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雲形函數與最佳求積公式之研究

A Note on Spline Functions & Intermediate Best Quadrature Formulas

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ABSTRACT — From the result of item [1] in the list of references concerning the incermediate best quadrature formulas (in the sence of Sard) an explicit expression has been developed. The formula arises from a class of spline functions which will be obtained by some polynomials that may be regarded as generalizations of the even -degree Bernoulli polynomials.

I. Introduction

There has been much recent work investigating the connection between numerical quadrature formulas and spline functions. Two of the oldest and best known formulas are Hermites' quadrature formula and the Euler-Maclaurin sum formula. I.J. Schoenberg showed that [2] these two classical formulas are best in the sense of Sard [3] . Peter R. Lipow showed [1] that there exists other best quadrature formulas, which are intermediate between the two formulas mentioned above, but he didn't find the explicit formulas.

In this paper we develope an explicit expression for the intermediate quadrature formulas.

II. Monosplines and Quadrature Formulas

By a cardinal spline function of degree m-1 or order m with r-fold ($1 \le r \le m$) integer knows we mean a function of the continuity class $C^{m-1-r}(-\infty,\infty)$ which is a polynomial of degree at most m-1 in each interval (v, v+1). By a monospline of degree m we mean a function K(x) of the form

$$K(x) = \frac{x^m}{m!} + S(x) \tag{2.1}$$

where S(x) is a cardinal spline function of degree m-1.

Following from [3] and [1] there is a known one -to-one correspondence between monosplines and quadrature formulas. Thus, assuming $K(x)$ to be a monospline function of degree m in C^{m-1-r} and $f(x)$ in C^m, we have the general quadrature formula

$$(2.2)\quad \int_o^n f(x)dx = \sum_{v=0}^{n} A_v^o f(v) + \sum_{v=0}^{n} A'_v f'(v) + \cdots + \sum_{v=0}^{n} A_v^{r-1} f^{(r-1)}(v)$$

$$+ \sum_{j=r}^{m-1} B_j f^{(j)}(o) + \sum_{j=r}^{m-1} C_j f^{(j)}(n) + Rf$$

where

$$(2.3)\quad Rf = (-1)^m \int_o^n K(x) f^{(m)}(x)dx$$

and

$$(2.4)\quad A_o^k = (-1)^{k+1} K^{(m-1-k)}(o^+) \quad \text{for } k=o,1,\cdots, r-1,$$

$$A_n^k = (-1)^k K^{(m-1-k)}(n^-) \quad \text{for } k=o,1,\cdots, r-1,$$

$$A_v^k = (-1)^k (K^{(m-1-k)}(v^-) - K^{(m-1-k)}(v^+))$$

$$\text{for } 1 \le v \le n-1 \text{ and } k=o,1,\cdots, r-1,$$

$$B_j = (-1)^{j+1} K^{(m-1-j)}(o) \quad \text{for } j=r,r+1,\cdots, m-1,$$

$$C_j = (-1)^j K^{(m-1-j)}(n) \quad \text{for } j=r,r+1,\cdots, m-1.$$

The expression of the remainder (2.3) shows that (2.2) is exact for all polynomials of degree m-1 or less.

Given the natural numbers m and r, there are infinitely many monosplines of the form (2.1), each of which generates a quadrature formula (2.2) (2.3) which is exact for π_{m-1} (Here π_{m-1} denote the set of polynomials of degree not exceed m-1). Sard's idea was to determine the constants A_v^k, B_j, C_j of (2.2) so that

$$Rf=0 \qquad \text{if } f \in \pi_{m-1}$$

and

$$\int_o^n [K(x)]^2 dx = \text{minimum for all monosplines of degree m.}$$

If such a monospline exists, the corresponding formula is said to be best in the sense of Sard.

III. Interpolation of $X^{2m}/(2m)!$ by Cardinal Splines

Some notations and theorems from [1] and [4] are needed. We state the theorems without proofs.

THEOREM 1. (cf. [4], Theorem 1). There is a unique cardenal spline function of degreen 2m-1 with r-fold integer knots ($1\leq r\leq m$) which interpolates (at the integers) $f(x)=x^{2m}/(2m)!$ and its first r-1 derivatives and is in the growth class $0(|x|^{2m})$ as $|x|\to\infty$.

