{1}{n}}) + \frac{\frac{1}{n}(-\lambda^2)^{\frac{1}{n}-1}}{3-\frac{1}{n}}(k^{3-\frac{1}{n}} - \mathbb{R}^{3-\frac{1}{n}})$$

$$+ \frac{\frac{1}{n}(\frac{1}{n}-1)(-\lambda^2)^{\frac{1}{n}-2}}{(2!)(5-\frac{1}{n})}(k^{5-\frac{1}{n}} - \mathbb{R}^{5-\frac{1}{n}}) + \frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)(-\lambda^2)^{\frac{1}{n}-3}}{(3!)(7-\frac{1}{n})}$$

$$(k^{7-\frac{1}{n}}-\mathbb{R}^{7-\frac{1}{n}})+\cdots]\frac{\partial T}{\partial z}$$

$$=\frac{k}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) \qquad (47)$$

因接觸時間短受熱流影響不會很深，取自內管外壁向外座標 $w=r-kR$，而 $\frac{w}{kR}$ 為小值，於是

$$\mathbb{R}^{1-\frac{1}{n}}=(k+\frac{w}{R})^{1-\frac{1}{n}}=k^{1-\frac{1}{n}}(1+\frac{w}{kR})^{1-\frac{1}{n}}\simeq k^{1-\frac{1}{n}}[1+(1-\frac{1}{n})\frac{w}{kR}]$$

$$k^{1-\frac{1}{n}}-\mathbb{R}^{1-\frac{1}{n}}\simeq-(1-\frac{1}{n})(\frac{w}{kR})k^{1-\frac{1}{n}}$$

$$k^{3-\frac{1}{n}}-\mathbb{R}^{3-\frac{1}{n}}\simeq-(3-\frac{1}{n})(\frac{w}{kR})k^{3-\frac{1}{n}}$$

$$k^{5-\frac{1}{n}}-\mathbb{R}^{5-\frac{1}{n}}\simeq-(5-\frac{1}{n})(\frac{w}{kR})k^{5-\frac{1}{n}}$$

$$k^{7-\frac{1}{n}}-\mathbb{R}^{7-\frac{1}{n}}\simeq-(7-\frac{1}{n})(\frac{w}{kR})k^{7-\frac{1}{n}}$$

....

而 (47) 式簡略為

$$-\frac{\rho c_P(P_0-P_L)^{\frac{1}{n}}R^{\frac{1}{n}}}{(2Lm)^{1/n}}(\frac{-\lambda^2}{k})^{\frac{1}{n}}w[1+\frac{1}{n}(\frac{k^2}{-\lambda^2})+\frac{\frac{1}{n}(\frac{1}{n}-1)}{2!}(\frac{k^2}{-\lambda^2})^2+\frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)}{3!}$$

$$(\frac{k^2}{-\lambda^2})^3+\cdots]\frac{\partial T}{\partial z}=k\frac{\partial^2 T}{\partial w^2}$$

即 $$-\rho c_P(\frac{\tau_{KR}}{m})^{\frac{1}{n}}w\frac{\partial T}{\partial z}=k\frac{\partial^2 T}{\partial w^2} \qquad (48)$$

設下列無單位量群為新變數：

$$\theta=\frac{T-T_0}{T_1-T_0},\quad Z=\frac{z}{R},\quad W=\frac{w}{R},\quad N=-\frac{\rho c_P}{k}(\frac{\tau_{KR}}{m})^{\frac{1}{n}}(R^2)$$

則 $$N\frac{\partial\theta}{\partial Z}=\frac{1}{W}\frac{\partial^2\theta}{\partial W^2} \qquad (49)$$

與 (37) 式相同，亦得與 (39) 式相同之溫度分佈。單位面積熱流率則為

$$q = \frac{k(T_1-T_0)}{\psi(\eta_0)}\left(\frac{N}{9Z}\right)^{1/3}\left(\frac{1}{R}\right) = k^{2/3}(T_1-T_0)\frac{1}{\psi(\eta_0)}\left[-\frac{\rho c_P}{9z}\left(\frac{\tau_{KR}}{m}\right)^{\frac{1}{n}}\right]^{1/3} \tag{50}$$

夾層內面總熱流率為

$$Q = 3\pi R k^{2/3}(T_1-T_0)\frac{1}{\psi(\eta_0)}\left[-\frac{\rho c_P}{9}\left(\frac{\tau_{KR}}{m}\right)^{\frac{1}{n}}\right]^{1/3} \ell^{2/3} \tag{51}$$

2. 在 $\lambda \leqslant \mathbb{R} \leqslant 1$ 範圍內：將(22)式代入(33)式：

$$\frac{\rho c_P(P_0-P_L)^{\frac{1}{n}} R^{\frac{1}{n}} R}{(2Lm)^{1/n}}\Big[\frac{1}{\frac{1}{n}+1}(1-\mathbb{R}^{\frac{1}{n}+1}) + \frac{(-\lambda^2)(\frac{1}{n})}{\frac{1}{n}-1}(1-\mathbb{R}^{\frac{1}{n}-1})$$

$$+\frac{(-\lambda^2)^2(\frac{1}{n})(\frac{1}{n}-1)}{(2!)(\frac{1}{n}-3)}(1-\mathbb{R}^{\frac{1}{n}-3}) + \frac{(-\lambda^2)^3(\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)}{(3!)(\frac{1}{n}-5)}(1-\mathbb{R}^{\frac{1}{n}-5})$$

$$+\frac{(-\lambda^2)^4(\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)(\frac{1}{n}-3)}{(4!)(\frac{1}{n}-7)}(1-\mathbb{R}^{\frac{1}{n}-7}) + \cdots\Big]\frac{\partial T}{\partial z}$$

$$= \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{52}$$

同理取自外管內壁向內座標 $s = R - r$，而 $\frac{s}{R}$ 為小值，則

$$\mathbb{R}^{\frac{1}{n}+1} = \left(1-\frac{s}{R}\right)^{\frac{1}{n}+1} \simeq 1-\left(\frac{1}{n}+1\right)\frac{s}{R}$$

$$1-\mathbb{R}^{\frac{1}{n}+1} \simeq \left(\frac{1}{n}+1\right)\frac{s}{R}$$

$$1-\mathbb{R}^{\frac{1}{n}-1} = 1-\left(1-\frac{s}{R}\right)^{\frac{1}{n}-1} \simeq 1-\left[1-\left(\frac{1}{n}-1\right)\frac{s}{R}\right] = \left(\frac{1}{n}-1\right)\frac{s}{R}$$

$$1-\mathbb{R}^{\frac{1}{n}-3} \simeq \left(\frac{1}{n}-3\right)\frac{s}{R}$$

$$1-\mathbb{R}^{\frac{1}{n}-5} \simeq \left(\frac{1}{n}-5\right)\frac{s}{R}$$

$$1-\mathbb{R}^{\frac{1}{n}-7} \simeq \left(\frac{1}{n}-7\right)\frac{s}{R}$$

$\cdots$

而(52)式簡略為

$$\frac{\rho c_P (P_0-P_L)^{\frac{1}{n}} R^{\frac{1}{n}} R}{(2Lm)^{1/n}} \left[\frac{s}{R} + \frac{(-\lambda^2)}{n} (\frac{s}{R}) + \frac{(-\lambda^2)^2 (\frac{1}{n})(\frac{1}{n}-1)}{2!} (\frac{s}{R}) \right.$$

$$+ \frac{(-\lambda^2)^3 (\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)}{3!} (\frac{s}{R}) + \frac{(-\lambda^2)^4 (\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)(\frac{1}{n}-3)}{4!} (\frac{s}{R})$$

$$\left. + \cdots \right] \frac{\partial T}{\partial z} = k \frac{\partial^2 T}{\partial s^2}$$

即 $$\rho c_P (\frac{\tau_R}{m})^{\frac{1}{n}} (s)(\frac{\partial T}{\partial z}) = k \frac{\partial^2 T}{\partial s^2} \qquad (53)$$

設下列無單位量群為新變數：

$$\theta = \frac{T - T_0}{T_2 - T_0}, \quad Z = \frac{z}{R}, \quad S = \frac{s}{R}, \quad N = \frac{\rho c_P}{k} (\frac{\tau_R}{m})^{\frac{1}{n}} (R^2)$$

則 $$N \frac{\partial \theta}{\partial S} = \frac{1}{S} \frac{\partial^2 \theta}{\partial S^2} \qquad (54)$$

與(44)式相同，亦得與(39)式相同之溫度分佈。單位面積熱流率則為

$$q = \frac{k(T_2-T_0)}{\phi(\eta_0)} (\frac{N}{9Z})^{1/3} (\frac{1}{R}) = k^{2/3} (T_2-T_0) \frac{1}{\phi(\eta_0)} \left[\frac{\rho c_P}{9z} (\frac{\tau_R}{m})^{\frac{1}{n}} \right]^{1/3} \qquad (55)$$

夾層外面總熱流率為

$$Q = 3\pi R k^{2/3} (T_2-T_0) \frac{1}{\phi(\eta_0)} \left[\frac{\rho c_P}{9} (\frac{\tau_R}{m})^{\frac{1}{n}} \right]^{1/3} (\ell^{2/3}) \qquad (56)$$

(三)意立司型：以流速分佈代入(33)式解之。

1.在 $k \leq \mathbb{R} \leq \lambda$ 範圍內：將(27)式代入(33)式

$$\rho c_P \left\{ \frac{\varphi_0 (P_0-P_L) R^2}{2L} (\frac{k^2-\mathbb{R}^2}{2} + \lambda^2 \ell n \frac{\mathbb{R}}{k}) + \frac{\varphi_1 (P_0-P_L)^{\alpha} R^{\alpha+1}}{(2L)^{\alpha}} \left[\frac{(-\lambda^2)^{\alpha}}{1-\alpha} (k^{1-\alpha} - \mathbb{R}^{1-\alpha}) \right. \right.$$

$$\left. \left. + \frac{\alpha(-\lambda^2)^{\alpha-1}}{3-\alpha} (k^{3-\alpha} - \mathbb{R}^{3-\alpha}) + \frac{\alpha(\alpha-1)(-\lambda^2)^{\alpha-2}}{(2!)(5-\alpha)} (k^{5-\alpha} - \mathbb{R}^{5-\alpha}) + \cdots \right] \right\} \frac{\partial T}{\partial z}$$

$$= \frac{k}{r} \frac{\partial}{\partial r} (r \frac{\partial T}{\partial r}) \qquad (57)$$

