The Journal of National Chiao Tung University
Vol. 8, July 1980
pp. 31-54

九階十四級顯郎基一顧塔法之研究
On the (14, 9) Explicit Runge-Kutta Method

李家生 Jia-Sheng Lee

Institute of Computer Science, N. C. T. U.

倪維城 Wei-Chen Ni

Institute of Applied Mathematics, N. C. T. U.

(Received September 30, 1978)

Abstract — In this paper, we obtain a reduced system from the original system of the (14, 9) exclicit Runge-Kutta method. A set of coefficients for this method is derived in some optimal sense from the reduced system and a numerical example is provided and compared with different order Runge-Kutta methods.

I. Introduction

We shall consider the system of n first order simultaneous differential equations

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2, \ldots, y_n), \quad i = 1, 2, \ldots, n. \tag{1}$$

where $y_1, y_2, \ldots, y_n$ are dependent variables and x is an independent variable. For the sake of convenience, we write (1) in a vector form as

$$\frac{dy_i}{dx} = f_i\,(x, Y) \quad \text{for} \quad i = 1\,(1)\,n,$$

where $Y=(y_1, \ldots, y_n)$ and the notation $i=k(t)s$ ($s=k+nt$ for some positive integer n, and t, k are integers) stands for $i=k$, $k+t$, $k+2t$, m, $k+nt$. Among the many approaches which can be applied to approximate the solution of (1), the Runge-Kutta method is most frequently used.

An explicit v-stage p^{th} order Runge-Kutta method (abbreviated (v, p) E-R-K method) can be described as follows. Suppose $Y(x_0)$ is given. The idea is to obtain the approximation $Y_p(x_0+h)$ for $Y(x_0+h)$, where h is the current step size, by approximating the integral in

$$Y(x) = Y(x_0) + \int_{x_0}^{x} ff\,dx,$$

where

$$ff = (f_1, f_2, \ldots, f_n)$$

is a vector function. With the choice of points being $\{x_0+c_ih\}_{i=0}^{v}$ and their weights $\{b_i\}_{i=1}^{v}$, we intend to use the form

$$\hat{Y}_p(x) = Y(x_o) + \psi(x_o, Y(x_o), h)$$

where $\psi(x_o, Y(x_o), h) = h\,(\sum_{i=1}^{v} b_i g^{(i)})$ (2)

and $g^{(i)} = ff(x_o + c_i h, Y(x_o) + h \sum_{j=1}^{v} a_{ij} g^{(j)}$

to approximate the exact solution $Y(x)$ of (1) with error $E=Q(h^p)$.

The coefficients a_{ij}, b_i and c_i for $i, j=l(1)v$ are parameters which characterize theprocess. a_{ij} and b_i depend on the order and stage of the method, and for every i, c_i is defined by

$$c_i = \sum_{j=1}^{v} a_{ij} \,. \tag{3}$$

The coefficients of various (v, p) explicit Runge-Kutta methods such as (6, 5), (8, 6), (7, 6), (9, 7), (12, 8), (11, 8), (17, 10) and (18, 10) have been studied by Luther and Koner [12, 13], Huta, Butcher [3] and Lawson [ll], Butcher, Shanks, Curtis [7] and Cooper and Verner [6], Hairer [9], and Curtis [8] respectively. It is the objective of the present paper to investigate on the coefficients of the (14, 9) explicit Runge-Kutta method and compare numerical results of an example with that done by explicit Runge-Kutta method of different orders.

II. A Reduced System of (14, 9) Explicit Runge-Kutta (abbreviated E-R-K) Method

Following what is done in Butcher [1], one has to solve a non-linear algebraic system of 486 equations [9] in order to calculate a Runge-Kutta method of order 9. reduced system is established by converting some independent relations of the original system into dependent relations. This possibility is best illustrated by an example.

(i) Let us consider two equations of the original system

$$\Phi([[_6\tau]_6\,\tau]) = \Sigma\, b_i c_i a_{i,j} a_{j,k} a_{k,\ell} a_{\ell,m} a_{m,n} a_{n,p} c_p = \frac{1}{45360} \tag{4}$$

$$\Phi([[_5\,\tau^2]_5\,\tau]) = \Sigma\, b_i c_i a_{i,j} a_{j,k} a_{k,\ell} a_{\ell,m} a_{m,n} c_n^{\,2} = \frac{-1}{22680} \tag{5}$$

If we assumed that $\sum_{j=1}^{i-1} a_{i,j} c_j = \frac{1}{2} c_i^2$ for $i \geq 3$

hold, then for (3), (4) and (5), we have

$$\Phi([[_5\tau^2]_5\tau]) - 2\cdot\Phi([[_6\tau]_6\tau]) = -2\,\Sigma b_i c_i a_{i,j} a_{j,k} a_{k,\ell} a_{\ell,m} a_{m,2} c_2^2$$

Now, if it is further assumed that

$$b_i c_i a_{i,j} a_{j,k} a_{k,\ell} a_{\ell,m} a_{m,2} = 0 \tag{6}$$

then, if either (4) or (5) is satisfied, so is the other.

If we assume that

$$a_{m,2} = 0 \quad \text{for } m \geq 4 \tag{7}$$

then (6) will hold, leaving for the case m=3

$$\Sigma b_i c_i a_{i,j} a_{j,k} a_{k,\ell} a_{\ell,3} = 0 \tag{8}$$

If $\quad a_{\ell,3} = 0 \quad \text{for } \ell \geq 6 \quad (9)$

and $\quad a_{m,4} = 0 \quad \text{for } m \geq 8 \quad (10)$

then (8) will hold, provided that

$$b_i c_i a_{j,k} a_{k,\ell} = 0 \quad \text{for } \ell \geq 5 \tag{11}$$

and $\quad b_i c_i a_{i,j} a_{j,k} = 0 \quad \text{for } k=5(1)7 \quad (12)$

Thus, we add the equations (7), (9)-(12) to our reduced system. Some equations of the original system and dependent; (for example, $\Phi([_8\tau]_8)$ and $\Phi([_7\tau^2]_7]$ become mutually dependent.)

(ii) Another idea of establishing our reduced system is described as follows:

Butcher proposed [2] that k and ℓ always denote positive integers. Let the symbols $A(\xi)$, $B(\xi)$, $C(\xi)$, $D(\xi)$, $E(\xi)$, where ξ is any given integer, represent certain statements about the R-K coefficients $a_{i,j}$, b_i and c_i.

$A(\xi)$: $\quad \Phi(t) = \dfrac{1}{R(t)} \quad$ whenever $r(t) \leq \xi$

where $\Phi(t)$, $R(t)$ are the elementary weight and elementary weight and elementary constant of rooted tree t respectively.

$B(\xi)$: $\quad \Phi([\tau^{k-1}]) = \sum_{i=1}^{v} b_i c_i^{k-1} = \dfrac{1}{k} \quad$ for $k \leq \xi$

$C(\xi)$: $\quad \sum_{j=1}^{v} a_{i,j} c_j^{k-1} = \dfrac{c_i^k}{k} \quad$ for $i=1(1)v$, and $k \leq \xi$

$D(\xi, \eta)$: $\quad \Phi([\tau^{k-1}[\tau^{\ell-1}]]) = \sum_{i=1}^{v} \sum_{j=1}^{v} b_i c_i^{k-1} a_{i,j} c_j^{\ell-1} = \dfrac{1}{\ell(k+\ell)}$

for $k \leq \xi$ and $\ell \leq \eta$.

We get the following propositions

<u>Proposition 1</u> If $A(\xi)$ holds then so does $B(\xi)$.

<u>Proposition 2</u> If $A(\xi+\eta)$ holds then so does $D(\xi, \eta)$.

<u>Proposition 3</u> If $A(\xi+\eta)$ and $C(\xi)$ hold then so does $D(\xi, \eta)$.