We then obtain the identity

$$x^{2m}/(2m)! = S(x) + R_{2m,r}(x) \tag{3.1}$$

where S(x) is the interpolating spline function and $R_{2m,r}(x)$ is the remainder. Let $P_{2m,r}(x)$ be the restriction of $R_{2m,r}(x)$ to [0,1], clearly $R_{2m,r}(x)=x^{2m}/(2m)!-S(x)$ is a monospline of degree 2m in the continuity class C^{2m-1-r}, while $P_{2m,r}(x)\epsilon\pi_{2m}$.

These two functions $R_{2m,r}(x)=R(x)$, $P_{2m,r}(x)=P(x)$ have the following properties:

THEOREM 2. (cf. [4], Theorems 7, 8)

(3.2) $R(v)=R'(v)=\cdots=R^{(r-1)}(v)=0$ for all integers v;

(3.3) $P^{(j)}(0)=P^{(j)}(1)$ for $j=r,r+1,\cdots,2m-r-1$;

(3.4) $R(x+1)=R(x)=R(-x)$;

(3.5) $P(1-x)=P(x)$;

(3.6) $\int_0^1 P(x)dx=(-1)^m \int_0^1 [P^{(m)}(x)]^2 dx$;

(3.7) if r is odd and <m, then
$R_{2m,r}(x)=R_{2m,r+1}(x)$ and hence similarly
$P_{2m,r}(x)=P_{2m,r+1}(x)$.

THEOREM 3. Differentiating a monospline of degree m produces a monospline of degree m-1 in C^{m-2-r}.

PROOF: Let $P(x)=\frac{x^m}{m!}+S(x)$ be a monospline of degree m in the class C^{m-1-r}, then clearly $P'(x)=\frac{x^{m-1}}{(m-1)!}+S'(x)$ is a monospline of degree m-1 in the class C^{m-2-r}.

THEOREM 4. [4]

(3.8) $$R_{2m,1}(x)=\frac{1}{(2m)!}(\bar{\beta}_{2m}(x)-\beta_{2m})$$

and

(3.9) $$R_{2m,m}(x)=\begin{cases}\frac{1}{(2m)!}(x^m(x-1)^m) & \text{if } 0\le x\le 1\\ R_{2m,m}(x-v) & \text{if } v\le x<v+1\end{cases}$$

where $\beta_{2m}(x)$ and β_{2m} are Bernoulli polynomials and Bernoulli numers of degree 2m [5], and $\bar{\beta}_{2m}(x)$ is the periodic extension (period 1) of the Bernoulli polynomial $\beta_{2m}(x)$.

THEOREM 5. [1]

(3.10) $$R^{(m)}_{2m,1}(x)=\frac{1}{m!}\bar{\beta}_m(x)\ \varepsilon\ C^{m-2},$$

(3.11) $$R^{(m)}_{2m,m}(x)=\begin{cases}\frac{1}{(2m)!}\frac{d^m}{dx^m}(x^m(x-1)^m) & \text{if } 0\le x\le 1\\ R^{(m)}_{2m,m}(x-v) & \text{if } v\le x<v+1\end{cases}$$

I.J. Schoenberg showed that the best quadrature formula corresponding to the monospline (3.10) is the Euler-Maclaurin sum formula (3.12), and the best quadrature formula corresponding to (3.11) is the classical Hermite's quadrature formula (3.13) [2].

(3.12) $$\int_0^n f(x)dx=T_nf+\sum_{o<2i\le m}\frac{\beta_{2i}}{(2i)!}(f^{(2i-1)}(o)-f^{(2i-1)}(n))+Rf$$

where $$Rf=\frac{(-1)^m}{m!}\int_0^n\bar{\beta}_m(x)f^{(m)}(x)dx$$

(3.13) $$\int_0^n f(x)dx=T_nf+\frac{1}{2!}\frac{m(m-1)}{2m(2m-1)}(f'(o)-f'(n))$$

$$+\frac{2}{3!}\frac{m(m-1)(m-2)}{2m(2m-1)(2m-2)}T_nf''$$

$$+\frac{1}{4!}\frac{m(m-1)\ (m-2)\ (m-3)}{2m(2m-1)\ (2m-2)\ (2m-3)}(f'''(o)-f'''(n))+\cdots$$