亦因接觸時間短，受熱流影響不會很深，取自內管外壁向外座標$w = r - kR$，而$\frac{w}{kR}$為小值，則

$$\frac{1}{2}(k^2-\mathbb{R}^2) = \frac{1}{2}[k^2-k^2(1+\frac{w}{kR})^2] \simeq -\frac{w}{kR}$$

$$\lambda^2 \ell n \frac{\mathbb{R}}{k} = \lambda^2 \ell n (1+\frac{w}{kR}) = \lambda^2 [\frac{w}{kR} - \frac{1}{2}(\frac{w}{kR})^2 + \frac{1}{3}(\frac{w}{kR})^3 - \cdots] \simeq \lambda^2 (\frac{w}{kR})$$

$$k^{1-\alpha} - \mathbb{R}^{1-\alpha} = k^{1-\alpha} - [k(1+\frac{w}{kR})]^{1-\alpha} \simeq - k^{1-\alpha}(1-\alpha)\frac{w}{kR}$$

$$k^{3-\alpha} - \mathbb{R}^{3-\alpha} \simeq - k^{3-\alpha}(3-\alpha)\frac{w}{kR}$$

$$k^{5-\alpha} - \mathbb{R}^{5-\alpha} \simeq - k^{5-\alpha}(5-\alpha)\frac{w}{kR}$$

$$\cdots\cdots$$

而(57)式簡略為

$$-\rho c_P(\varphi_0 \tau_{KR} + \varphi_1 \tau_{KR}^{\alpha}) w \frac{\partial T}{\partial z} = k \frac{\partial^2 T}{\partial w^2} \tag{58}$$

設下列無單位量群為新變數：

$$\theta = \frac{T - T_0}{T_1 - T_0}, \quad Z = \frac{z}{R}, \quad W = \frac{w}{R}, \quad N = -(\varphi_0 \tau_{KR} + \varphi_1 \tau_{KR}^{\alpha}) \frac{\rho c_P R^2}{k}$$

則 $$N \frac{\partial \theta}{\partial Z} = \frac{1}{W} \frac{\partial^2 \theta}{\partial W^2} \tag{59}$$

亦與(37)式相同，而得與(39)式相同之溫度分佈。單位面積熱流率則為

$$q = \frac{k(T_1-T_0)}{\psi(\eta_0)} (\frac{N}{9Z})^{1/3} (\frac{1}{R}) = k^{2/3} (T_1-T_0) \frac{1}{\psi(\eta_0)} [-(\varphi_0 \tau_{KR} + \varphi_1 \tau_{KR}^{\alpha}) \frac{\rho c_P}{9z}]^{1/3} \tag{60}$$

夾層內面總熱流率為

$$Q = 3\pi R k^{2/3} (T_1-T_0) \frac{1}{\psi(\eta_0)} [-(\varphi_0 \tau_{KR} + \varphi_1 \tau_{KR}^{\alpha}) \frac{\rho c_P}{9}]^{1/3} \ell^{2/3} \tag{61}$$

2. 在 $\lambda \leqslant \mathbb{R} \leqslant 1$ 範圍內：將(29)式代入(33)式

$$\rho c_P \{ \frac{\varphi_0 (P_0-P_L) R^2}{2L} [\frac{1}{2}(1-\mathbb{R}^2) + \lambda^2 \ell n \mathbb{R}] + \frac{\varphi_1 (P_0-P_L)^{\alpha} R^{\alpha} R}{(2L)^{\alpha}} [\frac{1}{\alpha+1}(1-\mathbb{R}^{\alpha+1})$$
$$+ \frac{(-\lambda^2)(\alpha)}{\alpha-1}(1-\mathbb{R}^{\alpha-1}) + \frac{(-\lambda^2)^2(\alpha)(\alpha-1)}{(2!)(\alpha-3)}(1-\mathbb{R}^{\alpha-3}) + \frac{(-\lambda^2)^3(\alpha)(\alpha-1)(\alpha-2)}{(3!)(\alpha-5)}$$
$$(1-\mathbb{R}^{\alpha-5}) + \cdots] \} \frac{\partial T}{\partial z} = \frac{k}{r} \frac{\partial}{\partial r}(r \frac{\partial T}{\partial r}) \tag{62}$$

同理取自外管內壁向內座標 $s = R - r$，而 $\frac{s}{R}$ 為小值，則

$$1-\mathbb{R}^2 = 1-(1-\frac{s}{R})^2 \simeq \frac{2s}{R}$$

$$\ell n\mathbb{R} = \ell n(1-\frac{s}{R}) = -\frac{s}{R}-\frac{1}{2}(-\frac{s}{R})^2+\frac{1}{3}(-\frac{s}{R})^3-\cdots \simeq -\frac{s}{R}$$

$$1-\mathbb{R}^{\alpha+1} = 1-(1-\frac{s}{R})^{\alpha+1} \simeq (\alpha+1)\frac{s}{R}$$

$$1-\mathbb{R}^{\alpha-1} \simeq (\alpha-1)\frac{s}{R}$$

$$1-\mathbb{R}^{\alpha-3} \simeq (\alpha-3)\frac{s}{R}$$

$$1-\mathbb{R}^{\alpha-5} \simeq (\alpha-5)\frac{s}{R}$$

....

而(62)式簡略為

$$\rho c_P(\varphi_0\tau_R+\varphi_1\tau_R^{\alpha})\, s\frac{\partial T}{\partial z} = k\frac{\partial^2 T}{\partial s^2} \qquad (63)$$

設下列無單位量群為新變數：

$$\theta = \frac{T-T_0}{T_2-T_0}, \quad Z = \frac{z}{R}, \quad S = \frac{s}{R}, \quad N = \frac{\rho c_P}{k}(\varphi_0\tau_R+\varphi_1\tau_R^{\alpha})R^2$$

則 $$N\frac{\partial\theta}{\partial Z} = \frac{1}{S}\frac{\partial^2\theta}{\partial S^2} \qquad (64)$$

亦與(44)式相同，而得與(39)式相同之解。單位面積熱流率為

$$q = \frac{k(T_2-T_0)}{\phi(\eta_0)}(\frac{N}{9Z})^{1/3}(\frac{1}{R}) = k^{2/3}(T_2-T_0)\frac{1}{\phi(\eta_0)}[\frac{\rho c_P}{9z}(\varphi_0\tau_R+\varphi_1\tau_R^{\alpha})]^{1/3} \qquad (65)$$

夾層外面總熱流率為

$$Q = 3\pi Rk^{2/3}(T_2-T_0)\frac{1}{\phi(\eta_0)}[\frac{\rho c_P}{9}(\varphi_0\tau_R+\varphi_1\tau_R^{\alpha})]^{1/3}\ell^{2/3} \qquad (66)$$

四、等強熱源對非牛頓系流體之熱傳

非牛頓系液體在長圓套管夾層內流動已達充分發展層流後，自 $z=0$ 處起，內管內有等強熱源 q_1，外管外繞以等強熱源 q_2。使管壁與液體在 $z<0$ 處溫度仍為 T_0（無熱流影響）。有熱源部份，即 $z>0$ 處，液體始受熱流影響，溫度遂有變化。因係等強熱源，其熱流率可保持一定，q_1 ＝常數，q_2 ＝常數。又因流體性質視為常數[1][2]，而有熱源部份管亦很長，使流體溫度在軸向呈線性變化[2]。由穩態能量平衡，在 z 很大處：（圖4）

$$2\pi R(kq_1+q_2)dz=\rho\pi R^2(1-k^2)v_m c_P\frac{\partial T}{\partial z}dz$$

故 $$\frac{\partial T}{\partial z}=\frac{2(kq_1+q_2)}{\rho R(1-k^2)v_m c_P}=\text{常數} \qquad (67)$$

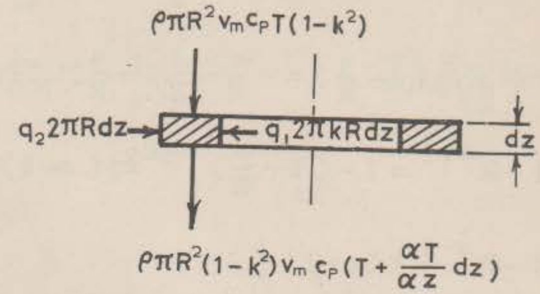

圖 4

現按各型非牛頓系液體，分別分析其溫度分佈情況如下：

(一)賓漢型：其流速分佈以$\mathbb{R}=\lambda_1$及$\mathbb{R}=\lambda_2$兩處為分界，而在$\lambda_1\leqslant\mathbb{R}\leqslant\lambda_2$範圍內為等速流動。若溫度分佈亦以$\mathbb{R}=\lambda_1$及$\mathbb{R}=\lambda_2$兩處為分界，令此兩處之溫度為$T_{\lambda_1 R}$及$T_{\lambda_2 R}$，均僅為z之函數。可分析距z = 0相當遠處沿徑向之溫度分佈：

1. 在$kR\leqslant r\leqslant\lambda_1 R$範圍內：將(4)式及(67)式代入(33)式

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{2(kq_1+q_2)r}{k(1-k^2)v_m}\left\{\frac{(P_o-P_L)R}{2\mu_o L}\left[\frac{1}{2}k^2-\frac{1}{2}\left(\frac{r}{R}\right)^2+\lambda^2\ell n\frac{r}{kR}\right]-\frac{\tau_o}{\mu_o}\left(\frac{r}{R}-k\right)\right\} \qquad (68)$$

邊界條件(1)在$r=kR$處，$-k\dfrac{\partial T}{\partial r}=-q_1=$常數；(2)在$r=\lambda_1 R$處，$T=T_{\lambda_1 R}$。得其溫度分佈為

$$T-T_{\lambda_2 R}=\frac{2(kq_1+q_2)}{\mu_o k v_m(1-k^2)}\left\{\frac{\tau_R}{1-\lambda_2}\left[\frac{k^2\lambda_1^2R^2}{8}\left(\frac{r^2}{\lambda_1^2R^2}-1\right)-\frac{\lambda_1^4R^2}{32}\left(\frac{r^4}{\lambda_1^4R^4}-1\right)+\frac{\lambda^2}{4}\lambda_1^2R^2\left(\frac{r^2}{\lambda_1^2R^2}\ell n\frac{r}{kR}-\ell n\frac{\lambda_1}{k}+1-\frac{r^2}{\lambda_1^2R^2}\right)\right]-\tau_o\left[\frac{\lambda_1^3R^2}{9}\left(\frac{r^3}{\lambda_1^3R^3}-1\right)-\frac{k\lambda_1^2R^2}{4}\left(\frac{r^2}{\lambda_1^2R^2}-1\right)\right]\right\}+\frac{(kq_1+q_2)kR}{k}\left\{\frac{q_1}{kq_1+q_2}-\frac{kR}{\mu_o v_m(1-k^2)}\left[\frac{\tau_{KR}}{k-\frac{\lambda^2}{k}}\left(\frac{k^2}{4}-\frac{\lambda^2}{2}\right)+\frac{\tau_o k}{3}\right]\right\}\ell n\frac{r}{\lambda_1 R}+T_{\lambda_1 R}-T_{\lambda_2 R} \qquad (69)$$