In searching for a set of the (14, 9) E-R-K coefficients a_{ij}, b_i, c_i, we are guided by the analysis of the original system A(9) of 486 equations. Since A(9) implies B(9), D(2, 7), D(3, 6). If B(9) and C(5) are satisfied, then D(4, 5) holds.

According to the above rules (i) and (ii) of adding some extra relations to our reduced system, and if some survivors of the original system such as B(9), D(2, 6), D(2, 7) and D(3, 6) are added to our reduced system, we can construct a reduced system of 8v-8 equations for p=9. We obtain a reduced system consisting of the following equations, in which v stands for the stage of E-R-K method.

$$\bar{\Phi}([\tau^{r-1}]) = \Sigma b_i c_i^{r-1} = \frac{1}{r} \quad \text{for} \quad r = 1(1)9 \tag{13}$$

$$\bar{\Phi}([[\tau^5]\tau]) = \Sigma b_i c_i a_{ij} c_j^5 = \frac{1}{48} \tag{14}$$

$$\bar{\Phi}([[\tau^6]\tau]) = \Sigma b_i c_i a_{ij} c_j^6 = \frac{1}{63} \tag{15}$$

$$\bar{\Phi}([[_2\tau^5]_2\tau]) = \Sigma b_i c_i a_{ij} a_{jk} c_k^5 = \frac{1}{378} \tag{16}$$

$$\bar{\Phi}([[\tau^5]\tau^2]) = \Sigma b_i c_i^2 a_{ij} c_j^5 = \frac{1}{54} \tag{17}$$

$$\Sigma b_i a_{ij} = b_j(1-c_j) \qquad \text{for} \quad j = 1(1)v \tag{18}$$

$$\Sigma b_i c_i a_{ij} a_{jk} a_{k\ell} = 0 \qquad \ell = 5 \tag{19}$$

$$\Sigma b_i c_i^r a_{ij} = 0 \qquad \text{for} \quad r = 1,2, \quad j = 5(1)7 \tag{20}$$

$$\Sigma b_i c_i^3 a_{ij} = 0 \qquad j = 5 \tag{21}$$

$$\Sigma b_i c_i a_{ij} a_{jk} = 0 \quad \text{for} \quad k = 5(1)7 \tag{22}$$

$$\Sigma b_i c_i^2 a_{ij} a_{jk} = 0 \qquad k = 5 \tag{23}$$

$$\Sigma b_i c_i a_{ij} c_j a_{jk} = 0 \qquad k = 5 \tag{24}$$

$$a_{i2} = 0 \qquad \text{for} \quad i = 4(1)v \tag{25}$$

$$a_{i3} = 0 \qquad \text{for} \quad i = 6(1)v \tag{26}$$

$$a_{i4} = 0 \qquad \text{for} \quad i = 8(1)v \tag{27}$$

$$\Sigma a_{ij} c_j = \frac{1}{2} c_i^2 \qquad \text{for} \quad i = 3(1)v \tag{28}$$

$$\Sigma a_{ij} c_j^2 = \frac{1}{3} c_i^3 \qquad \text{for} \quad i = 4(1)v \tag{29}$$

$$\Sigma a_{ij} c_j^{\,3} = \tfrac{1}{4} c_i^{\,4} \quad \text{for} \quad i = 6(1)v \tag{30}$$

$$\Sigma a_{ij} c_j^{\,4} = \tfrac{1}{5} c_i^{\,5} \quad \text{for} \quad i = 8(1)v \tag{31}$$

$$b_2 = b_3 = b_4 = b_5 = b_6 = b_7 = 0 \tag{32}$$

In the reduced system Φ, equations (13) - (17) are the only survivors of the original system and equations (18) - (32) are the additional relations. It is easy to see that the reduced system is by no means unique.

Definition 1 A reduced system is said to be a good reduced system if its solution is a solution of the original system.

Theorem 1 The reduced system ψ is a good reduced system.

Proof:

Solution of Φ of 486 equations satisfies all the equations of the original system can be proved by showing that each of the 486 equations can be derived from the reduced system. Since it is tedious and space-consuming to list them all, we shall only do so for a couple of equations here as examples.

$$\Phi([_2\tau^2]_2) = \Sigma b_i a_{ij} c_j^{\,2} = \frac{1}{12}$$

by (13), (29) and (32).

$$\Phi([_4\tau]_4) = \Sigma b_i a_{ij} a_{j,k} a_{k,\ell} c_\ell = \frac{1}{120}$$

by (13), (28), (29), (30) and (32). The rest of the equations can be checked one by one in exactly the same fashion.

Theorem 2 The reduced system ψ of 8v equations includes eight redundant equations.

Proof.

First, by (32) equation (18) becomes

$$\sum_{i=1}^{v} b_i a_{ij} = b_j(1 - c_j) = 0 \qquad \text{for } j=2(1)4$$

and by (25) - (27) and (32) equation (18) becomes

$$\sum_{i=8}^{v} b_i a_{i2} = \sum_{i=8}^{v} b_i a_{i,3} = \sum_{i=8}^{v} b_i a_{i,4} = 0 .$$

Next, for $r = 1(1)5$,

$$\sum_{j=1}^{v} c^{r-1} \{ \sum_{i=1} b_i a_{ij} - b_j (1 - c_j) \}$$

$$= \sum_{i=8}^{v} b_i (\sum_{j=1}^{v} a_{ij} c^{r-1}) - \sum_{j=1}^{v} b_j c^{r-1} + \sum_{j=1}^{v} b_j c_j$$

$$= \frac{1}{r+1}\ \frac{1}{r+2} - \frac{1}{r+1} + \frac{1}{r+2}$$

$= 0$ by (13), (28), (29) and (30).

Since $v \geq 0$ (see [3] and [4]),(13) implies that at least 5 of $c_5, c_6, \ldots, c_v$ are nonzero, and any two of them are unequal. For this, at least 5 of the equations with $j \geq 5$ in (18) can be derived from (13), (28), (29) and (30). Hence there are all together eight redundant equations in the reduced system.

For a (v, 9) E-R-K process, there are 8(v-1) equations to be satisfied with v(v+1)/2 coefficients. Therefore, some flexibility in the choice of a solution is expectable.

In a (v, p) R-K process, if $c_2=0$ then $g^{(2)}=g^{(1)}$ by (2) and $Y(x)=Y(x_o)+(b_1+b_2)g^{(1)}+\sum_{i=3}^{v} b_i\, g^{(i)} = Y(x_o) + \sum_{j=1}^{v-1} \omega_j\, g^{(j)}$, where $b_{i+j} = \omega_i$ for i=2(1) and $b_1+b_2= \omega_1$. That is, a new (v-1) stage R-K method can be obtained. Hence in (v, p) R-K process, $c_2 \neq 0$. For the same reason, we shall assume that $b_v \neq 0$ so that we can get $c_v=1$ by (18). In our reduced system Ψ, it is reasonable to try v=14, since $8(v-1) < v(v+1)/2$ for v = 14.

III. Derivation of the Undetermined Coefficients of the (14, 9) Explicit Runge-Kutta Method

Since the number of coefficients involved is numerous, one may easily be confused with the solved and unsolved coefficients. We suggest the reader to refer to the summary given at the end of this section, which describes the procedures for finding all the coefficients in brief.

Let $h_r \equiv h_r\,(c_8, c_9, c_{10}, c_{11})$ be the sum of all homogeneous products of order r in the quantities c_8, c_9, c_{10}, c_{11}.