$$+\frac{1}{m!}\frac{m(m-1)\cdots 2.1}{2m(2m-1)\cdots(m+1)}(f^{(m-1)}(o)+(-1)^{m-1}f^{(m-1)}(1)$$

$$+f^{(m-1)}(1)+(-1)^{m-1}f^{(m-1)}(2)+\cdots+(-1)^{m-1}f^{(m-1)}(n))+Rf$$

where $\quad Rf=(-1)^m \int_o^n R_{2m,m}^{(m)}(x) f^{(m)}(x)dx,$

(3.14) $\quad T_n f^{(j)} = \frac{1}{2} f^{(j)}(o) + f^{(j)}(1) + \cdots + f^{(j)}(n-1) + \frac{1}{2} f^{(j)}(n).$

IV. The Intermediate Quadrature Formulas

Here we obtain the best quadrature formulas corresponding to $R_{2m,r}^{(m)}(x)$. Because of $R_{2m,r}(x)=R_{2m,r+1}(x)$ for odd $r<m$, we shall always assume that r is odd and $\le m$.

LEMMA 1. $\quad (-1)^j P^{(j)}(1) = P^{(j)}(0) \quad$ for $j=0,1,\cdots,2m.$

PROOF: For $P(1-x) = P(x)$,

then $(-1)^j P^{(j)}(1-x) = P^{(j)}(x)$, substituting $x=0$

gives $\quad (-1)^j P^{(j)}(1) = P^{(j)}(0).$

LEMMA 2. For odd $j \le 2m-1-r$, $P^{(j)}(0)=P^{(j)}(1)=0.$

PROOF: Combining Lemma 1 with (3.2),(3.3) completes the proof.

If we write

(4.1) $$P_{2m,r}(x) = \frac{1}{(2m)!}\left(x^{2m} + \binom{2m}{1} A_1 x^{2m-1} + \cdots + \binom{2m}{2m-1} A_{2m-1} x + A_{2m}\right)$$

then by Lemma 1,

(4.2) $$(-1)^j P^{(j)}(1) = P^{(j)}(0), \text{ for } j=0,1,\cdots,2m$$

we get

(4.3) $$(-1)^j \left(1 + \binom{2m-j}{1} A_1 + \binom{2m-j}{2} A_2 + \cdots + \binom{2m-j}{2m-j} A_{2m-j}\right) = A_{2m-j}$$

for $\quad j=0,1,\cdots,2m.$

Thus, we obtain the following system of 2m linear equations:

(4.4) $$1+A_1 = -A_1,$$
$$1+2A_1+A_2 = A_2,$$
$$1+3A_1+3A_2+A_3 = -A_3,$$

$$\vdots$$

$$1+\binom{j}{1}A_1+\binom{j}{2}A_2+\cdots+\binom{j}{j-1}A_{j-1}+A_j = (-1)^j A_j,$$

$$\vdots$$

$$1+\binom{2m}{2}A_1+\binom{2m}{2}A_2+\cdots+\binom{2m}{2m-1}A_{2m-1}+A_{2m} = A_{2m}$$

Using a symbolic notation, we can write the system in a very elegant way, namely $(1+A)^j=(-1)^j A^j$ $(j=1,2,\cdots,2m)$ with the convention that all powers A^i should be replaced by A_i. It is easy to prove that the system contains only m linearly independent equations $(1+A)^{2i} =A^{2i}$, for $i=1,2,\cdots,m$. Combining this system with $A_{2m} = A_{2m-1} = \cdots =A_{2m-r+1}=0$ (3.2) and $A_{2m-r}=A_{2m-r-2}= \cdots = A_{r+2}=0$ (Lemma 2), we can determine the A_i's. The polynomials $P_{2m,r}(x)$ may be regarded as generalizations of the even-degree Bernoulli polynomials [5]. Now, we reprove and find the explicit expression of the CLAIM in [1].