取 $$\theta_1=\frac{T-T_{\lambda_2 R}}{(kq_1+q_2)\tau_R R^2/(\mu_o k v_m)}, \qquad \mathbb{R}=\frac{r}{R}$$

得溫度分佈為

$$\theta_1=\frac{1}{(1-k^2)(1-\lambda^2)}\left\{\frac{k^2\lambda_1^2}{4}\left(\frac{\mathbb{R}^2}{\lambda_1^2}-1\right)-\frac{\lambda_1^4}{16}\left(\frac{\mathbb{R}^4}{\lambda_1^4}-1\right)+\frac{\lambda^2\lambda_1^2}{2}\left(\frac{\mathbb{R}^2}{\lambda_1^2}\ell n\frac{\mathbb{R}}{k}-\frac{\mathbb{R}^2}{\lambda_1^2}+1-\ell n\frac{\lambda_1}{k}\right)-(\lambda_2-\lambda_1)\left[\frac{2\lambda_1^3}{9}\left(\frac{\mathbb{R}^3}{\lambda_1^3}-1\right)-\frac{k\lambda_1^2}{2}\left(\frac{\mathbb{R}^2}{\lambda_1^2}-1\right)\right]\right\}+\frac{\mu_o k v_m}{\tau_R R}\left\{\frac{q_1}{kq_1+q_2}\right.$$

$$-\frac{kR}{\mu_o v_m(1-k^2)}\left[\frac{\tau_{KR}}{k-\frac{\lambda^2}{k}}\left(\frac{k^2}{4}-\frac{\lambda^2}{2}\right)+\frac{\tau_o k}{3}\right]\}\ell n\frac{\mathbb{R}}{\lambda_1}+\frac{(T_{\lambda_1 R}-T_{\lambda_2 R})\mu_o k v_m}{(kq_1+q_2)\tau_R R^2} \quad (70)$$

2.在 $\lambda_1 R \leqslant r \leqslant \lambda_2 R$ 範圍內：將(7)式及 (67) 式代入 (33) 式

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{2(kq_1+q_2)r}{k(1-k^2)v_m}\left[\frac{\tau_R}{\mu_o(1-\lambda^2)}\left(\lambda^2\ell n\lambda^2-\frac{\lambda_2^2-1}{2}\right)+\frac{\tau_o}{\mu_o}(\lambda^2-1)\right] \quad (71)$$

邊界條件(1)在 $r=\lambda_1 R$ 處，$T=T_{\lambda_1 R}$；(2)在 $r=\lambda_2 R$ 處，$T=T_{\lambda_2 R}$。得其溫度分佈為

$$T-T_{\lambda_2 R}=\frac{(kq_1+q_2)R^2}{2\mu_o v_m k(1-k^2)}\left[\frac{\tau_R}{1-\lambda^2}\left(\lambda^2\ell n\lambda^2-\frac{\lambda_2^2-1}{2}\right)+\tau_o(\lambda_2-1)\right]\left(\frac{r^2}{R^2}-\lambda_2^2\right)$$

$$+\left\{\frac{T_{\lambda_2 R}-T_{\lambda_1 R}}{\ell n\frac{\lambda_2}{\lambda_1}}-\frac{(kq_1+q_2)R^2(\lambda_2^2-\lambda_1^2)}{2\mu_o v_m k(1-k^2)\ell n\frac{\lambda_2}{\lambda_1}}\left[\frac{\tau_R}{1-\lambda^2}\left(\lambda^2\ell n\lambda_2-\frac{\lambda_2^2-1}{2}\right)\right.\right.$$

$$\left.\left.+\tau_o(\lambda_2-1)\right]\right\}\ell n\frac{r}{\lambda_2 R} \quad (72)$$

取 $\theta_3=\dfrac{T-T_{\lambda_2 R}}{(kq_1+q_2)\tau_R R^2/(\mu_o k v_m)}$，$\mathbb{R}=\dfrac{r}{R}$，則 (72) 式化為

$$\theta_3=\frac{1}{2(1-k^2)}\left[\frac{1}{1-\lambda^2}\left(\lambda^2\ell n\lambda^2-\frac{\lambda_2^2-1}{2}\right)+\frac{\tau_o}{\tau_R}(\lambda_2-1)\right](\mathbb{R}^2-\lambda_2^2)$$

$$+\left\{\frac{(T_{\lambda_2 R}-T_{\lambda_1 R})\mu_o k v_m}{(kq_1+q_2)\tau_R R^2\ell n\frac{\lambda_2}{\lambda_1}}-\frac{\lambda_2^2-\lambda_1^2}{2(1-k^2)\ell n\frac{\lambda_2}{\lambda_1}}\left[\frac{1}{1-\lambda^2}\left(\lambda^2\ell n\lambda^2-\frac{\lambda_2^2-1}{2}\right)\right.\right.$$

$$\left.\left.+\frac{\tau_o}{\tau_R}(\lambda_2-1)\right]\right\}\ell n\frac{\mathbb{R}}{\lambda_2} \quad (73)$$

3.在 $\lambda_2 R \leqslant r \leqslant R$ 範圍內：將(6)式及 (67) 式代入 (33) 式

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\frac{2(kq_1+q_2)r}{k v_m(1-k^2)}\left\{\frac{(P_o-P_L)R}{4\mu_o L}\left[1-\left(\frac{r}{R}\right)^2+2\lambda^2\ell n\frac{r}{R}\right]+\frac{\tau_o}{\mu_o}\left(\frac{r}{R}-1\right)\right\}$$

(74)

邊界條件(1)在 $r=R$ 處，$-k\dfrac{\partial T}{\partial r}=q_2=$ 常數；(2)在 $r=\lambda_2 R$ 處，$T=T_{\lambda_2 R}$。得其溫度分佈為

$$T-T_{\lambda_2 R}=\frac{2(kq_1+q_2)}{k v_m(1-k^2)}\left\{\frac{(P_o-P_L)R^3}{4\mu_o L}\left[\frac{1}{4}\left(\frac{r^2}{R^2}-\lambda_2^2\right)-\frac{1}{16}\left(\frac{r^4}{R^4}-\lambda_2^4\right)\right.\right.$$

$$+\frac{\lambda^2}{2}\left(\frac{r^2}{R^2}\ell n\frac{r}{R}+\lambda_2^2-\frac{r^2}{R^2}-\lambda_2^2\ell n\lambda_2\right)\Big]+\frac{\tau_o R^2}{\mu_o}\Big[\frac{1}{9}\left(\frac{r^3}{R^3}-\lambda_2^3\right)$$

$$-\frac{1}{4}\left(\frac{r^2}{R^2}-\lambda_2^2\right)\Big\}-\frac{(kq_1+q_2)R}{k}\Big\{\frac{q_2}{kq_1+q_2}+\frac{R}{\mu_o v_m(1-k^2)}\Big[\frac{(P_o-P_L)R}{2L}$$

$$\left(\frac{1}{4}-\frac{\lambda^2}{2}\right)-\frac{\tau_o}{3}\Big]\Big\}\ell n\frac{r}{\lambda_2 R} \tag{75}$$

取 $\theta_2=\dfrac{T-T_{\lambda 2R}}{(kq_1+q_2)\tau_R R^2/(\mu_o v_m k)}$，$\mathbb{R}=\dfrac{r}{R}$，則 (75) 式化為

$$\theta_2=\frac{1}{(1-k^2)(1-\lambda^2)}\Big[\frac{1}{4}(\mathbb{R}^2-\lambda_2^2)-\frac{1}{16}(\mathbb{R}^4-\lambda_2^4)+\frac{\lambda^2}{2}(\mathbb{R}^2\ell n\mathbb{R}-\mathbb{R}^2+\lambda_2^2-\lambda_2^2\ell n\lambda_2)\Big]$$

$$+\frac{2\tau_o(1-\lambda^2)}{\tau_R(1-k^2)}\Big[\frac{1}{9}(\mathbb{R}^3-\lambda_2^3)-\frac{1}{4}(\mathbb{R}^2-\lambda_2^2)-\frac{v_m\mu_o}{\tau_R R}\Big\{\frac{q_2}{kq_1+q_2}+\frac{R}{\mu_o v_m(1-k^2)}$$

$$\Big[\frac{\tau_R}{1-\lambda^2}\left(\frac{1}{4}-\frac{\lambda^2}{2}\right)-\frac{\tau_o}{3}\Big]\Big\}\ell n\frac{\mathbb{R}}{\lambda_2} \tag{76}$$

(二)奧斯瓦第瓦爾型：將流速分佈代入 (33) 式解之。

1. 在 $k\leqslant\mathbb{R}\leqslant\lambda$ 範圍內：將 (20) 式代入 (33) 式

$$\rho c_P R\left[\frac{\tau_R}{m(1-\lambda^2)}\right]^{\frac{1}{n}}\Big[\frac{(-\lambda^2)^{\frac{1}{n}}}{1-\frac{1}{n}}\left(k^{1-\frac{1}{n}}-\mathbb{R}^{1-\frac{1}{n}}\right)+\frac{\frac{1}{n}(-\lambda^2)^{\frac{1}{n}}}{3-\frac{1}{n}}\left(k^{3-\frac{1}{n}}-\mathbb{R}^{3-\frac{1}{n}}\right)$$

$$+\frac{\frac{1}{n}(\frac{1}{n}-1)(-\lambda^2)^{\frac{1}{n}-2}}{(2!)(5-\frac{1}{n})}\left(k^{5-\frac{1}{n}}-\mathbb{R}^{5-\frac{1}{n}}\right)+\frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)(-\lambda^2)^{\frac{1}{n}-3}}{(3!)(7-\frac{1}{n})}$$

$$\left(k^{7-\frac{1}{n}}-\mathbb{R}^{7-\frac{1}{n}}\right)+\cdots\Big]\frac{\partial T}{\partial z}=\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{77}$$

設下列無單位量群為新變數：

$$\theta = \frac{T-T_0}{(kq_1+q_2)R/k}\ ,\ \mathbb{R}=\frac{r}{R}\ ,\ Z=\frac{zk}{\rho c_P\left[\frac{\tau_R}{m(1-\lambda^2)}\right]^{1/n}(R^3)}$$