Lemma 1

$$(10h_0 - 15h_1 + 24h_2 - 42h_3 + 84h_4) - 3c_{12}(5h_0 - 8h_1 + 14h_2 - 28h_3 + 70h_4) = 0 \tag{33}$$

Proof. Let $S_1 = \sum_{i,j=1}^{14} (1-c_i)a_{ij}\,(c_j-c_8)(c_j-c_9)(c_j-c_{10})(c_j-c_{11})(c_j-c_{12})c_j$

$$= \sum_{j=1}^{14} \Delta_j \sum_{j=1}^{14} b_j(1-c_i)a_{i,j} \tag{34}$$

where $\Delta_j = c_j(c_j-c_8)(c_j-c_9)(c_j-c_{10})(c_j-c_{11})(c_j-c_{12})$

By considering the values of j, it can be verified that the reduced system of equations implies $S_1=0$ as follows

For j=1 and 8(1)12, $\Delta j=0$ by inspection.

For j=13 and 14, $\sum_{i=1}^{14} b_i(1-c_j)a_{i,j}=0$ since $c_{14}=1$ and $a_{i,14}=0$.

For j=2(1)4, $\sum_{i=1}^{14} b_i(1-c_i)a_{i,j}=0$ by (25), (26), (27) and (32).

For j=5(1)7, $\sum_{i=1}^{14} b_i(1-c_i)a_{i,j}=0$ by (20) and (32).

For simplicity, let $H_r \equiv H_r(c_8,c_9,c_{10},c_{11},c_{12})$ be the sum of the homogeneous products in the quantities indicated in the parenthesis. Then by expanding the terms in (34) and reducing the middle term of (34) we have

$$S_1 = \sum_{i,j=1}^{14} b_i(1-c_i)a_{ij}(c_j-c_8)(c_j-c_9)(c_j-c_{10})(c_j-c_{11})(c_j-c_{12})c_j$$

$$= [\sum_{i,j=1}^{14} b_i a_{ij} c_j^6 - \sum_{i,j=1}^{14} b_i a_{ij} c_j^5 H_1 + \sum_{i,j=1}^{14} b_i a_{ij} c_j^4 H_2 - \sum_{i,j=1}^{14} b_i a_{ij} c_j^3 H_3 + \sum_{i,j=1}^{14} b_i a_{ij} c_j^2 H_4 - \sum_{i,j=1}^{14} b_i a_{ij} c_j H_5] - \sum_{i,j=1}^{14} b_i c_i a_{ij} c_j^6 + \sum_{i,j=1}^{14} b_i c_i a_{ij} c_j^5 H_1 - \sum_{i,j=1}^{14} b_i c_i a_{ij} c_j^4 H_2 + \sum_{i,j=1}^{14} b_i c_i a_{ij} c_j^3 H_3 - \sum_{i,j=1}^{14} b_i c_i a_{ij} c_j^2 H_4 + \sum_{i,j=1}^{14} b_i c_i a_{ij} c_j H_5$$

$$= \frac{1}{504} - \frac{H_1}{336} + \frac{H_2}{210} - \frac{H_3}{120} + \frac{H_4}{60} - \frac{H_5}{24}$$

Hence we have

$$10H_0 - 15H_1 + 24H_2 - 42H_3 + 84H_4 - 210H_5 = 0 \tag{34}$$

i.e., we have (33).

Lemma 2

$$(35h_0 - 54h_1 + 90h_2 + 168h_3 + 378h_4) - 9c_{13}(5h_0 - 8h_1 + 14h_2 - 28h_3 + 70h_4) = 0 \tag{35}$$

Proof.

Let $S_2 = \sum_{i,j=1}^{14} b_i(1-c_i)(c_i-c_{13})a_{ij}(c_j-c_8)(c_j-c_9)(c_j-c_{10})(c_j-c_{11})c_j$

$$= \sum_{j=2}^{14} \Delta_j \sum_{i=1}^{14} b_i(1-c_i)(c_i-c_{13})a_{ij} \tag{36}$$

where $\Delta_j = c_j(c_j-c_8)(c_j-c_9)(c_j-c_{10})(c_j-c_{11})$

It is easy to see that $\Delta_j=0$ for j=1 and 8(1)11.

For j=2(1)4 and 12(1)14, $\sum_{i=1}^{14} b_i(1-c_i)(c_i-c_{13})a_{i,j} = 0$ by (25) - (27) and (32).

For j=5(1)7, expanding the terms on the right hand sides in (35) by (18) - (26), we get

$$S_2 = [\frac{1}{432} - \frac{h_1}{280} + \frac{h_2}{168} - \frac{h_3}{90} + \frac{h_4}{40}] - c_{13}[\frac{1}{336} - \frac{h_1}{210} + \frac{h_2}{120} - \frac{h_3}{60} + \frac{h_4}{24}].$$

Thus, we have (35).

In a simitar manner, we can prove the following lemmas.

Lemma 3

$$5h_0 - 9h_1 + 18h_2 - 42h_3 + 126h_4 = 0 \tag{37}$$

Lemma 4

$$5h_0 - 8h_1 + 14h_2 - 28h_3 + 70h_4 = 1680 b_{13}(1-c_{13})a_{13,12}\Delta_{12}$$

$$\text{where } \Delta_j = (c_j-c_8)(c_j-c_9)(c_j-c_{10})(c_j-c_{11})c_j \tag{38}$$

Lemma 5

$$[18 - 30(c_8+c_9+c_{10}) + 56(c_8c_9 + c_8c_{10} + c_9c_{10}) - 126 c_8c_9c_{10}] - 3c_{13}[8 - 14(c_8+c_9+c_{10})$$

$$+ 28(c_8c_9+c_8c_{10}+c_9c_{10}) - 70 c_8c_9c_{10}] = 5040 b_{12}(1-c_{12})(c_{12}-c_{13}) a_{12,11}\Delta_{11}$$

$$\text{where } \Delta_j = c_j(c_j-c_8)(c_j-c_9)(c_j-c_{10}) \tag{39}$$

Lemma 6

$$3h_0 - 6(c_8+c_9+c_{10}) + 14(c_8c_9+c_8c_{10}+c_9c_{10}) - 42c_8c_9c_{10}$$

$$= 5040 b_{13}(1-c_{13}) a_{13,12} a_{12,11}\Delta_{11}$$

$$\text{where } \Delta_k = c_k(c_k-c_8)(c_k-c_9)(c_k-c_{10}) \tag{40}$$

Suppose c_{12} is distinct from any of c_8, c_9, c_{10}, c_{11} then from (38) we get $b_{13}=0$ or $c_{13}=1$ or $a_{13,12}=0$ or $c_{12}=0$, which is inconsistent with (13) and (40). Thus c_{12} is equal to one of c_8, c_9, c_{10}, c_{11}. In order to find c_{13}, first we assume that the determinant

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ c_8 & c_9 & c_{10} & c_{11} & c_{13} & 1 \\ c_8^2 & c_9^2 & c_{10}^2 & c_{11}^2 & c_{13}^2 & 1 \\ c_8^3 & c_9^3 & c_{10}^3 & c_{11}^3 & c_{13}^3 & 1 \\ c_8^4 & c_9^4 & c_{10}^4 & c_{11}^4 & c_{13}^4 & 1 \\ c_8^5 & c_9^5 & c_{10}^5 & c_{11}^5 & c_{13}^5 & 1 \end{vmatrix} \neq 0 \tag{41}$$

