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On the Study of Curtis' (11, 8) Explicit Runge-Kutta Method

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ABSTRACT — The proposition that any solution to the reduced system is a solution to the original system is proved. The existence of the solution of the reduced system is verified. Some numerical examples are studied, and the results are compared with that derived from a (4 , 4) method. The results that are derived from different (b_9, b_{10}) are presented.

I. Introduction

The notation (and terminology) used is substantially that of Butcher [1-3].

Let

(1.1) $$dy_i/dx = f_i(x_0, y_1, y_2, \ldots, y_n), \quad Y_i(x_0) = y_{i,0} \qquad i=1,2,\ldots,n$$

be a system of n first order differential equations. An explicit Runge-Kutta method with ν stages gives a numerical result

(1.2) $$y_i(x_0+h) = y_{i,0} + h \sum_{I=1}^{\nu} b_I g_i^{(I)}, \qquad i=1,2,\ldots,n$$

where, $g_i^{(I)}$ is defined by

(1.3) $$g_i^{(I)} = f_i\left(x_0 + c_I h,\ y_{1,0} + h\sum_{J=1}^{I-1} a_{IJ} g_1^{(J)},\ \ldots,\ y_{n,0} + h\sum_{J=1}^{I-1} a_{IJ} g_n^{(J)}\right)$$
$$i=1,2,\ldots,n; \quad J=1,2,\ldots,\nu,$$

the $\frac{1}{2}\nu(\nu+1)$ constants b_I's, a_{IJ}'s characterize the process and

(1.4) $$c_1 = 0, \qquad c_I = \sum_{J=1}^{I-1} a_{IJ}$$

If each f_i is differentiable arbitrarily often with respect to x and each y_i, the true (theoretical) solution and the numerical solution can be expanded in powers of h (Taylor series). It has been shown [1], that these two expansions agree up to terms in h^p (i.e., the process is of order p, and we call it an explicit (ν,p) Runge-Kutta process.), if

(1.5) $\Phi=1/\gamma$ whenever $r\leq p$

where Φ is a typical elementary weight of order r. For $p=\nu\leq 4$, there are at least as many parameters b_I, a_{IJ} to be chosen as there are equations of the form (1.5). However, for $p=\nu>4$, there are more equations (1.5) than there are parameters to choose. From [2] and [3], we see that the explicit (ν,ν) Runge-Kutta process doesn't exist if $\nu\geq 5$; and if $p^*(\nu)$ is the highest order explicit Runge-Kutta method attainable with ν stages, then

$$p^*(\nu) = \nu \quad \text{if } \nu\leq 4$$
$$p^*(5) = 4$$
$$p^*(6) = 5$$
$$p^*(7) = 6$$
$$p^*(8) = 6$$
$$p^*(9) = 7$$
$$p^*(\nu) \leq \nu-2 \quad \text{if } \nu\geq 9$$

For p=8, it leads to a system of 200 equations of the form (1.5), which have to be satisfied [1]. In this case, Shanks [5] gave explicit processes with 12 stages, but A.R. Curtis[4] derives a process with 11 stages. As we know, it is still the only (11,8) method and there is no (10,8) method yet. In this article, we study the system of 200 equations (we call it the original system) and the reduced system (of 6ν+4 equations) constructed by A.R. Curtis, prove that the solutions of the reduced system exist and are available. Finally, using Curtis' data and the computer of Chiao-Tung University, we study some numerical examples, and compare them with the results derived by using 4th order Runge-Kutta method. The results derived from different (b_9, b_{10}) are presented too.