則(77)式變為

$$\left[\frac{(-\lambda^2)^{\frac{1}{n}}}{1-\frac{1}{n}}(k^{1-\frac{1}{n}}-\mathbb{R}^{1-\frac{1}{n}})+\frac{\frac{1}{n}(-\lambda^2)^{\frac{1}{n}-1}}{3-\frac{1}{n}}(k^{3-\frac{1}{n}}-\mathbb{R}^{3-\frac{1}{n}})+\frac{\frac{1}{n}(\frac{1}{n}-1)(-\lambda^2)^{\frac{1}{n}-2}}{(2!)(5-\frac{1}{n})}\right.$$

$$\left.(k^{5-\frac{1}{n}}-\mathbb{R}^{5-\frac{1}{n}})+\frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)(-\lambda^2)^{\frac{1}{n}-3}}{(3!)(7-\frac{1}{n})}(k^{7-\frac{1}{n}}-\mathbb{R}^{7-\frac{1}{n}})+\cdots\right]\frac{\partial\theta}{\partial Z}$$

$$=\frac{1}{\mathbb{R}}\frac{\partial}{\partial\mathbb{R}}\mathbb{R}\frac{\partial\theta}{\partial\mathbb{R}} \tag{78}$$

由(67)式可設

$$\theta = C_0 Z+\psi(\mathbb{R})$$

則 $\frac{\partial\theta}{\partial Z}=C_0=$常數，$\frac{\partial\theta}{\partial\mathbb{R}}=\frac{d\psi}{d\mathbb{R}}$，$\frac{\partial}{\partial\mathbb{R}}(\mathbb{R}\frac{\partial\theta}{\partial\mathbb{R}})=\frac{d}{d\mathbb{R}}(\mathbb{R}\frac{d\psi}{d\mathbb{R}})$

(78)式化為

$$C_0\left[\frac{(-\lambda^2)^{\frac{1}{n}}}{1-\frac{1}{n}}(k^{1-\frac{1}{n}}-\mathbb{R}^{1-\frac{1}{n}})+\frac{\frac{1}{n}(-\lambda^2)^{\frac{1}{n}-1}}{3-\frac{1}{n}}(k^{3-\frac{1}{n}}-\mathbb{R}^{3-\frac{1}{n}})+\frac{\frac{1}{n}(\frac{1}{n}-1)(-\lambda^2)^{\frac{1}{n}-2}}{(2!)(5-\frac{1}{n})}\right.$$

$$\left.(k^{5-\frac{1}{n}}-\mathbb{R}^{5-\frac{1}{n}})+\frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)(-\lambda^2)^{\frac{1}{n}-3}}{(3!)(7-\frac{1}{n})}(k^{7-\frac{1}{n}}-\mathbb{R}^{7-\frac{1}{n}})+\cdots\right]$$

$$=\frac{1}{\mathbb{R}}\frac{d}{d\mathbb{R}}(\mathbb{R}\frac{d\psi}{d\mathbb{R}}) \tag{79}$$

邊界條件(1)在$\mathbb{R}=k$處，$\frac{\partial\theta}{\partial\mathbb{R}}\Big|_{\mathbb{R}=k}=\frac{d\psi}{d\mathbb{R}}\Big|_{\mathbb{R}=k}=\frac{q_1}{kq_1+q_2}$；(2)在$\mathbb{R}=\lambda$，$Z=Z_0$（$z=$

z_0'）處，$\theta=\theta_{\lambda_0}=\dfrac{T_{\lambda_0}-T_0}{(kq_1+q_2)R/k}$。得其溫度分佈為

$$\theta = C_0Z + C_0\left[\frac{(-\lambda^2)^{\frac{1}{n}}}{1-\frac{1}{n}}\left(\frac{k^{1-\frac{1}{n}}\mathrm{R}^2}{4}-\frac{\mathrm{R}^{3-\frac{1}{n}}}{(3-\frac{1}{n})^2}\right)+\frac{\frac{1}{n}(-\lambda^2)^{\frac{1}{n}-1}}{3-\frac{1}{n}}\left(\frac{k^{3-\frac{1}{n}}\mathrm{R}^2}{4}\right.\right.$$

$$\left.-\frac{\mathrm{R}^{5-\frac{1}{n}}}{(5-\frac{1}{n})^2}\right)+\frac{\frac{1}{n}(\frac{1}{n}-1)(-\lambda^2)^{\frac{1}{n}-2}}{(2!)(5-\frac{1}{n})}\left(\frac{k^{5-\frac{1}{n}}\mathrm{R}^2}{4}-\frac{\mathrm{R}^{7-\frac{1}{n}}}{(7-\frac{1}{n})^2}\right)$$

$$\left.+\frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)(-\lambda^2)^{\frac{1}{n}-3}}{(3!)(7-\frac{1}{n})}\left(\frac{k^{7-\frac{1}{n}}\mathrm{R}^2}{4}-\frac{\mathrm{R}^{9-\frac{1}{n}}}{(9-\frac{1}{n})^2}\right)+\cdots\right]+\left\{\frac{kq_1}{kq_1+q_2}\right.$$

$$-C_0\left[\frac{(-\lambda^2)^{\frac{1}{n}}}{1-\frac{1}{n}}(k^{3-\frac{1}{n}})\left(\frac{1}{2}-\frac{1}{3-\frac{1}{n}}\right)+\frac{\frac{1}{n}(-\lambda^2)^{\frac{1}{n}-1}}{3-\frac{1}{n}}(k^{5-\frac{1}{n}})\left(\frac{1}{2}-\frac{1}{5-\frac{1}{n}}\right)\right.$$

$$+\frac{\frac{1}{n}(\frac{1}{n}-1)(-\lambda^2)^{\frac{1}{n}-2}}{(2!)(5-\frac{1}{n})}(k^{7-\frac{1}{n}})\left(\frac{1}{2}-\frac{1}{7-\frac{1}{n}}\right)+\frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)(-\lambda^2)^{\frac{1}{n}-3}}{(3!)(7-\frac{1}{n})}$$

$$\left.\left.(k^{9-\frac{1}{n}})\left(\frac{1}{2}-\frac{1}{9-\frac{1}{n}}\right)+\cdots\right]\right\}\ell n\mathrm{R}+\theta_{\lambda_0}-\frac{kq_1\ell n\lambda}{kq_1+q_2}-C_0Z_0-C_0\left\{\frac{(-\lambda^2)^{\frac{1}{n}}}{1-\frac{1}{n}}\right.$$

$$\left[\lambda^2\left(\frac{k^{1-\frac{1}{n}}}{4}-\frac{\lambda^{1-\frac{1}{n}}}{(3-\frac{1}{n})^2}\right)-k^{3-\frac{1}{n}}\left(\frac{1}{2}-\frac{1}{3-\frac{1}{n}}\right)\ell n\lambda\right]+\frac{\frac{1}{n}(-\lambda^2)^{\frac{1}{n}-1}}{3-\frac{1}{n}}\left[\lambda^2\right.$$

$$\left.\left(\frac{k^{3-\frac{1}{n}}}{4}-\frac{\lambda^{3-\frac{1}{n}}}{(5-\frac{1}{n})^2}\right)-k^{5-\frac{1}{n}}\left(\frac{1}{2}-\frac{1}{5-\frac{1}{n}}\right)\ell n\lambda\right]+\frac{\frac{1}{n}(\frac{1}{n}-1)(-\lambda^2)^{\frac{1}{n}-2}}{(2!)(5-\frac{1}{n})}$$

$$[\lambda^2(\frac{k^{5-\frac{1}{n}}}{4} - \frac{\lambda^{5-\frac{1}{n}}}{(7-\frac{1}{n})^2} - k^{7-\frac{1}{n}}(\frac{1}{2}-\frac{1}{7-\frac{1}{n}})\ell n\lambda] + \frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)(-\lambda^2)^{\frac{1}{n}-3}}{(3!)(7-\frac{1}{n})}$$

$$[\lambda^2(\frac{k^{7-\frac{1}{n}}}{4} - \frac{\lambda^{7-\frac{1}{n}}}{(9-\frac{1}{n})^2}) - k^{9-\frac{1}{n}}(\frac{1}{2}-\frac{1}{9-\frac{1}{n}})\ell n\lambda] + \cdots\} \qquad (80)$$

2. 在 $\lambda \leq \mathbb{R} \leq 1$ 範圍內：將(22)式代入(33)式

$$\rho c_P R\left[\frac{\tau_R}{m(1-\lambda^2)}\right]^{\frac{1}{n}}\left[\frac{1}{\frac{1}{n}+1}(1-\mathbb{R}^{\frac{1}{n}+1}) + \frac{(-\lambda^2)(\frac{1}{n})}{\frac{1}{n}-1}(1-\mathbb{R}^{\frac{1}{n}-1}) + \frac{(-\lambda^2)^2(\frac{1}{n})(\frac{1}{n}-1)}{(2!)(\frac{1}{n}-3)}\right.$$

$$(1-\mathbb{R}^{\frac{1}{n}-3}) + \frac{(-\lambda^2)^3(\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)}{(3!)(\frac{1}{n}-5)}(1-\mathbb{R}^{\frac{1}{n}-5}) + \frac{(-\lambda^2)^4(\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)(\frac{1}{n}-3)}{(4!)(\frac{1}{n}-7)}$$

$$\left.(1-\mathbb{R}^{\frac{1}{n}-7}) + \cdots\right]\frac{\partial T}{\partial z} = \frac{k}{r}\frac{\partial}{\partial r}(r\frac{\partial T}{\partial r}) \qquad (81)$$

設下列無單位量群為新變數：

$$\theta = \frac{T-T_0}{(kq_1+q_2)R/k}, \quad \mathbb{R} = \frac{r}{R}, \quad Z = \frac{zk}{\rho c_P\left[\frac{\tau_R}{m(1-\lambda^2)}\right]^{1/n}(R^3)}$$

則(81)式變為

$$\left[\frac{1}{\frac{1}{n}+1}(1-\mathbb{R}^{\frac{1}{n}+1}) + \frac{(-\lambda^2)(\frac{1}{n})}{\frac{1}{n}-1}(1-\mathbb{R}^{\frac{1}{n}-1}) + \frac{(-\lambda^2)^2(\frac{1}{n})(\frac{1}{n}-1)}{(2!)(\frac{1}{n}-3)}(1-\mathbb{R}^{\frac{1}{n}-3})\right.$$

$$+\frac{(-\lambda^2)^3(\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)}{(3!)(\frac{1}{n}-5)}(1-\mathbb{R}^{\frac{1}{n}-5}) + \frac{(-\lambda^2)^4(\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)(\frac{1}{n}-3)}{(4!)(\frac{1}{n}-7)}$$

$$(1-\mathbb{R}^{\frac{1}{n}-7})+\cdots]\frac{\partial\theta}{\partial Z}=\frac{1}{\mathbb{R}}\frac{\partial}{\partial\mathbb{R}}(\mathbb{R}\frac{\partial\theta}{\partial\mathbb{R}}) \quad (82)$$