Let

$$D_1 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1/2 & 1 \\ c_8 & c_9 & c_{10} & c_{11} & 1/3 & 1 \\ c_8^2 & c_9^2 & c_{10}^2 & c_{11}^2 & 1/4 & 1 \\ c_8^3 & c_9^3 & c_{10}^3 & c_{11}^3 & 1/5 & 1 \\ c_8^4 & c_9^4 & c_{10}^4 & c_{11}^4 & 1/6 & 1 \\ c_8^5 & c_9^5 & c_{10}^5 & c_{11}^5 & 1/7 & 1 \end{vmatrix} \tag{42}$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1/3 & 1 \\ c_8 & c_9 & c_{10} & c_{11} & 1/4 & 1 \\ c_8^2 & c_9^2 & c_{10}^2 & c_{11}^2 & 1/5 & 1 \\ c_8^3 & c_9^3 & c_{10}^3 & c_{11}^3 & 1/6 & 1 \\ c_8^4 & c_9^4 & c_{10}^4 & c_{11}^4 & 1/7 & 1 \\ c_8^5 & c_9^5 & c_{10}^5 & c_{11}^5 & 1/8 & 1 \end{vmatrix} \tag{43}$$

$$D_3 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1/4 & 1 \\ c_8 & c_9 & c_{10} & c_{11} & 1/5 & 1 \\ c_8^2 & c_9^2 & c_{10}^2 & c_{11}^2 & 1/6 & 1 \\ c_8^3 & c_9^3 & c_{10}^3 & c_{11}^3 & 1/7 & 1 \\ c_8^4 & c_9^4 & c_{10}^4 & c_{11}^4 & 1/8 & 1 \\ c_8^5 & c_9^5 & c_{10}^5 & c_{11}^5 & 1/9 & 1 \end{vmatrix} \tag{44}$$

then from (13) and (41) - (44) we get

$$b_{13} = \frac{D_1}{D} = \frac{c_{13}D_2}{D} = \frac{c_{13}^2 D_3}{D} \tag{45}$$

Calculations by the computer reveal that $D_1=0$, $D_2=0$ and $D_3=0$, hence by (45) $c_{13}=D_1/D_2$ and $c_{13}=D_2/D_3$ which is a contradiction. Thus D=0. That is c_{13} is one of the c_8, c_9, c_{10}, c_{11} and c_{14}.

Secondly, it can be found from equations (28) - (31) that

for i = 3

$$a_{3,2} = \frac{c_3^2}{2c_2}$$

for i = 4

$$c_3 = \frac{2}{3} c_4$$

$$a_{4,3} = \frac{3}{4} c_4$$

and

$$a_{4,1} = \frac{1}{4} c_4$$

for i = 5

$$a_{5,3} = \frac{9c_5^2(\frac{1}{2}c_4 - \frac{1}{3}c_5)}{2c_4^2}$$

$$a_{5,4} = \frac{c_5^2(c_5 - c_4)}{c_4^2}$$

for i = 6

$$c_4 = \left(\frac{4c_5 - 3c_6}{6c_5 - 4c_6}\right) c_6$$

$$a_{6,4} = \frac{c_6^2(\frac{1}{2}c_5 - \frac{1}{3}c_6)}{c_4(c_5 - c_4)}$$

$$a_{6,5} = \frac{c_6^2(\frac{1}{3}c_6 - \frac{1}{2}c_4)}{c_5(c_5 - c_4)}$$

for i = 7

$$a_{7,4} = \begin{vmatrix} \frac{1}{2}c_7^2 & c_5^2 & c_6^2 \\ \frac{1}{3}c_7^3 & c_5^3 & c_6^3 \\ \frac{1}{4}c_7^4 & c_5^4 & c_6^4 \end{vmatrix} \Bigg/ \begin{vmatrix} c_4 & c_5 & c_6 \\ c_4^2 & c_5^2 & c_6^2 \\ c_4^3 & c_5^3 & c_6^3 \end{vmatrix}$$

$$a_{7,5} = \frac{\begin{vmatrix} c_4 & \frac{1}{2}c_7^2 & c_6 \\ c_4^2 & \frac{1}{3}c_7^3 & c_6^2 \\ c_4^3 & \frac{1}{4}c_7^4 & c_6^3 \end{vmatrix}}{\begin{vmatrix} c_4 & c_5 & c_6 \\ c_4^2 & c_5^2 & c_6^2 \\ c_4^3 & c_5^3 & c_6^3 \end{vmatrix}}$$

$$a_{7,6} = \frac{\begin{vmatrix} c_4 & c_5 & \frac{1}{2}c_7 \\ c_4^2 & c_5^2 & \frac{1}{3}c_7^2 \\ c_4^3 & c_5^3 & \frac{1}{4}c_7^3 \end{vmatrix}}{\begin{vmatrix} c_4 & c_5 & c_6 \\ c_4^2 & c_5^2 & c_6^2 \\ c_4^3 & c_5^3 & c_6^3 \end{vmatrix}}$$

for i = 8

$$\begin{vmatrix} c_5 & c_6 & c_7 & \frac{1}{2}c_8^2 \\ c_5^2 & c_6^2 & c_7^2 & \frac{1}{3}c_8^3 \\ c_5^3 & c_6^3 & c_7^3 & \frac{1}{4}c_8^4 \\ c_5^4 & c_6^4 & c_7^4 & \frac{1}{5}c_8^5 \end{vmatrix} = 0$$

$$c_5 = \left[\frac{\frac{1}{5}c_8^2 + \frac{1}{3}c_6c_7 - \frac{1}{4}c_7c_8 - \frac{1}{4}c_6c_8}{\frac{1}{4}c_8^2 + \frac{1}{2}c_6c_7 - \frac{1}{2}c_7c_8 - \frac{1}{3}c_6c_8}\right] c_8$$

$$a_{8,5} = \frac{\begin{vmatrix} \frac{1}{2}c_8^2 & c_6 & c_7 \\ \frac{1}{3}c_8^3 & c_6^2 & c_7^2 \\ \frac{1}{4}c_8^4 & c_6^3 & c_7^3 \end{vmatrix}}{\begin{vmatrix} c_5 & c_6 & c_7 \\ c_5^2 & c_6^2 & c_8^2 \\ c_5^3 & c_6^3 & c_7^3 \end{vmatrix}}$$

$$a_{8,6} = \frac{\begin{vmatrix} c_5 & \frac{1}{2}c_8{}^2 & c_7 \\ c_5{}^2 & \frac{1}{3}c_8{}^3 & c_7{}^2 \\ c_5{}^3 & \frac{1}{4}c_8{}^4 & c_7{}^3 \end{vmatrix}}{\begin{vmatrix} c_5 & c_6 & c_7 \\ c_5{}^2 & c_6{}^2 & c_7{}^2 \\ c_5{}^3 & c_6{}^3 & c_7{}^3 \end{vmatrix}}$$

$$a_{8,7} = \frac{\begin{vmatrix} c_5 & c_6 & \frac{1}{2}c_8{}^2 \\ c_5{}^2 & c_6{}^2 & \frac{1}{3}c_8{}^3 \\ c_5{}^3 & c_6{}^3 & \frac{1}{4}c_8{}^4 \end{vmatrix}}{\begin{vmatrix} c_5 & c_6 & c_7 \\ c_5{}^2 & c_6{}^2 & c_7{}^2 \\ c_5{}^3 & c_6{}^3 & c_7{}^3 \end{vmatrix}}$$

From the above set of equations, which se shall label as (46), we see that c_2, c_6 and c_7 can be arbitrary, and c_3, c_4, c_5 are determined in terms of c_6, c_7, c_8.

From Lemma 5 and Lemma 6, we can get the values of $a_{12,\,11}$, $a_{13,\,12}$ repectively.

Lemma 7.

$$9-15(c_8+c_9+c_{11})+28(c_8c_9+c_8c_{11}+c_9c_{11})-63c_8c_9c_{11}$$

$$-3c_{13}[4-7(c_8+c_9+c_{11})+14(c_8c_9+c_8c_{11}+c_9c_{11})-35c_8c_9c_{11}]$$

$$=2520c_{10}(c_{10}-c_8)(c_{10}-c_9)(c_{10}-c_{11})[b_{11}(1-c_{11})(c_{11}-c_{13})a_{11,10}$$

$$+b_{12}(1-c_{12})(c_{12}-c_{13})a_{12,10}] \qquad (47)$$

Proof. Let $S_7 = \sum_j b_i(1-c_i)(c_i-c_{13})a_{ij}(c_j-c_8)(c_j-c_9)$.

$$(c_j-c_{11})c_j = \sum_j \Delta_j \sum_j b_i(1-c_i)(c_i-c_{13})a_{i,j} \qquad (48)$$

where $\Delta_j = c_j(c_j-c_8)(c_j-c_9)(c_j-c_{11})$.