CLAIM Write $P_{2m,r}(x)= \frac{1}{(2m)!} \sum_{k=0}^{2m} \binom{2m}{k} A_k x^{2m-k}$ $(A_o=1)$, then for fixed m and r, the corresponding quadrature formula to $R_{2m,r}^{(m)}(x)$ is

$$\text{(4.5)} \quad \int_o^n f(x)dx=T_n f -\frac{2}{3!} A_3 T_n f'' - \cdots - \frac{2}{r!} A_r T_n f^{(r-1)}$$

$$+ \frac{1}{2!} A_2 (f'(o)-f'(n))+ \frac{1}{4!} A_4 (f'''(o)-f'''(n))+\cdots$$

$$+ \frac{1}{(t+1)!} A_{t+1} (f^{(t)}(o)-f^{(t)}(n))$$

$$+(-1)^m \int_o^n R_{2m,r}^{(m)}(x) f^{(m)}(x)dx$$

where $t=\begin{cases} m-1 & \text{if } m \text{ is even,} \\ m-2 & \text{if } m \text{ is odd,}\end{cases}$

and $T_n f^{(j)}=\frac{1}{2}f^{(j)}(o)+f^{(j)}(1)+\cdots+f^{(j)}(n-1)+\frac{1}{2}f^{(j)}(n)$.

PROOF: According to (2.2) with $K(x)=R_{2m,r}^{(m)}(x)$

$$\int_o^n f(x)dx=\sum_{v=0}^{n} A_v^o f(v) +\sum_{v=0}^{n} A'_v f'(v)+\cdots+\sum_{v=0}^{n} A_v^{r-1} f^{(r-1)}(v)$$

$$+ \sum_{j=r}^{m-1} B_j f^{(j)}(o)+\sum_{j=r}^{m-1} C_j f^{(j)}(n)+Rf$$

then $A_o^{\ell}=(-1)^{\ell+1}P^{(2m-1-\ell)}(o^+)$

$$=(-1)^{\ell+1}\frac{1}{(\ell+1)!}\sum_{k=o}^{\ell+1}\binom{\ell+1}{k}A_k x^{1+\ell-k}\Big|_{x=o^+}$$

$$=(-1)^{\ell+1}\frac{A_{\ell+1}}{(\ell+1)!} \quad \text{for } \ell=o,1,\cdots,r-1.$$

$$A_n^{\ell}=(-1)^{\ell}R^{(2m-1-\ell)}(n^-)$$
$$=(-1)^{\ell}R^{(2m-1-\ell)}(1^-)$$

$$=(-1)^{\ell}\frac{1}{(\ell+1)!}\sum_{k=o}^{\ell+1}\binom{\ell+1}{k}A_k x^{1+\ell-k}\Big|_{x=1^-}$$

$$=(-1)^{\ell}\frac{1}{(\ell+1)!}\sum_{k=o}^{\ell+1}\binom{\ell+1}{k}A_k \quad \text{for } \ell=o,1,\cdots,r-1$$

$$A_v^{\ell}=(-1)^{\ell}(R^{(2m-1-\ell)}(v^-)-R^{(2m-1-\ell)}(v^+))$$

$$=(-1)^{\ell}(R^{(2m-1-\ )}(1^-)-R^{(2m-1-\ell)}(o^+))$$

$$=(-1)^{\ell}\left(\frac{1}{(\ell+1)!}\sum_{k=o}^{\ell+1}\binom{\ell+1}{k}A_k-\frac{A_{\ell+1}}{(\ell+1)!}\right)$$

$$=(-1)^{\ell}\frac{1}{(\ell+1)!}\sum_{k=o}^{\ell}\binom{\ell+1}{k}A_k \quad \text{for } 1\le v\le n-1,\ o\le\ell\le r-1$$

$$B_{\ell}=(-1)^{\ell+1}R^{(2m-1-\ell)}(o)$$
$$=(-1)^{\ell+1}\frac{A_{\ell+1}}{(\ell+1)!} \quad \text{for } \ell=r,r+1,\cdots,m-1$$

$$C_{\ell}=(-1)^{\ell}R^{(2m-1-\ell)}(n)$$
$$=(-1)^{\ell}R^{(2m-1-\ell)}(1)$$
$$=(-1)^{\ell}\frac{1}{(\ell+1)!}\sum_{k=o}^{\ell+1}\binom{\ell+1}{k}A_k \quad \text{for } \ell=r,r+1,\cdots,m-1.$$