由(67)式可設(因在 $Z=Z$，$\mathbb{R}=\lambda$ 處，$\theta_{\kappa\leq\mathbb{R}\leq\lambda}=\theta_{\lambda\leq\mathbb{R}\leq 1}$，故 Z之係數亦必為 C_0)

$$\theta=C_0Z+\psi(\mathbb{R})$$

則 $\frac{\partial\theta}{\partial Z}=C_0=$常數，$\frac{\partial\theta}{\partial\mathbb{R}}=\frac{d\psi}{d\mathbb{R}}$，$\frac{\partial}{\partial\mathbb{R}}(\mathbb{R}\frac{\partial\theta}{\partial\mathbb{R}})=\frac{d}{\partial\mathbb{R}}(\mathbb{R}\frac{d\psi}{d\mathbb{R}})$

(82)式化為

$$C_0[\frac{1}{\frac{1}{n}+1}(1-\mathbb{R}^{\frac{1}{n}+1})+\frac{(-\lambda^2)(\frac{1}{n})}{\frac{1}{n}-1}(1-\mathbb{R}^{\frac{1}{n}-1})+\frac{(-\lambda^2)^2(\frac{1}{n})(\frac{1}{n}-1)}{(2!)(\frac{1}{n}-3)}(1-\mathbb{R}^{\frac{1}{n}-3})$$

$$+\frac{(-\lambda^2)^3(\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)}{(3!)(\frac{1}{n}-5)}(1-\mathbb{R}^{\frac{1}{n}-5})+\frac{(-\lambda^2)^4(\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)(\frac{1}{n}-3)}{(4!)(\frac{1}{n}-7)}$$

$$(1-\mathbb{R}^{\frac{1}{n}-7})+\cdots]=\frac{1}{\mathbb{R}}\frac{d}{d\mathbb{R}}(\mathbb{R}\frac{d\psi}{d\mathbb{R}}) \quad (83)$$

邊界條件(1)在 $\mathbb{R}=1$ 處，$\frac{\partial\theta}{\partial\mathbb{R}}\Big|_{\mathbb{R}=1}=\frac{d\psi}{d\mathbb{R}}\Big|_{\mathbb{R}=1}=-\frac{q_2}{kq_1+q_2}$；(2)在 $\mathbb{R}=\lambda$，$Z=Z_0$($z=z_0'$)處，$\theta=\theta_{\lambda_0}=\frac{T_{\lambda_0}-T_0}{(kq_1+q_2)\mathbb{R}/k}$。

得其溫度分佈為

$$\theta=C_0Z+C_0[\frac{1}{\frac{1}{n}+1}(\frac{\mathbb{R}^2}{4}-\frac{\mathbb{R}^{\frac{1}{n}+3}}{(\frac{1}{n}+3)^2})+\frac{(-\lambda^2)(\frac{1}{n})}{\frac{1}{n}-1}(\frac{\mathbb{R}^2}{4}-\frac{\mathbb{R}^{\frac{1}{n}+1}}{(\frac{1}{n}+1)^2})$$

$$+\frac{(-\lambda^2)^2(\frac{1}{n})(\frac{1}{n}-1)}{(2!)(\frac{1}{n}-3)}(\frac{\mathbb{R}^2}{4}-\frac{\mathbb{R}^{\frac{1}{n}-1}}{(\frac{1}{n}-1)^2})+\frac{(-\lambda^2)^3(\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)}{(3!)(\frac{1}{n}-5)}(\frac{\mathbb{R}^2}{4}$$

$$-\frac{\mathrm{IR}^{\frac{1}{n}-3}}{(\frac{1}{n}-3)^2})+\frac{(-\lambda^2)^4(\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)(\frac{1}{n}-3)}{(4!)(\frac{1}{n}-7)}(\frac{\mathrm{IR}^2}{4}-\frac{\mathrm{IR}^{\frac{1}{n}-5}}{(\frac{1}{n}-5)^2})+\cdots]$$

$$-\{\frac{q_2}{kq_1+q_2}+C_o[\frac{1}{\frac{1}{n}+1}(\frac{1}{2}-\frac{1}{\frac{1}{n}+3})+\frac{(-\lambda^2)(\frac{1}{n})}{\frac{1}{n}-1}(\frac{1}{2}-\frac{1}{\frac{1}{n}+1})+\frac{(-\lambda^2)^2(\frac{1}{n})(\frac{1}{n}-1)}{(2!)(\frac{1}{n}-3)}$$

$$(\frac{1}{2}-\frac{1}{\frac{1}{n}-1})+\cdots]\}\ \ell n\mathrm{IR}+\frac{q_2\ell n\lambda}{kq_1+q_2}+\theta_{\lambda_o}-C_oZ_o-C_o\{\frac{1}{\frac{1}{n}+1}[\frac{\lambda^2}{4}-\frac{\lambda^{\frac{1}{n}+3}}{(\frac{1}{n}+3)^2}$$

$$-(\frac{1}{2}-\frac{1}{\frac{1}{n}+3})\ell n\lambda]+\frac{(-\lambda^2)(\frac{1}{n})}{\frac{1}{n}-1}[\frac{\lambda^2}{4}-\frac{\lambda^{\frac{1}{n}+1}}{(\frac{1}{n}+1)^2}-(\frac{1}{2}-\frac{1}{\frac{1}{n}+1})\ell n\lambda]$$

$$+\frac{(-\lambda^2)^2(\frac{1}{n})(\frac{1}{n}-1)}{(2!)(\frac{1}{n}-3)}[\frac{\lambda^2}{4}-\frac{\lambda^{\frac{1}{n}-1}}{(\frac{1}{n}-1)^2}-(\frac{1}{2}-\frac{1}{\frac{1}{n}-1})\ell n\lambda]+\cdots\} \tag{84}$$

3. 由下列邊界條件可解出 (80) 式及 (84) 式中 C_o 之值以及 Z_o 與 θ_{λ_o} 之關係：在距離 $z=z$ 處（距 $z=0$ 處已相當遠），管壁已傳熱量必等於對流傳熱量[1]:

$$-2\pi Rz(kq_1+q_2)=\int_o^{2\pi}\int_{\lambda R}^{R}\rho c_P(T-T_o)Vz_2\,rdrd\phi+\int_o^{2\pi}\int_{kR}^{\lambda R}\rho c_P(T-T_o)Vz_1\,rdrd\phi \tag{85}$$

將流速分佈代入 (85) 式，並作變數變換得

$$-Z=\int_k^{\lambda}\theta[\frac{(-\lambda^2)^{\frac{1}{n}}}{1-\frac{1}{n}}(k^{1-\frac{1}{n}}-\mathrm{IR}^{1-\frac{1}{n}})+\frac{(-\lambda^2)^{\frac{1}{n}-1}(\frac{1}{n})}{3-\frac{1}{n}}(k^{3-\frac{1}{n}}-\mathrm{IR}^{3-\frac{1}{n}})$$

$$+\frac{\frac{1}{n}(\frac{1}{n}-1)(-\lambda^2)^{\frac{1}{n}-2}}{(2!)(5-\frac{1}{n})}(k^{5-\frac{1}{n}}-\mathrm{IR}^{5-\frac{1}{n}})+\frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)(-\lambda^2)^{\frac{1}{n}-3}}{(3!)(7-\frac{1}{n})}$$

$$\left(k^{7-\frac{1}{n}}-\mathbb{R}^{7-\frac{1}{n}}\right)+\cdots\Big]\mathbb{R}\,d\mathbb{R}+\int_{\lambda}^{1}\theta\Big[\frac{1}{\frac{1}{n}+1}\left(1-\mathbb{R}^{\frac{1}{n}+1}\right)+\frac{(-\lambda^{2})(\frac{1}{n})}{\frac{1}{n}-1}$$

$$\left(1-\mathbb{R}^{\frac{1}{n}-1}\right)+\frac{(-\lambda^{2})^{2}(\frac{1}{n})(\frac{1}{n}-1)}{(2!)(\frac{1}{n}-3)}\left(1-\mathbb{R}^{\frac{1}{n}-3}\right)+\frac{(-\lambda^{2})^{3}(\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)}{(3!)(\frac{1}{n}-5)}$$

$$\left(1-\mathbb{R}^{\frac{1}{n}-5}\right)+\frac{(-\lambda^{2})^{4}(\frac{1}{n})(\frac{1}{n}-1)(\frac{1}{n}-2)(\frac{1}{n}-3)}{(4!)(\frac{1}{n}-7)}\left(1-\mathbb{R}^{\frac{1}{n}-7}\right)+\cdots\Big]\mathbb{R}\,d\mathbb{R} \quad (86)$$

將溫度分佈(80)式及(84)式代入(86)式，積分並比較等式兩邊係數而得 C_o 值及 Z_o 與 θ_{λ_o} 之關係。例如取 $n=0.554$[1]，$k=0.600$，由(21)式與(23)式相等得 $\lambda=0.636$，故

$$C_o=-18.197 \quad (87)$$

又因內外等強熱源均為定值，Z_o 與 θ_{λ_o} 呈直線性關係。例如當 $q_1=q_2=1$（單位面積單位時間熱流率）時

$$0.20381\,Z_o+0.01190\,\theta_{\lambda_o}+0.01997=0 \quad (88)$$

如圖5。

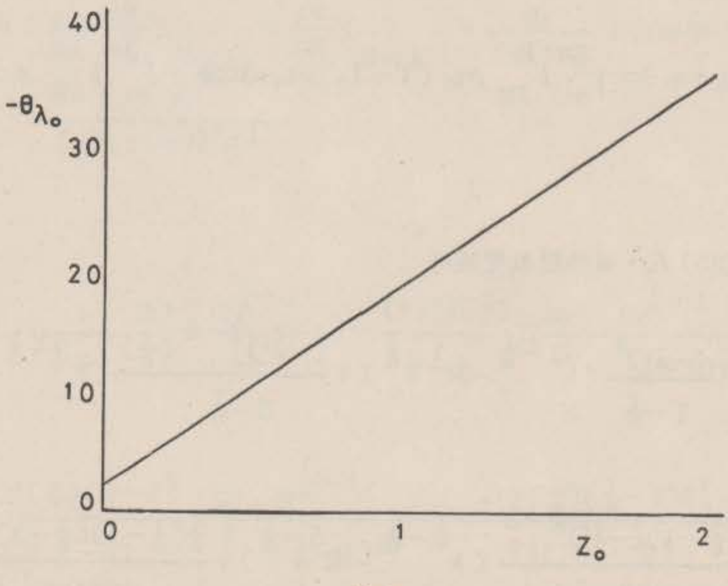

圖5

(三)意立司型：將流速分佈代入(33)式解之。

1.在 $k \leqslant \mathbb{R} \leqslant \lambda$ 範圍內：將(27)式代入(33)式

$$\rho c_P \left(\frac{\tau_R}{1-\lambda^2}\right) R \left\{ \varphi_0 \left(\frac{k^2-\mathbb{R}^2}{2} + \lambda^2 \ell n \frac{\mathbb{R}}{k}\right) + \varphi_1 \left(\frac{\tau_R}{1-\lambda^2}\right)^{\alpha-1} \left[\frac{(-\lambda^2)^{\alpha}}{1-\alpha} (k^{1-\alpha} - \mathbb{R}^{1-\alpha}) \right.\right.$$

$$\left.\left. + \frac{\alpha(-\lambda^2)^{\alpha-1}}{3-\alpha} (k^{3-\alpha} - \mathbb{R}^{3-\alpha}) + \frac{\alpha(\alpha-1)(-\lambda^2)^{\alpha-2}}{(2!)(5-\alpha)} (k^{5-\alpha} - \mathbb{R}^{5-\alpha}) + \cdots \right] \right\} \frac{\partial T}{\partial z}$$

$$= \frac{k}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \tag{89}$$

設下列無單位量群為新變數：

$$\theta = \frac{T - T_0}{(kq_1 + q_2)\mathbb{R}/k}, \quad \mathbb{R} = \frac{r}{R}, \quad Z = \frac{zk}{\rho c_P \left(\frac{\tau_R}{1-\lambda^2}\right) \mathbb{R}^3}$$