For $j = 1, 8, 9, 11$, $\Delta_j = 0$.

For $j = 12(1)14$, $\sum_i b_i(1 - c_i)(c_i - c_{13})\, a_{i,j} = 0.$

For $j = 5(1)7$, $\sum_i b_i(1 - c_i)(c_i - c_{13})\, a_{i,j} = 0$

by (18), (20) and (32).

For $j = 2(1)4$ we have $\sum_i b_i(1 - c_i)(c_i - c_{13})\, a_{i,j} = 0$

by (25) - (27) and (32).

Hence

$$S_8 = c_{10}(c_{10} - c_8)(c_{10} - c_9)(c_{10} - c_{11})\,[b_{11}(1 - c_{11})\cdot (c_{11} - c_{13})\, a_{11,10} + b_{12}(1 - c_{12})(c_{12} - c_{13})\, a_{12,10}] \tag{49}$$

On the other hand, if we expand all the terms in (46) and use (13), (14), (18), (28) - (32) then

$$S_8 = \frac{1}{2520}\{[9 - 15(c_8 + c_9 + c_{11}) + 28(c_8 c_9 + c_8 c_{11} + c_9 c_{11}) - 63 c_8 c_9 c_{11}] - 3c_{13}[4 - 7(c_8 + c_9 + c_{11}) + 12(c_8 c_9 + c_8 c_{11} + c_9 c_{11}) - 35 c_8 c_9 c_{11}]\} \tag{50}$$

Hence from (49) and (50) we obtain (47).

Similary, lemmas 8 - 10 stated below hold.

Lemma 8.

$$1680[b_{13}(1-c_{13})a_{13,11}+b_{12}(1-c_{12})a_{12,11}](c_{11}-c_8)(c_{11}-c_9)\cdot(c_{11}-c_{10})(c_{11}-c_{12})c_{11} = 5-8(c_8+c_9+c_{10}+c_{12})+14(c_8c_9+c_8c_{10}+c_8c_{12}+c_9c_{10}+c_9c_{12}+c_{10}c_{12})-28(c_8c_9c_{10}+c_8c_9c_{12}+c_8c_{10}c_{12}+c_9c_{10}c_{12})+70c_8c_9c_{10}c_{12} \tag{51}$$

Lemma 9.

$$3-6(c_8+c_9+c_{11})+14(c_8c_9+c_8c_{11}+c_9c_{11})-42c_8c_9c_{11} = 5040[b_{13}(1-c_{13})a_{13,11}a_{11,10}+b_{12}(1-c_{12})a_{12,11}a_{11,10}+b_{13}(1-c_{13})a_{13,12}a_{12,10}]c_{10}(c_{10}-c_8)(c_{10}-c_9)(c_{10}-c_{11}) \tag{52}$$

Lemma 10.

$$[9-15(c_8+c_9+c_{11})+28(c_8c_9+c_8c_{11}+c_9c_{11})-63c_8c_9c_{11}]$$
$$-3c_{12}[4-7(c_8+c_9+c_{11})+14(c_8c_9+c_8c_{11}+c_9c_{11})-35c_8c_9c_{11}]$$
$$= 2520\{[b_{11}(1-c_{11})(c_{11}-c_{12})a_{11,10}$$
$$+b_{13}(1-c_{13})(c_{13}-c_{12})a_{13,10}]c_{10}(c_{10}-c_8)(c_{10}-c_9)\cdot$$
$$(c_{10}-c_{11})c_{10}\} \qquad (53)$$

From (42), (51), (52) and (53), we get the values of $a_{13,\,11}$, $a_{11,\,10}$, $a_{12,\,10}$ and $a_{13,\,10}$ respectively. Thus we obtain the values of $a_{14,\,10}$, $a_{14,\,11}$, $a_{14,\,12}$, $a_{14,\,13}$ by using (18) for j=10(1)13. Similarly, if we set

$$S_{11} = \Sigma\, b_i(1-c_i)a_{ij}a_{jk}c_k(c_k-c_8)(c_k-c_{10})(c_k-c_{11}) ,$$

$$S_{12} = \Sigma\, b_i(c_i-c_{13})a_{ij}a_{jk}c_k(c_k-c_8)(c_k-c_{10})(c_k-c_{11}),$$

$$S_{13} = \Sigma\, b_i(c_i-c_{12})a_{ij}a_{jk}c_k(c_k-c_8)(c_k-c_{10})(c_k-c_{11}),$$

$$S_{14} = \Sigma\, b_i(1-c_i)a_{ij}c_j(c_j-c_8)(c_j-c_{10})(c_j-c_{11}),$$

then we obtain Lemma 11 - Lemma 14 as follows.

Lemma 11.

$$\{[b_{11}(1-c_{11})a_{11,10} + b_{12}(1-c_{12})a_{12,10} + b_{13}(1-c_{13})a_{13,10}]a_{10,9}$$
$$+ [b_{13}(1-c_{13})a_{13,11} + b_{12}(1-c_{12})a_{12,11}]a_{11,9} + b_{13}(1-c_{13})a_{13,12}$$
$$a_{12,9}\} = \frac{1}{1680} - \frac{c_8+c_{10}+c_{11}}{840} + \frac{c_8c_{10}+c_8c_{11}+c_{10}c_{11}}{360} - \frac{c_8c_{10}c_{11}}{120}$$

Lemma 12.

$$\{[b_{11}(c_{11}-c_{13})a_{11,10} + b_{12}(c_{12}-c_{13})a_{12,10} + b_{14}(c_{14}-c_{13})$$
$$a_{14,10}]a_{10,9} + [b_{12}(c_{12}-c_{13})a_{12,11} + b_{14}(c_{14}-c_{13})a_{14,11}]a_{11,9}$$
$$+ [b_{14}(c_{14}-c_{13})a_{14,12}]a_{12,9} + [b_{14}(c_{14}-c_{13})a_{14,13}]a_{13,9}\}$$

$$\cdot c_9(c_9-c_8)(c_9-c_{10})(c_9-c_{11})$$
$$= (\tfrac{1}{240} - \tfrac{c_{13}}{210}) - (\tfrac{1}{140} - \tfrac{c_{13}}{120})(c_8+c_{10}+c_{11}) + (\tfrac{1}{72} - \tfrac{c_{13}}{60})$$
$$(c_8c_{10}+c_8c_{11}+c_{10}c_{11}) - (\tfrac{1}{30} - \tfrac{c_{13}}{24})c_8c_{10}c_{11} -$$
$$(c_{14}-c_{13})a_{14,13}a_{13,12}c_{12}\ c_{14}\ (c_{12}-c_8)(c_{12}-c_{10})(c_{12}-c_{11})$$

Lemma 13

$$\{[b_{11}(c_{11}-c_{12})a_{11,10} + b_{13}(c_{13}-c_{12})a_{13,10} + b_{14}(c_{14}-c_{12}) a_{14,10}]a_{10,9} + [b_{13}(c_{13}-c_{12})a_{13,11} + b_{14}(c_{14}-c_{12})a_{14,11}]a_{11,9} + [b_{14}(c_{14}-c_{12})a_{14,12} + b_{13}(c_{13}-c_{12})a_{13,12}]a_{12,9} + [b_{14}(c_{14}-c_{12})a_{14,13}]a_{13,9}\} c_9(c_9-c_8)(c_9-c_{10})(c_9-c_{11})$$
$$= (\frac{1}{240} - \frac{c_{12}}{120}) - (c_8+c_{10}+c_{11})(\frac{1}{140} - \frac{c_{12}}{120}) + (c_8c_{10}+c_8c_{11} + c_{10}c_{11})(\frac{1}{72} - \frac{c_{12}}{60}) - (\frac{1}{30} - \frac{c_{12}}{24})c_8c_{10}c_{11} - b_{14}(c_{14}-c_{12}) a_{14,13}a_{13,12}c_{12}(c_{12}-c_8)(c_{12}-c_{10})(c_{12}-c_{11})$$