II. Main Theorem

To find a solution for the original system of 200 equations, A.R. Curtis [4] constructed a reduced system of 6ν+4 equations:

(2.1) $$\sum_{I=1}^{\nu} b_I c_I^r = 1/(r+1), \quad r=0,1,2,\ldots,7.$$

(2.2) $$\sum_{I,J=1}^{\nu} b_I c_I a_{IJ} c_J^4 = 1/35,$$

(2.3) $\sum_{I,J,K=1}^{\nu} b_I c_I a_{IJ} a_{JK} c_K^4 = 1/240$

(2.4) $\sum_{I,J=1}^{\nu} b_I c_I a_{IJ} c_J^5 = 1/48$

(2.5) $\sum_{I,J=1}^{\nu} b_I c_I^2 a_{IJ} c_J^4 = 1/40$

(2.6) $\sum_{I=1}^{\nu} b_I a_{IJ} = b_J(1-c_J),$ $J=1,2,\ldots,\nu$

(2.7) $\sum_{J=1}^{\nu} a_{IJ} c_J = (1/2)c_I^2$ $I=3,4,\ldots,\nu$

(2.8) $\sum_{J=1}^{\nu} a_{IJ} c_J^2 = (1/3)c_I^3$ $I=4,5,\ldots,\nu$

(2.9) $\sum_{J=1}^{\nu} a_{IJ} c_J^3 = (1/4)c_I^4$ $I=6,7,\ldots,\nu$

(2.10) $\sum_{I=1}^{\nu} b_I c_I^r a_{IJ} = 0$ $r=1,2;\ J=4,5$

(2.11) $\sum_{I,J=1}^{\nu} b_I c_I a_{IJ} a_{JK} = 0$ $K=4,5$

(2.12) $a_{I,2} = 0$ $I=4,5,\ldots,\nu$

(2.13) $a_{I,3} = 0$ $I=6,7,\ldots,\nu$

(2.14) $b_2 = b_3 = b_4 = b_5 = 0$

THEOREM 1. Any solution of the reduced system (2.1) - (2.14) is available (i.e., is a solution of the original system).

PROOF: We prove this theorem by exhibiting the original 200 equations and verifying that each of them can be derived from the reduced system. Each of the following summation is from 1 up to ν.

Since $c_1=0$ and $a_{IJ}=0$ if $J \geq I$, $I,J=1,2,\ldots,\nu$. therefore:

1. $\phi=\Sigma b_I=1$ holds by (2.1).

2. $[\phi]=\Sigma b_I c_I=1/2$ holds by (2.1).

3. $[_2\phi]_2=\Sigma b_I a_{IJ} c_J=1/6$ by (2.7),(2.1) this holds.

4. $[\phi^2]=\Sigma b_I c_I^2=1/3$ holds by (2.1).

5. $[_3\phi]_3=\Sigma b_I a_{IJ} a_{JK} c_K=1/24$ holds by (2.7), (2.8) and (2.1).

6. $[[\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J=1/8$ holds by (2.7), (2.1).

7. $[_2\phi^2]_2=\Sigma b_I a_{IJ} c_J^2=1/12$ holds by (2.14), (2.8) and (2.1).

8. $[\phi^3]=\Sigma b_I c_I^3=1/4$ holds by (2.1).

9. $[_4\phi]_4=\Sigma b_I a_{IJ} a_{JK} a_{KL} c_L=1/120$ holds by (2.7), (2.8), (2.14), (2.9), (2.1).

10. $[_3\phi^2]_3=\Sigma b_I a_{IJ} a_{JK} c_K^2=1/60$ holds by (2.14), (2.13), (2.8), (2.9), (2.1).

11. $[_2[\phi]\phi]_2=\Sigma b_I a_{IJ} c_J a_{JK} c_K=1/40$ holds by (2.7), (2.1).

12. $[_2\phi^3]_2=\Sigma b_I a_{IJ} c_J^3=1/20$ holds by (2.14), (2.9), (2.1).

13. $[[_2\phi]_2\phi]=\Sigma b_I c_I a_{IJ} a_{JK} c_K=1/30$ holds by (2.7), (2.8), (2.14) (2.13), (2.9), (2.1).

14. $[[\phi^2]\phi]=\Sigma b_I c_I a_{IJ} c_J^2=1/15$ by (2.14), (2.8), (2.1).