則(89)式化為

$$\left\{ \varphi_0 \left(\frac{k^2-\mathbb{R}^2}{2} + \lambda^2 \ell n \frac{\mathbb{R}}{k}\right) + \varphi_1 \left(\frac{\tau_R}{1-\lambda^2}\right)^{\alpha-1} \left[\frac{(-\lambda^2)^{\alpha}}{1-\alpha} (k^{1-\alpha} - \mathbb{R}^{1-\alpha}) + \frac{\alpha(-\lambda^2)^{\alpha-1}}{3-\alpha} \right.\right.$$

$$\left.\left. (k^{3-\alpha} - \mathbb{R}^{3-\alpha}) + \frac{\alpha(\alpha-1)(-\lambda^2)^{\alpha-2}}{(2!)(5-\alpha)} (k^{5-\alpha} - \mathbb{R}^{5-\alpha}) + \cdots \right] \right\} \frac{\partial \theta}{\partial Z}$$

$$= \frac{1}{\mathbb{R}} \frac{\partial}{\partial \mathbb{R}} \left(\mathbb{R} \frac{\partial \theta}{\partial \mathbb{R}} \right) \tag{90}$$

因軸向溫度分佈呈直線性變化，可設 $\theta = C_0 Z + \psi(\mathbb{R})$，則

$$\frac{\partial \theta}{\partial Z} = C_0 = \text{常數}, \quad \frac{\partial \theta}{\partial \mathbb{R}} = \frac{d\psi}{d\mathbb{R}}, \quad \frac{\partial}{\partial \mathbb{R}} \left(\mathbb{R} \frac{\partial \theta}{\partial \mathbb{R}} \right) = \frac{d}{d\mathbb{R}} \left(\mathbb{R} \frac{d\psi}{d\mathbb{R}} \right)$$

(90)式化為

$$\frac{d}{d\mathbb{R}} \left(\mathbb{R} \frac{d\psi}{d\mathbb{R}} \right) = C_0 \left\{ \varphi_0 \left(\frac{k^2-\mathbb{R}^2}{2} + \lambda^2 \ell n \frac{\mathbb{R}}{k}\right) + \varphi_1 \left(\frac{\tau_R}{1-\lambda^2}\right)^{\alpha-1} \left[\frac{(-\lambda^2)^{\alpha}}{1-\alpha} \right.\right.$$

$$(k^{1-\alpha}-\mathbb{R}^{1-\alpha})+\frac{\alpha(-\lambda^2)^{\alpha-1}}{3-\alpha}(k^{3-\alpha}-\mathbb{R}^{3-\alpha})+\frac{\alpha(\alpha-1)(-\lambda^2)^{\alpha-2}}{(2!)(5-\alpha)}(k^{5-\alpha}-\mathbb{R}^{5-\alpha})$$

$$+\cdots]\}\,\mathbb{R} \tag{91}$$

邊界條件(1)在$\mathbb{R}=k$處，$\left.\frac{\partial\theta}{\partial\mathbb{R}}\right|_{\mathbb{R}=k}=\left.\frac{d\psi}{d\mathbb{R}}\right|_{\mathbb{R}=k}=\frac{q_1}{kq_1+q_2}$；(2)在$\mathbb{R}=\lambda$，$Z=Z_0\,(z=z_0')$

處，$\theta=\theta_{\lambda_0}=\dfrac{T_{\lambda_0}-T_0}{(kq_1+q_2)R/k}$

得其溫度分佈為

$$\theta=\theta_{\lambda_0}-\frac{kq_1\,\ell n\lambda}{kq_1+q_2}+C_0(Z-Z_0)+C_0\left\{\varphi_0\left(\frac{1}{8}\left(k^2\mathbb{R}^2-\frac{\mathbb{R}^4}{4}\right)+\frac{\lambda^2\mathbb{R}^2}{4}(\ell n\mathbb{R}-1-\ell nk\right.\right.$$

$$)]+\varphi_1\left(\frac{\tau_R}{1-\lambda^2}\right)^{\alpha-1}\left[\frac{(-\lambda^2)^{\alpha}}{1-\alpha}\left(\frac{k^{1-\alpha}\mathbb{R}^2}{4}-\frac{\mathbb{R}^{3-\alpha}}{(3-\alpha)^2}\right)+\frac{\alpha(-\lambda^2)^{\alpha-1}}{3-\alpha}\right.$$

$$\left(\frac{k^{3-\alpha}\mathbb{R}^2}{4}-\frac{\mathbb{R}^{5-\alpha}}{(5-\alpha)^2}\right)+\frac{\alpha(\alpha-1)(-\lambda^2)^{\alpha-2}}{(2!)(5-\alpha)}\left(\frac{k^{5-\alpha}\mathbb{R}^2}{4}-\frac{\mathbb{R}^{7-\alpha}}{(7-\alpha)^2}\right)+\cdots]\}$$

$$+\left\{\frac{kq_1}{kq_1+q_2}-C_0\varphi_0\left(\frac{k^4}{8}-\frac{\lambda^2k^2}{8}\right)-C_0\varphi_1\left(\frac{\tau_R}{1-\lambda^2}\right)^{\alpha-1}\left[\frac{(-\lambda^2)^{\alpha}}{1-\alpha}\left(\frac{1}{2}-\frac{1}{3-\alpha}\right)\right.\right.$$

$$k^{3-\alpha}+\frac{\alpha(-\lambda^2)^{\alpha-1}}{3-\alpha}\left(\frac{1}{2}-\frac{1}{5-\alpha}\right)k^{5-\alpha}+\frac{\alpha(\alpha-1)(-\lambda^2)^{\alpha-2}}{(2!)(5-\alpha)}\left(\frac{1}{2}-\frac{1}{7-\alpha}\right)$$

$$k^{7-\alpha}+\cdots]\}\,\ell n\mathbb{R}-C_0\left\{\varphi_0\left[\frac{1}{8}\left(k^2\lambda^2-\frac{\lambda^4}{4}\right)+\frac{\lambda^4}{4}\left(\ell n\frac{\lambda}{k}-1\right)-\left(\frac{k^4}{8}-\frac{k^2\lambda^2}{4}\right)\right.\right.$$

$$\ell n\lambda]+\varphi_1\left(\frac{\tau_R}{1-\lambda^2}\right)^{\alpha-1}\left[\frac{(-\lambda^2)^{\alpha}}{1-\alpha}\left(\frac{k^{1-\alpha}\lambda^2}{4}-\frac{\lambda^{3-\alpha}}{(3-\alpha)^2}-\left(\frac{1}{2}-\frac{1}{3-\alpha}\right)k^{3-\alpha}\right.\right.$$

$$\left.\ell n\lambda\right)+\frac{\alpha(-\lambda^2)^{\alpha-1}}{3-\alpha}\left(\frac{k^{3-\alpha}\lambda^2}{4}-\frac{\lambda^{5-\alpha}}{(5-\alpha)^2}-\left(\frac{1}{2}-\frac{1}{5-\alpha}\right)k^{5-\alpha}\,\ell n\lambda\right)$$

$$+\frac{\alpha(\alpha-1)(-\lambda^2)^{\alpha-2}}{(2!)(5-\alpha)}\left(\frac{k^{5-\alpha}\lambda^2}{4}-\frac{\lambda^{7-\alpha}}{(7-\alpha)^2}-\left(\frac{1}{2}-\frac{1}{7-\alpha}\right)k^{7-\alpha}\ \ell n\lambda\right)$$

$$+\cdots]\} \tag{92}$$

2. 在 $\lambda\leqslant \mathbb{R}\leqslant 1$ 範圍內：將 (29) 式代入 (33) 式

$$\rho c_P R\left(\frac{\tau_R}{1-\lambda^2}\right)\{\varphi_0\left[\frac{1}{2}(1-\mathbb{R}^2)+\lambda^2\ell n\mathbb{R}\right]+\varphi_1\left(\frac{\tau_R}{1-\lambda^2}\right)^{\alpha-1}\left[\frac{1}{\alpha+1}(1-\mathbb{R}^{\alpha+1})\right.$$

$$+\frac{\alpha(-\lambda^2)}{\alpha-1}(1-\mathbb{R}^{\alpha-1})+\frac{\alpha(\alpha-1)(-\lambda^2)^2}{(2!)(\alpha-3)}(1-\mathbb{R}^{\alpha-3})+\frac{\alpha(\alpha-1)(\alpha-2)(-\lambda^2)^3}{(3!)(\alpha-5)}$$

$$(1-\mathbb{R}^{\alpha-5})+\cdots]\}\frac{\partial T}{\partial z}=\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{93}$$

設下列無單位量群為新變數：

$$\theta=\frac{T-T_0}{(kq_1+q_2)R/k},\ \mathbb{R}=\frac{r}{R},\ Z=\frac{zk}{\rho R^3 c_P\left(\frac{\tau_R}{1-\lambda^2}\right)}$$