Lemma 14

$$\{b_{10}(1-c_{10})a_{10,9} + b_{11}(1-c_{11})a_{11,9} + b_{12}(1-c_{12})a_{12,9} + b_{13}(1-c_{13})a_{13,9}\}c_9(c_9-c_8)(c_9-c_{10})(c_9-c_{11})$$
$$= \frac{1}{210} - \frac{c_8+c_{10}+c_{11}}{120} + \frac{c_8c_{10}+c_8c_{11}+c_{10}c_{11}}{60} - \frac{c_8c_{10}c_{11}}{24} - b_{13}(1-c_{13})a_{13,12}c_{12}(c_{12}-c_8)(c_{12}-c_{10})(c_{12}-c_{11})$$

From Lemmas 11 - 14 we can get the values of $a_{10,9}$, $a_{11,9}$, $a_{12,9}$, $a_{13,9}$ and using (8) we can obtain the value of $a_{14,9}$. The rest of the R-K coefficients can be found from the rest of equations (25) - (31), for example

$$\left.\begin{aligned} a_{9,5}c_5 + a_{9,6}c_6 + a_{9,7}c_7 + a_{9,8}c_8 &= \tfrac{1}{2}c_9^2 \\ a_{9,5}c_5 + a_{9,6}c_6^2 + a_{9,7}c_7^2 + a_{9,8}c_8^2 &= \tfrac{1}{3}c_9^3 \\ a_{9,5}c_5^3 + a_{9,6}c_6^3 + a_{9,7}c_7^3 + a_{9,8}c_8^3 &= \tfrac{1}{4}c_9^4 \\ a_{9,5}c_5^4 + a_{9,6}c_6^4 + a_{9,7}c_7^4 + a_{9,8}c_8^4 &= \tfrac{1}{5}c_9^5 \end{aligned}\right\} \qquad (54)$$

For convenience, it can be written in the form

$$Ax = C \qquad (55)$$

where

$$A = \begin{pmatrix} c_5 & c_6 & c_7 & c_8 \\ c_5^2 & c_6^2 & c_7^2 & c_8^2 \\ c_5^3 & c_6^3 & c_7^3 & c_8^3 \\ c_5^4 & c_6^4 & c_7^4 & c_8^4 \end{pmatrix}$$

$$X = \begin{pmatrix} a_{9,5} \\ a_{9,6} \\ a_{9,7} \\ a_{9,8} \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \frac{1}{2}c_9^2 \\ \frac{1}{2}c_9^3 \\ \frac{1}{2}c_9^4 \\ \frac{1}{2}c_9^5 \end{pmatrix}$$

by direct computation, we get

$$A^{-1} = \begin{pmatrix} \frac{c_6c_7c_8}{g_1} & \frac{c_7c_8+c_6c_7+c_6c_8}{g_1} & \frac{c_6+c_7+c_8}{g_1} & \frac{1}{g_1} \\ \frac{c_5c_7c_8}{g_2} & \frac{c_7c_8+c_5c_7+c_5c_8}{g_2} & \frac{c_5+c_7+c_8}{g_2} & \frac{1}{g_2} \\ \frac{c_5c_6c_8}{g_3} & \frac{c_5c_6+c_5c_8+c_6c_8}{g_3} & \frac{c_5+c_6+c_8}{g_3} & \frac{1}{g_3} \\ \frac{c_5c_6c_7}{g_4} & \frac{c_5c_6+c_5c_7+c_6c_7}{g_4} & \frac{c_5+c_6+c_7}{g_4} & \frac{1}{g_4} \end{pmatrix}.$$

where

$$g_1 = c_5(c_6-c_5)(c_7-c_5)(c_8-c_5)$$

$$g_2 = c_6(c_5-c_6)(c_7-c_6)(c_8-c_6)$$

$$g_3 = c_7(c_5-c_7)(c_6-c_7)(c_8-c_7)$$

$$g_4 = c_8(c_5-c_8)(c_6-c_8)(c_7-c_8)$$

By using the same techniques, we can get the values of $a_{i,j}$ for $i=10(1)13$, $j=5(1)8$. Finally $a_{14,9}$ is obtained for (13) and $a_{i,j}=c_i - \sum_{j=2}^{v} a_{i,j}$. Hence we get all the values of $a_{i,j}$ (for $i > j$) of the Explicit R-K process.

A more satisfactory check on the correctness of the R-K coefficients can be made by direct substitution in the original system. However, this would be a very difficult task, since there are 486 equations, many of which contain hundreds of terms.

To summarize the above discussion, we express the process of finding all coefficients of the (14, 9) Runge-Kutta method by the following steps in which the corresponding values of $a_{i,j}$ are obtained and illustrated in table 1.

step 1: From equations (25) - (27) of the reduced system, we can obtain the $a_{i,j}$'s which are equal to zero.

step 2: For the lower value of i we can obtain the corresponding $a_{i,j}$ from (46).

step 3: From Lemma 1-Lemma 10 and (18) we can get the corresponding value $a_{i,j}$ that appeared in the lemmas.

step 4: From Lemma 11-Lemma 14, and (18) we can obtain the values $a_{i,j}$ $\forall i \geq 10$.

step 5: As in (55) we can use the inverse matrix A^{-1} to get $a_{i,j}$ $\forall i \geq 9$ and $j=5(1)8$.

step 6: We can use (3) to obtain the values of $a_{i,j}$, $\forall i \geq 2$, $j=1$.

1													
2	VI												
3	VI	II											
4	VI	I	II										
5	VI	I	II	II									
6	VI	I	I	II	II								
7	VI	I	I	II	II	II							
8	VI	I	I	I	II	II	II						
9	VI	I	I	I	V	V	V	V					
10	VI	I	I	I	V	V	V	V	IV				
11	VI	I	I	I	V	V	V	V	IV	III			
12	VI	I	I	I	V	V	V	V	IV	III	III		
13	VI	I	I	I	V	V	V	V	IV	III	III	III	
14	VI	I	I	I	V	V	V	V	IV	III	III	III	III
	1	2	3	4	5	6	7	8	9	10	11	12	13

Table 1

The following is suggested in [10] for determining the arbitrary quantities C_2, C_6 and C_7. These three constants are chosen so that

1) $\Sigma\, b_i^2$ is minimized

2) $\Sigma\, | a_{i,j} |$ is minimized

3) the values of c_i are restricted to be in the range [0, 1].