Substituting these coefficients into (2.2) gives

$$\int_o^n f(x)\,dx=-A_1 f(o)+f(1)+\cdots+f(n-1)+(1+A_1)f(n)$$
$$+\frac{A_2}{2!}f'(o)-\frac{1}{2!}(1+2A_1)(f'(1)+\cdots+f'(n-1))-\frac{1}{2!}(1+2A_1+A_2)f'(n)$$

$$-\frac{A_3}{3!}f''(o)+\frac{1}{3!}(1+3A_1+3A_2)(f''(1)+\cdots+f''(n-1))+\frac{1}{3!}(1+3A_1+3A_2+A_3)f''(n)$$

$$+\cdots-\frac{1}{r!}A_r f^{(r-1)}(o)+\frac{1}{r!}(1+\binom{r}{1}A_1+\cdots+\binom{r}{r-1}A_{r-1})(f^{(r-1)}(1)$$

$$+\cdots+f^{(r-1)}(n-1))+\frac{1}{r!}(1+\binom{r}{1}A_1+\cdots+\binom{r}{r-1}A_{r-1}+A_r)f^{(r-1)}(n)$$

$$+\sum_{j=r}^{m-1}(-1)^{j+1}\frac{A_{j+1}}{(j+1)!}f^{(j)}(o)+\sum_{j=r}^{m-1}(-1)^j\frac{1}{(j+1)!}(\sum_{k=0}^{\ell+1}A_k)f^{(j)}(n)$$

$$+(-1)^m\int_o^n R_{2m,r}^{(m)}(x)\ f^{(m)}(x)\,dx$$

From (4.4) $(1+A)^j=(-1)^jA^j$ with A^i replaced by A_i we get

$$\int_o^n f(x)dx=T_nf-\frac{2}{3!}A_3T_nf''-\frac{2}{5!}A_5T_nf^{(4)}-\cdots\frac{2}{r!}A_rT_nf^{(r-1)}$$

$$+\frac{1}{2!}A_2(f'(o)-f'(o)-f'(n))+\frac{1}{4!}A_4(f'''(o)-f'''(n))+\cdots$$

$$+\frac{1}{(t+1)!}A_{t+1}(f^{(t)}(o)-f^{(t)}(n))+(-1)^m\int_o^n R_{2m,r}^{(m)}(x)f^{(m)}(x)dx$$

where $t=\begin{cases} m-1 & \text{if } m \text{ is even,}\\ m-2 & \text{if } m \text{ is odd.}\end{cases}$

which completes the proof.

We call (4.5) an intermediate formula because of the following simple observation. In going from r=m (3.13) to r=r (4.5) ($1\le r\le m$) the terms involving $T_nf^{(j)}$ for even j r-1 drop out, while going from r=r (4.5) to r=1 (3.12) eliminates $T_nf^{(j)}$ for even j, $2\le j\le r-1$.

Peter R. Lipow showed that the quadrature formula corresponding to $R_{2m,r}^{(m)}(x)$ is best in the sense of Sard (see [1]), that is, if K(x) is any other monospline of degree m, whose quadrature formula involving the same f-data, then

$$\int_o^n [R_{2m,r}^{(m)}(x)]^2\,dx<\int_o^n K^2(x)dx$$

Using Schwarz's inequality to estimate $|Rf|$ in (2.2), we are interested in $\int_o^n [R_{2m,m}^{(m)}(x)]^2dx$. In [2], Schoenberg gives

$$\int_o^n [R_{2m,1}^{(m)}(x)]^2\,dx=\frac{|B_{2m}|}{(2m)!}\cdot n$$

$$\int_o^n [R_{2m,m}^{(m)}(x)]^2\, dx = \frac{1}{(2m+1)!}\left[\frac{m!}{(2m)!}\right]^2 \cdot n$$

for $1 < r < m$, we find that

$$\int_o^n [R_{2m,r}^{(m)}(x)]^2 dx = (-1)^m \int_o^n R_{2m,r}(x)\,dx$$

$$= (-1)^m \cdot n \int_o^1 P_{2m,r}(x)\,dx$$

$$= (-1)^m \cdot n \int_o^1 \frac{1}{(2m)!} \sum_{k=o}^{2m-r} \binom{2m}{k} A_k x^{2m-k} dx$$