則 (93) 式化為

$$\{\varphi_0\left[\frac{1}{2}(1-\mathbb{R}^2)+\lambda^2\ell n\mathbb{R}\right]+\varphi_1\left(\frac{\tau_R}{1-\lambda^2}\right)^{\alpha-1}\left[\frac{1}{\alpha+1}(1-\mathbb{R}^{\alpha+1})+\frac{\alpha(-\lambda^2)}{\alpha-1}(1-\mathbb{R}^{\alpha-1})\right.$$

$$+\frac{\alpha(\alpha-1)(-\lambda^2)^2}{(2!)(\alpha-3)}(1-\mathbb{R}^{\alpha-3})+\frac{\alpha(\alpha-1)(\alpha-2)(-\lambda^2)^3}{(3!)(\alpha-5)}(1-\mathbb{R}^{\alpha-5})+\cdots]\}\frac{\partial\theta}{\partial Z}$$

$$=\frac{1}{\mathbb{R}}\frac{\partial}{\partial\mathbb{R}}\left(\mathbb{R}\frac{\partial\theta}{\partial\mathbb{R}}\right) \tag{94}$$

同理可設 $\theta=C_0Z+\psi(\mathbb{R})$，則 (94) 式化為

$$C_0\ \{\varphi_0\left[\frac{1}{2}(1-\mathbb{R}^2)+\lambda^2\ell n\mathbb{R}\right]+\varphi_1\left(\frac{\tau_R}{1-\lambda^2}\right)^{\alpha-1}\left[\frac{1}{\alpha+1}(1-\mathbb{R}^{\alpha+1})+\frac{\alpha(-\lambda^2)}{\alpha-1}\right.$$

$$(1-\mathbb{R}^{\alpha-1}) + \frac{\alpha(\alpha-1)(-\lambda^2)^2}{(2!)(\alpha-3)}(1-\mathbb{R}^{\alpha-3}) + \frac{\alpha(\alpha-1)(\alpha-2)(-\lambda^2)^3}{(3!)(\alpha-5)}(1-\mathbb{R}^{\alpha-5}) + \cdots$$

$$]\} = \frac{1}{\mathbb{R}}\frac{d}{d\mathbb{R}}(\mathbb{R}\frac{d\psi}{d\mathbb{R}}) \tag{95}$$

邊界條件(1)在$\mathbb{R}=1$處，$\frac{\partial\theta}{\partial\mathbb{R}}\Big|_{\mathbb{R}=1} = \frac{d\psi}{d\mathbb{R}}\Big|_{\mathbb{R}=1} = -\frac{q_2}{kq_1+q_2}$；(2)在$\mathbb{R}=\lambda$，$Z=Z_o(z=z_o')$處，$\theta=\theta_{\lambda_o}=\frac{T_{\lambda_o}-T_o}{(kq_1+q_2)R/k}$

得其溫度分佈為

$$\theta = \theta_{\lambda_o} + \frac{q_2\ell n\lambda}{kq_1+q_2} + C_o(Z-Z_o) + C_o\{\varphi_o[\frac{1}{2}(\frac{\mathbb{R}^2}{4}-\frac{\mathbb{R}^4}{16}) + \frac{\lambda^2\mathbb{R}^2}{4}(\ell n\mathbb{R}-1)]$$

$$+\varphi_1(\frac{\tau_R}{1-\lambda^2})^{\alpha-1}[\frac{1}{\alpha+1}(\frac{\mathbb{R}^2}{4}-\frac{\mathbb{R}^{\alpha+3}}{(\alpha+3)^2}) + \frac{\alpha(-\lambda^2)}{\alpha-1}(\frac{\mathbb{R}^2}{4}-\frac{\mathbb{R}^{\alpha+1}}{(\alpha+1)^2})$$

$$+\frac{\alpha(\alpha-1)(-\lambda^2)^2}{(2!)(\alpha-3)}(\frac{\mathbb{R}^2}{4}-\frac{\mathbb{R}^{\alpha-1}}{(\alpha-1)^2}) + \frac{\alpha(\alpha-1)(\alpha-2)(-\lambda^2)^3}{(3!)(\alpha-5)}(\frac{\mathbb{R}^2}{4}-\frac{\mathbb{R}^{\alpha-3}}{(\alpha-3)^2})$$

$$+\cdots]\} - \{\frac{q_2}{kq_1+q_2} + C_o\varphi_o(\frac{1}{8}-\frac{\lambda^2}{4}) + C_o\varphi_1(\frac{\tau_R}{1-\lambda^2})^{\alpha-1}[\frac{1}{\alpha+1}(\frac{1}{2}-\frac{1}{\alpha+3})$$

$$+\frac{\alpha(-\lambda^2)}{\alpha-1}(\frac{1}{2}-\frac{1}{\alpha+1}) + \frac{\alpha(\alpha-1)(-\lambda^2)^2}{(2!)(\alpha-3)}(\frac{1}{2}-\frac{1}{\alpha-1}) + \frac{\alpha(\alpha-1)(\alpha-2)(-\lambda^2)^3}{(3!)(\alpha-5)}$$

$$(\frac{1}{2}-\frac{1}{\alpha-3}) + \cdots]\}\ell n\mathbb{R} - C_o\{\varphi_o[\frac{1}{2}(\frac{\lambda^2}{4}-\frac{\lambda^4}{16}-\frac{\ell n\lambda}{4}) + \lambda^2(\frac{\lambda^2}{4}\ell n\lambda-\frac{\lambda^2}{4}$$

$$+\frac{\ell n\lambda}{4})] + \varphi_1(\frac{\tau_R}{1-\lambda^2})^{\alpha-1}[\frac{1}{\alpha+1}(\frac{\lambda^2}{4}-\frac{\lambda^{\alpha+3}}{(\alpha+3)^2}-(\frac{1}{2}-\frac{1}{\alpha+3})\ell n\lambda) + \frac{\alpha(-\lambda^2)}{\alpha-1}$$

$$(\frac{\lambda^2}{4}-\frac{\lambda^{\alpha+1}}{(\alpha+1)^2}-(\frac{1}{2}-\frac{1}{\alpha+1})\ell n\lambda) + \frac{\alpha(\alpha-1)(-\lambda^2)^2}{(2!)(\alpha-3)}(\frac{\lambda^2}{4}-\frac{\lambda^{\alpha-1}}{(\alpha-1)^2}-(\frac{1}{2}-\frac{1}{\alpha-1})$$

$$\ell n\lambda) + \frac{\alpha(\alpha-1)(\alpha-2)(-\lambda^2)^3}{(3!)(\alpha-5)}\left(\frac{\lambda^2}{4} - \frac{\lambda^{\alpha-3}}{(\alpha-3)^2} - \left(\frac{1}{2} - \frac{1}{\alpha-3}\right)\ell n\lambda\right) + \cdots \Big]\Big\} \quad (96)$$

3. 由下列邊界條件可解出(92)式及(96)式中 C_o 之值及 Z_o 與 θ_{λ_o} 之關係：在距 $z=0$ 已相當遠，$z=z$ 處（$Z=Z$），管壁已傳熱量必等於對流傳熱量[1]，即

$$-2\pi Rz(kq_1+q_2) = \int_o^{2\pi}\int_{kR}^{\lambda R}\rho c_P(T-T_o)Vz_1 rdrd\phi + \int_o^{2\pi}\int_{\lambda R}^{R}\rho c_P(T-T_o)Vz_2\, rdrd\phi \quad (97)$$

將流速分佈代入(97)式，並變換為無單位量群新變數

$$-Z = \int_k^\lambda \theta\Big\{\varphi_o\Big(\frac{k^2-\mathbb{R}^2}{2} + \lambda^2\ell n\frac{\mathbb{R}}{k}\Big) + \varphi_1\Big(\frac{\tau_R}{1-\lambda^2}\Big)^{\alpha-1}\Big[\frac{(-\lambda^2)^\alpha}{1-\alpha}(k^{1-\alpha}-\mathbb{R}^{1-\alpha}) + \frac{\alpha(-\lambda^2)^{\alpha-1}}{3-\alpha}(k^{3-\alpha}-\mathbb{R}^{3-\alpha}) + \frac{\alpha(\alpha-1)(-\lambda^2)^{\alpha-2}}{(2!)(5-\alpha)}(k^{5-\alpha}-\mathbb{R}^{5-\alpha}) + \cdots\Big]\Big\}\mathbb{R}d\mathbb{R} + \int_\lambda^1\theta\Big\{\varphi_o\Big[\frac{1}{2}(1-\mathbb{R}^2) + \lambda^2\ell n\mathbb{R}\Big] + \varphi_1\Big(\frac{\tau_R}{1-\lambda^2}\Big)^{\alpha-1}\Big[\frac{1}{\alpha+1}(1-\mathbb{R}^{\alpha+1}) + \frac{\alpha(-\lambda^2)}{\alpha-1}(1-\mathbb{R}^{\alpha-1}) + \frac{\alpha(\alpha-1)(-\lambda^2)^2}{(2!)(\alpha-3)}(1-\mathbb{R}^{\alpha-3}) + \frac{\alpha(\alpha-1)(\alpha-2)(-\lambda^2)^3}{(3!)(\alpha-5)}(1-\mathbb{R}^{\alpha-5}) + \cdots\Big]\Big\}\mathbb{R}d\mathbb{R} \quad (98)$$

再將溫度分佈(92)式及(96)式代入(98)式，比較等式兩邊係數得 C_o 值及 Z_o 與 θ_{λ_o} 之關係。例如取 $\varphi_o=0.1377$，$\varphi_1=0.3211$，$\alpha=1.170$[1]，$\frac{\tau_R}{1-\lambda^2}=\frac{(P_o-P_L)R}{2L}=1$，$k=0.600$，由(28)式與(30)式相等得 $\lambda=0.677$，故

$$C_o = -152.94 \quad (99)$$

又當內外等強熱源 $q_1=q_2=1$ 時

$$0.8598\,Z_o + 0.006539\,\theta_{\lambda_o} + 0.3366 = 0 \quad (100)$$

如圖 6

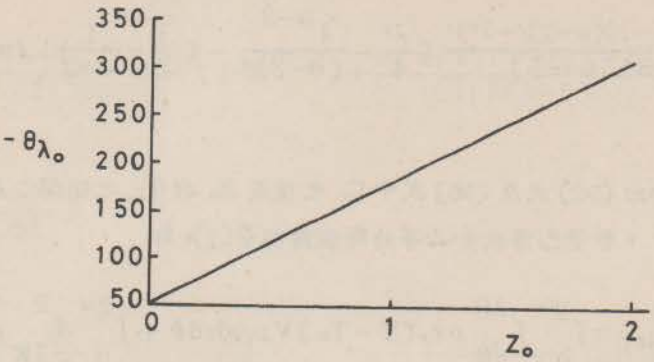

圖 6

五、結　　論

非牛頓系流體中賓漢型、奧斯瓦第瓦爾型、及意立司型在長圓套管夾層內流動，已達充分發展層流，其流速分佈與體積流量，均已得合理結果。自 z = 0 處起繞以等溫熱源作短時間熱傳，流體溫度分佈為軸向距離 z 與內管外壁向外距離 w 及外管內壁向內距離 s 之函數。其內外面單位面積熱流率均為 z 之函數。有熱源部份管長 ℓ 內外面之總熱流率均得合理結果。又自 z = 0 處起若繞以等強熱源，其溫度分佈沿軸向呈直線性變化，奧斯瓦第瓦爾型與意立司型均已得合理結果，而賓漢型僅能得徑向溫度分佈（因賓漢型流速分佈有三個區域）。現將各型之流速分佈、溫度分佈、及單位面積熱流率分佈繪圖如下：