If 3) holds, we set $w = \sum_j |\frac{a_{i,j}}{C_i}|$. It can be shown that $\Sigma\, |a_{i,j}|$ is minimized when w tends to 1. Sets of values of $a_{i,j}$, b_i and c_i can be obtained by assigning different values to C_2, C_6 and C_7 and following the procedures given above. The set of values for $a_{i,j}$, b_i and c_i which fits the above criteria most is listed as follows.

$$c_2 = 0.4121375829316104D+00$$
$$a_{21} = 0.4121375829316104D+00$$
$$c_3 = 0.4121375829316104D+00$$
$$a_{21} = 0.2060687914658053D+00$$
$$a_{32} = 0.2060687914658052D+00$$
$$c_4 = 0.6182063743974157D+00$$
$$a_{41} = 0.1545515935993539D+00$$
$$a_{42} = 0.0000000000000000D+00$$
$$a_{43} = 0.4636547807980617D+00$$
$$c_5 = 0.3089474901392525D+00$$
$$a_{51} = 0.1545321118556073D+00$$
$$a_{52} = 0.0000000000000000D+00$$
$$a_{53} = 0.2315622312219574D+00$$
$$a_{54} = 0.7723685293831210D-01$$
$$c_6 = 0.6100913727162194D+00$$
$$a_{61} = 0.1029829279441656D+00$$
$$a_{62} = 0.0000000000000000D+00$$
$$a_{63} = 0.0000000000000000D+00$$
$$a_{64} = 0.9518193910983065D-01$$
$$a_{65} = 0.4119265056622232D+00$$
$$c_7 = 0.7512075453643801D+00$$
$$a_{71} = 0.9998530851942300D-01$$
$$a_{72} = 0.0000000000000000D+00$$
$$a_{73} = 0.0000000000000000D+00$$
$$a_{74} = 0.1888343749808105D+01$$
$$a_{75} = 0.4332566606739650D+00$$
$$a_{76} = -0.1670378173137113D+01$$
$$c_8 = 0.8825276619647323D+00$$
$$a_{81} = 0.9700103082630174D-01$$
$$a_{82} = 0.0000000000000000D+00$$
$$a_{83} = 0.0000000000000000D+00$$
$$a_{84} = 0.0000000000000000D+00$$

$a_{8,5}$ = 0.4430270201218226D+00

$a_{8,6}$ = 0.3353861580672712D-01

$a_{8,7}$ = 0.3089609952098809D+00

c_9 = 0.6426157582403225D+00

$a_{9,1}$ = 0.9708425223543431D-01

$a_{9,2}$ = 0.0000000000000000D+00

$a_{9,3}$ = 0.0000000000000000D+00

$a_{9,4}$ = 0.0000000000000000D+00

$a_{9,5}$ = 0.4425947122909040D+00

$a_{9,6}$ = 0.1353109303957377D-01

$a_{9,7}$ = 0.1326470518640587D+00

$a_{9,8}$ =-0.4324135118964827D-01

c_{10} = 0.3573842417695775D+00

$a_{10,1}$ = 0.9861985284242530D-01

$a_{10,2}$ = 0.0000000000000000D+00

$a_{10,3}$ = 0.0000000000000000D+00

$a_{10,4}$ = 0.0000000000000000D+00

$a_{10,5}$ = 0.3384957102817448D+00

$a_{10,6}$ =-0.1174415669314042D+01

$a_{10,7}$ =-0.4369004873175694D-01

$a_{10,8}$ =-0.2604457073755758D-01

$a_{10,9}$ = 0.1114419078419973D+01

c_{11} = 0.1174723380352677D+00

$a_{11,1}$ = 0.7336163144449284D-01

$a_{11,2}$ = 0.0000000000000000D+00

$a_{11,3}$ = 0.0000000000000000D+00

$a_{11,4}$ = 0.0000000000000000D+00

$a_{11,5}$ = 0.2212026236453348D+00

$a_{11,6}$ = 0.3472821455008031D+00

$a_{11,7}$ = 0.1715065564990597D+00

$a_{11,8}$ =-0.3400582984103659D-01

$a_{11,9}$ =-0.4753902270297825D+00

$a_{11,10}$ =-0.1864845621836035D+00

c_{12} = 0.6426157582403225D+00

$a_{12,1}$ =-0.6444783119078880D+00

$a_{12,2}$ = 0.0000000000000000D+00

$a_{12,3}$ = 0.0000000000000000D+00

$a_{12,4}$ = 0.0000000000000000D+00

$a_{12,5}$ =-0.4299016132302245D+01

$a_{12,6}$ = 0.4192129557445641D+01

$a_{12,7}$ =-0.1168433154276576D+01

$a_{12,8}$ = 0.4427935336476432D+00

$a_{12,9}$ =-0.2694335937400000D+01

$a_{12,10}$= 0.1625713713737149D+01

$a_{12,11}$= 0.2188242488586599D+01

c_{13} = 0.8325276619647323D+00

$a_{13,11}$= 0.1720964265153169D+01

$a_{13,2}$ = 0.0000000000000000D+00

$a_{13,3}$ = 0.0000000000000000D+00

$a_{13,4}$ = 0.0000000000000000D+00

$a_{13,5}$ = 0.1079934673370318D+02

$a_{13,6}$ =-0.8530001636215179D+01

$a_{13,7}$ = 0.3353914493479719D+01

$a_{13,8}$ =-0.1093352809576339D+01

$a_{13,9}$ = 0.4495117187500001D+01

$a_{13,10}$=-0.5744250477195000D+01

$a_{13,11}$=-0.4787198579007157D+01

$a_{13,12}$= 0.6684884841223399D+00

c_{14} = 0.0000000000000000D+01

$a_{14,1}$ =-0.3627663087003784D+01

$a_{14,2}$ = 0.0000000000000000D+00

$a_{14,3}$ = 0.0000000000000000D+00

$a_{14,4}$ = 0.0000000000000000D+00

$a_{14,5}$ =-0.2357482805891780D+02

$a_{14,7}$ =-0.77732860164372780D+01
$a_{14,8}$ = 0.24351637965162990D+01
$a_{14,9}$ =-0.76479770174281360D+01
$a_{14,10}$ = 0.13578664083522240D+02
$a_{14,11}$ = 0.10986920246422100D+02
$a_{14,12}$ =-0.54708509452983930D+00
$a_{14,13}$ = 0.44460333004103810D+00
b_1 = 0.33333333333333330D-01
b_2 = 0.00000000000000000D+00
b_3 = 0.00000000000000000D+00
b_4 = 0.00000000000000000D+00
b_5 = 0.00000000000000000D+00
b_6 = 0.00000000000000000D+00
b_7 = 0.00000000000000000D+00
b_8 = 0.63079159382974460D-01
b_9 = 0.92476396172581040D-01
b_{10} = 0.27742918851774300D+00
b_{11} = 0.18923747814892380D+00
b_{12} = 0.18495279234516210D+00
b_{13} = 0.12615831876594890D+00
b_{14} = 0.33333333333333330D-01

IV. Numerical Example

We first make an investigation on the errors of the E-R-K method of different orders with the same step size. From the above discussion, we obtain a set of (14, 9) E-R-K coefficients which contains arbitrary coefficients with $\lambda = 0.6913$, $u=0.8512$, $b_{12}=2b_9$, $b_{13}=2b_8$. (where $\lambda=c_6/c_8$ and $u=c_7/c_8$). Let us consider the fourth-order differential equation

$$y^{(4)} = y^{(2)}(12y^2 + 8) \tag{56}$$

with initial value $(x_0, y_0, y_0', y_0'', y_0''') = (x_0, y_0, u_0, z_0, w_0) = (0, 0, 1, 0, 2)$. The true value (exact solution) is $y(x) = \tan x$. We transform the differential equation (56) into a system of 4 first-order equations as follows:

$$y^{(1)} = f_1(x, y, u, z, w) = u$$

$$y^{(2)} = f_2(x, y, u, z, w) = z$$

$$y^{(3)} = f_3\,(x, y, u, z, w) = w \tag{57}$$

$$y^{(4)} = f_4\,(x, y, u, z, w) = z\ \ (12y^2+8)$$

for § 1, we have

$$g_k^{(i)}=f_k(x_0+c_ih,\ y_0+h\sum_{j=1}^{v} a_{i,j}g_1^{(j)},\ u_0+h\sum_{j=1}^{v} a_{i,j}\,g_2^{(j)},$$

$$z_0+h\sum_{j=1}^{v} a_{i,j}g_3^{(j)},\ w_0+h\sum_{j=1}^{v} a_{i,j}g_4^{(j)})$$

for k=1(1)4 and i=1(1)v

(56) and (57), we get

$$g_1^{(i)} = f_1(x_0+c_ih,\ y_0+h\sum_{j=1}^{v} a_{i,j}\,g_1^{(j)},\ u_0+h\sum_{j=1}^{v} a_{i,j}\,g_2^{(j)}$$