$$= (-1)^m \cdot \frac{n}{(2m+1)!} \sum_{k=o}^{2m-r} \binom{2m+1}{k} A_k.$$

V. An Example

For m=6, r=3, by solving the system (4.4) with $A_{12}=A_{11}=A_{10}=A_9=A_7=A_5=0$, we find that $A_1=-\frac{1}{2}$, $A_2=\frac{43}{198}$, $A_3=-\frac{5}{66}$, $A_4=\frac{17}{990}$, $A_6=\frac{-1}{693}$, $A_8=\frac{1}{2970}$

and

$$\text{(5.1)}\quad P_{12,3}(x)=\frac{1}{12!}\left(x^{12}+\binom{12}{1}\left(-\frac{1}{2}\right)x^{11}+\binom{12}{2}\left(\frac{43}{198}\right)x^{10}+\binom{12}{3}\left(-\frac{5}{66}\right)x^9 + \binom{12}{4}\left(\frac{17}{990}\right)x^8+\binom{12}{6}\left(-\frac{1}{693}\right)x^6+\binom{12}{8}\left(\frac{1}{2970}\right)x^4\right)$$

so that

$$\text{(5.2)}\quad R_{12,3}^{(6)}(x)=\begin{cases}\frac{1}{6!}\left(x^6+\binom{6}{1}\left(-\frac{1}{2}\right)x^5+\binom{6}{2}\left(\frac{43}{198}\right)x^4+\binom{6}{3}\left(-\frac{5}{66}\right)x^3 + \binom{6}{4}\left(\frac{17}{990}\right)x^2-\frac{1}{693}\right) & \text{for } 0\le x\le 1\\ R_{12,3}^{(6)}(x-v) & \text{for } v\le x<v+1\end{cases}$$

and the corresponding formula is

$$\text{(5.3)}\quad \int_o^n f(x)\,dx=T_n f+\frac{5}{198}T_n f''+\frac{43}{296}(f'(o)-f'(n))+\frac{17}{4!\times 990}(f'''(o)-f'''(n)) - \frac{1}{6!}\times\frac{1}{693}(f^{(5)}(o)-f^{(5)}(n))+Rf$$

where

(5.4) $$Rf=(-1)^m \int_0^n R_{12,3}^{(6)}(x)f^{(6)}(x)dx.$$

For comparison we write down the Hermite and Euler-Maclaurin formulas for m=6

(5.5) $$\int_0^n f(x)dx=T_n f+\frac{2}{3!}\cdot\frac{6\cdot5\cdot4}{12\cdot11\cdot10}T_n f''+\frac{2}{5!}\cdot\frac{6\cdot5\cdot4\cdot3\cdot2}{12\cdot11\cdot10\cdot9\cdot8}T_n f^{(4)}$$

$$+\frac{1}{2!}\cdot\frac{6.5}{12\cdot11}(f'(o)-f'(n))+\frac{1}{4!}\cdot\frac{6\cdot5\cdot4\cdot3}{12\cdot11\cdot10\cdot9}(f'''(o)-f'''(n))$$

$$+\frac{1}{6!}\cdot\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{12\cdot11\cdot10\cdot9\cdot8\cdot7}(f^{(5)}(o)-f^{(5)}(n))$$

$$+(-1)^6\int_0^n R_{12,6}^{(6)}(x)f^{(6)}(x)dx$$

(5.6) $$\int_0^n f(x)dx=T_n f+\frac{1}{12}(f'(o)-f'(n)-\frac{1}{720}(f'''(o)-f'''(n)+\frac{1}{30240}(f^{(5)}(o)-$$

$$-f^{(5)}(n))+(-1)^6\int_0^n R_{12,1}^{(6)}(x)f^{(6)}(x)dx$$

and

(5.7) $$J_1\equiv\int_0^n [R_{12,1}^{(6)}(x)]^2dx=\frac{|\beta_{12}|}{12!}\cdot n=\frac{691n}{12!\cdot2730}$$

(5.8) $$J_3\equiv\int_0^n [R_{12,3}^{(6)}(x)]^2dx=\frac{1}{12!}\cdot\frac{53}{90090}n.$$

(5.9) $$J_6\equiv\int_0^n [R_{12,6}^{(6)}(x)]^2dx=\frac{1}{12!}\cdot\frac{1}{12012}n.$$

Observing that $J_1>J_3>J_6$, we may expect that J_r decreases for increasing r.

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