(一)流速分佈（圖 7）

——————— 賓漢型

- - - - - - - 奧斯瓦第瓦爾型

—·—·—·— 意立司型

(二)等溫熱源短時間熱傳

1. 溫度分佈

$$\theta = \frac{T - T_0}{T_1 - T_0}$$ 夾層內管外面附近（圖 8）流體溫差比

$$= \frac{T - T_0}{T_2 - T_0}$$ 夾層外管內面附近（圖 8）流體溫差比

$$Z = \frac{z}{R}$$

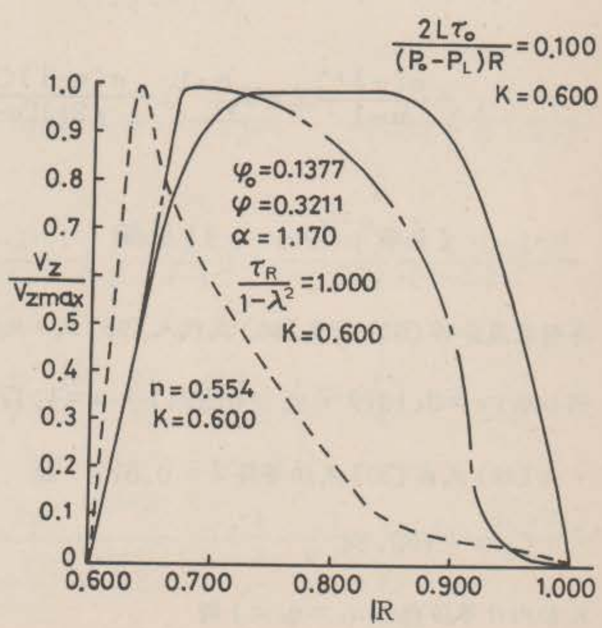

圖 7

$W=\frac{w}{R}$ 在夾層內管外面附近位置

$S=\frac{s}{R}$ 在夾層外管內面附近位置

$N=\frac{\rho c_P R^2}{k}\left(\frac{-\tau_{KR}-\tau_0}{\mu_0}\right)$ 賓漢型在內管外面附近性質

$=\frac{\rho c_P R^2}{k}\left(\frac{\tau_R-\tau_0}{\mu_0}\right)$ 賓漢型在外管內面附近性質

$=-\frac{\rho c_P R^2}{k}\left(\frac{\tau_{KR}}{m}\right)^{1/n}$ 奧斯瓦第瓦爾型在內管外面附近性質

$=\frac{\rho c_P R^2}{k}\left(\frac{\tau_R}{m}\right)^{1/n}$ 奧斯瓦第瓦爾型在外管內面附近性質

$=-(\varphi_0\tau_{KR}+\varphi_1\tau_{KR}^{\alpha})\frac{\rho c_P R^2}{k}$ 意立司型在內管外面附近性質

$=\frac{\rho c_P R^2}{k}(\varphi_0\tau_R+\varphi_1\tau_R^{\alpha})$ 意立司型在外管內面附近性質

$\eta=\left(\frac{N}{9Z}\right)^{1/3}W$ 夾層內管外面附近位置

$=\left(\frac{N}{9Z}\right)^{1/3}S$ 夾層外管內面附近位置

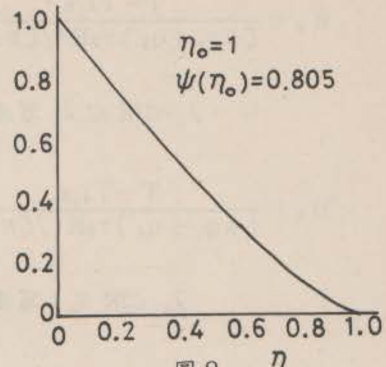

圖 8

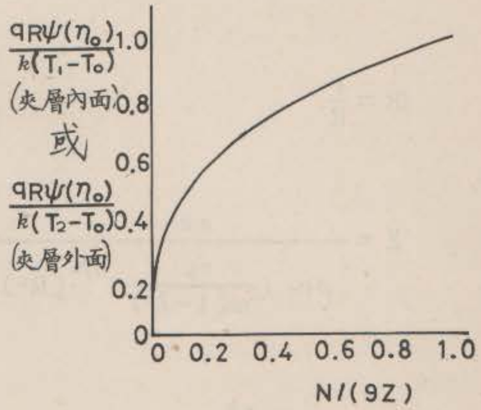

圖 9

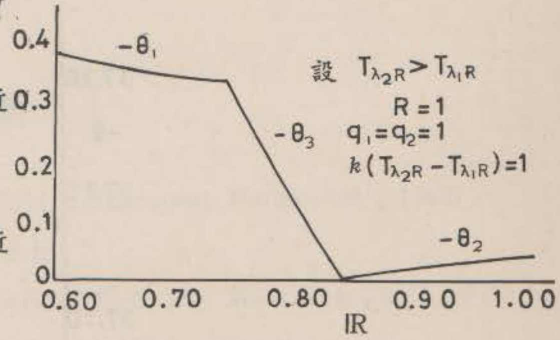

圖 10

2. 單位面積熱流率分佈（圖 9 ）

㈢等強熱源熱傳溫度分佈

$$\theta_1=\frac{T-T_{\lambda_2 R}}{(kq_1+q_2)\tau_R R^2/(\mu_0 k v_m)}$$

$k\leqslant IR\leqslant\lambda_1$ 賓漢型（圖 10 ）

$$\theta_3 = \frac{T - T_{\lambda_2 R}}{(kq_1 + q_2)\tau_R R^2/(\mu_0 k v_m)}$$

$\lambda_1 \leqslant \mathrm{IR} \leqslant \lambda_2$ 賓漢型（圖 10）

$$\theta_2 = \frac{T - T_{\lambda_2 R}}{(kq_1 + q_2)\tau_R R^2/(\mu_0 k v_m)}$$

$\lambda_2 \leqslant \mathrm{IR} \leqslant 1$ 賓漢型（圖 10）

$$\theta = \frac{T - T_0}{(kq_1 + q_2)R/k}$$ 奧斯瓦第瓦爾型（圖 11）及意立司型（圖 12）

$$\mathrm{IR} = \frac{r}{R}$$

$$Z = \frac{zk}{\rho c_P \left[\frac{\tau_R}{m(1-\lambda^2)}\right]^{1/n}(R^3)}$$ 奧斯瓦第瓦爾型

$$= \frac{zk}{\rho c_P R^3 \left(\frac{\tau_R}{1-\lambda^2}\right)}$$ 意立司型

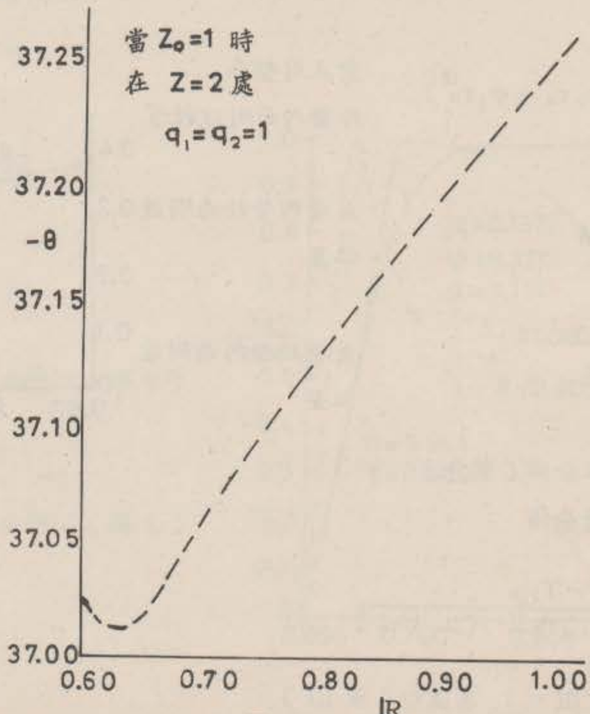

圖 11

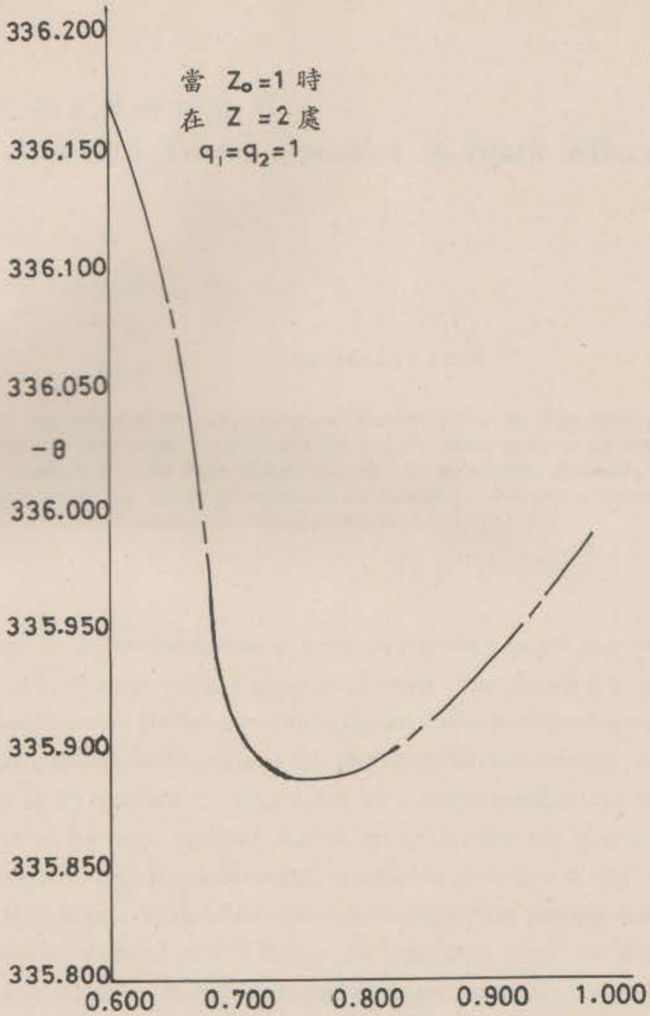

圖 12

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