$$z_0+h\sum_{j=1}^{v} a_{i,j}\ g_3^{(j)},\ w_0+h\sum_{j=1}^{v} a_{i,j}\,g_4^{(j)})$$

$$= u_0 + h\sum_{j=1}^{v} a_{i,j}\ g_2^{(j)}$$

$$g_2^{(i)} = z_0 + h\ \sum_{j=1}^{v} a_{i,j}\ g_3^{(j)}$$

$$g_3^{(i)} = w_0 + h\ \sum_{j=1}^{v} a_{i,j}\ g_4^{(j)}$$

$$g_4^{(i)} = (z_0 + h\ \sum_{j=1}^{v} a_{i,j}\,g_3^{(j)})\ [12\,(y_0 + h\ \Sigma\, a_{i,j}\ \ g_1^{(j)})^2\ + 8\ ]$$

and the R-K approximation

$$\hat{y}_p(x_0+h) = y(x_0) + (x_0, y_0, h)$$

$$= y(x_0) + h\ \sum_{j=1}^{v} b_i\ \ g_1^{(i)}$$

For comparison, some computations in the above example are performed on the Dec-10 system by using double precision to assure 16 significant digits. We list the error terms of the E-R-K method of different orders in table 2.

Secondly, we shall look at the errors of the E-R-K method of different orders with approximately equal number of operations excluding the number of operations in functional evaluation. We compare the errors of the numerical example by applying the E-R-K method of different orders with the same number of iterations. We also divide the step size h into m equal parts and get (m+1) points $\{x_0 + \frac{kh}{m}\}_{k=0}^{m}$. The corresponding (m+1) E-R-K approximations $\hat{y}^*(x_0 + \frac{ih}{m})$ for i=0(1)m are claculated following the procedures below.

Step 1. Replace the step size h by the new step size h/m

Step 2. $x \leftarrow x_0$ and $y \leftarrow y_0 = y(x_0)$

Step 3. Calculate the approximated value of $\hat{y}^*(x+ \frac{h}{m})$ by (4, 4) E-R-K method as in [14, p. 200, 5-6-49]

step size h	Error x 10^{12} of (4, 4) E-R-K[14,p.200,5-6-49]	Error x 10^{12} of (4, 4) E-R-K[14,p.200,5-6-49]	Error from (6, 5) E-R-K [12] x 10^{12}	Error from (7, 6) E-R-K) [11] x 10^{12}
0.04	-13811.27158125994	-13663.02342098447	23.92008813245639	-2.311927988182042
0.05	-41895.29108356016	-41710.51708190543	114.0445342329155	-11.35572885724301
0.06	-104055.0793793282	-103834.1398730547	408.5747018489627	-40.68004266777336
0.07	-224799.8554527619	-224543.1835190548	1201.749963430920	-119.6429899597007

step size h	Error from (11,8) E-R-K [7] x 10^{12}	Error from (14,9) E-R-K x 10^{13}	Error from (17,10) E-R-K x 10^{14}	Error from (18, 10) E-R-K x 10^{14}
0.04	-0.01883215805520422	-0.01196265309033606	-0.002775557561562891	-0.002775557561562891
0.05	-0.02984418268070499	-0.089060703256649280	-0.002081668171172169	-0.004163336342344337
0.06	-0.06580846978465615	-0.459660087704304	-0.002775557561562891	-0.01942890293094024
0.07	-0.1876554467372671	-1.841415908643285	0.001387778780781446	-0.08743006318923108

Table 2

The error discussion of different order Runge-Kutta methods

Step 4. $x \leftarrow x + \frac{h}{m}$ (58)

Step 5. $y \leftarrow \hat{y}^*(x)$

Step 6. Repeat approximating $\hat{y}^*$ by following step 3 to step 5 m times to obtain the approximated value $\hat{y}^*(x_0+h)$

Now we proceed to determine the value of m . First we compare the number of operations excluding the functional evaluations (multiplications and additions) of R-K method and list them in Table 3.

From (2), for a v stage R-K method we calculate the value $\hat{Y}_p(x)$ from $Y(x_0)$. By direct computation it requires $T=(v-1) + [2v + \frac{(v-1)v}{2} - w] \cdot n$ operations whre v stands for the stage of the E-R-K method, n is the order of the given differential equations and w is the total number of coefficients a_{ij}, b_i which are equal to zero. We list them as follows:

Table 3

Numbers / type	Number of multiplications	Number of additions
(4,4) E-R-K method shown in [14, p.200, 5-6-48]	3+11n	3+11n
(4,4) E-R-K method shown in [14, p.200, 5-6-49]	3+14n	3+14n
(11, 8) E-R-K method	10+59n	10+59n
(14,9) E-R-K method	13+86n	13+86n
(18,10) E-R-K method	17+113n	17+113n
(17,10) E-R-K method	16+115n	16+115n
(6, 5) E-R-K method	5+24n	5+24n
(7, 6) E-R-K method	6+32n	6+32n

From table 3, we know that the number of operations in (14, 9) E-R-K method is less than 8(7) folds of the (4,4) E-R-K method that is used in [14, p. 200, 5-6-48] ([14, p. 200, 5-6-49]) respectively. When we use the algorithm in (58), we get the approximations y(0, 1) of the (14, 9) E-R-K process and $\hat{y}^*(0, 1)$ of the (4, 4) E-R-K with m=8 in the following table

Table 4

type	error term
(14,9) E-R-K method with h=0.1, m=1	$-0.4572521528078966 \times 10^{-11}$
(4,4) E-R-K coef. shown in [14, p. 200, 5-6-48] h=0.1, m=8	$-0.293807603246598 \times 10^{-9}$
(4,4) E-R-K coef. shown in [14, p. 200, 5-6-49] h = 0.1, m = 7	$-0.5853988821469258 \times 10^{-9}$

References

1. J. C. Butcher, "Coefficients for the study of Runge-Kutta integration process", J. Austral Math. Soc. 3, pp. 185-201, (1963).
2. J. C. Butcher, "Implicit Runge-Kutta process", Math. Comp. 18, pp. 50-64, (1964).
3. J. C. Butcher, "On Runge-Kutta process of high order", J. Sustral Math. Soc. 4, pp. 179-194, (1964).
4. J. C. Butcher, "On the attainable order of Runge-Kutta method", Math. Comp. 19, pp. 408-417, (1965).
5. C. R. Cassity, "The complete solution of the fifth order Runge-Kutta Equations", SIAM. J. Numer. ANAL. 6, pp. 432-436, (1972).
6. G. J. Copper and J. H. Verner, "Some explicit Runge-Kutta method of high order", SIAM. J. Numer. ANAL. 9, pp. 389-405, (1972).
7. A. R. Curtis, "An eight order Runge-Kutta process with eleven function evaluations per step", Numer. Math. 16, pp. 268-277, (1970).
8. A. R. Curtis, "High-order explicit Runge-Kutta formulae, their uses and limitations", J. Inst. Maths. Applies 16, pp. 35-55, (1975).
9. E. Hairer, "A Runge-Kutta method of order 10", J. Inst. Maths. Applies. 21, pp. 47-59, (1978).
10. T. E. Hull and R. L. Johnston, "Optimal Runge-Kutta method", Math. Comp. 18, pp. 306-310, (1964).
11. J. Douglas Lawson, "An order Six Runge-Kutta process with extended region of stability", SIAM. J. Numer. Anal., 4, pp. 620-625, (1967).
12. H. A. Luther and H. P. Koner, "Some fifth order classical Runge-Kutta formula", SIAM Rev. 7, pp. 551-558, (1967).
13. H. A. Luther, "An explicit sixth-order Runge-Kutta formula", Math. Comp. 22, pp. 101-103, (1968).
14. A. Ralston, A first course in Numerical Analysis, McGraw-Hill Book Co, Ltd., New York, (1971).