15. $[[\phi]^2]=\Sigma b_I a_{IJ} c_J a_{IK} c_K=1/20$ by (2.7), (2.1).

16. $[[\phi]\phi^2]=\Sigma b_I c_I^2 a_{IJ} c_J=1/10$ by (2.7), (2.1).

17. $[\phi^4]=\Sigma b_I c_I^4=1/5$ by (2.1).

18. $[_5\phi]_5=\Sigma b_I a_{IJ} a_{JK} a_{KL} a_{LM} c_M=1/720$

$[_5\phi]_5=(1/6)\Sigma b_I a_{IJ} a_{JK} c_K^3$ by (2.7), (2.8)

$=(1/6)[\Sigma b_J a_{JK} c_K^3-\Sigma b_J c_J a_{JK} c_K^3]$ by (2.6)

$=(1/6)(1/20-1/24)=1/720$ by (2.9), (2.1).

19. $[_4\phi^2]_4=\Sigma b_I a_{IJ} a_{JK} a_{KL} c_L^2=1/360$ by (2.6), (2.14), (2.13), $K\geq 3$.

20. $[_3[\phi]\phi]_3=\Sigma b_I a_{IJ} a_{JK} c_K a_{KL} c_L=1/240$ by (2.7), (2.6), (2.14), (2.13).

21. $[_3\phi^3]_3=\Sigma b_I a_{IJ} a_{JK} c_K^3=1/120$ by (2.7), (2.8), (2.9), (2.6), (2.1).

22. $[_2[_2\phi]_2\phi]_2=\Sigma b_I a_{IJ} c_J a_{JK} a_{KL} c_L=1/180$ by (2.7), (2.8), (2.6), (2.9), (2.1)

23. $[_2[\phi^2]\phi]_2=\Sigma b_I a_{IJ} c_J a_{JK} c_K^2=1/90$ by (2.7), (2.8), (2.6), (2.9), (2.1).

24. $[_2[\phi]^2]_2=\Sigma b_I a_{IJ} a_{JK} c_K a_{JL} c_L=1/120$ by (2.7),(2.6),(2.1).

25. $[_2[\phi]\phi^2]_2=\Sigma b_I a_{IJ} c_J^2 a_{JK} c_K=1/60$ by (2.7),(2.6),(2.1).

26. $[_2\phi^4]_2=\Sigma b_I a_{IJ} c_J^4=1/30$ by (2.7),(2.6),(2.1).

27. $[[_3\phi]_3\phi]=\Sigma b_I c_I a_{IJ} a_{JK} a_{KL} c_L=1/144$ by (2.7),(2.8),(2.14),(2.1).

28. $[[_2\phi^2]_2\phi]=\Sigma b_I c_I a_{IJ} a_{JK} c_K^2=1/72$ by (2.7),(2.8),(2.14),(2.1).

29. $[[[\phi]\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J a_{JK} c_K=1/48$ by (2.7),(2.14),(2.1).

30. $[[\phi^3]\phi]=\Sigma b_I c_I a_{IJ} c_J^3=1/24$ by (2.7),(2.14),(2.1).

31. $[[_2\phi]_2[\phi]]=\Sigma b_I a_{IJ} c_J a_{JK} a_{KL} c_L=1/72$ by (2.7),(2.8),(2.1).

32. $[[\phi^2][\phi]]=\Sigma b_I a_{IJ} c_J a_{IK} c_K^2=1/36$ by (2.7),(2.8),(2.1).

33. $[[_2\phi]_2\phi^2]=\Sigma b_I c_I^2 a_{IJ} a_{JK} c_K=1/36$ by (2.7),(2.8),(2.1).

34. $[[\phi^2]\phi^2]=\Sigma b_I c_I^2 a_{IJ} c_J^2=1/18$ by (2.7),(2.8),(2.1).

35. $[[\phi]^2\phi]=\Sigma b_I c_I a_{IJ} c_J a_{IK} c_K=1/24$ by (2.7),(2.1).

36. $[[\phi]\phi^3]=\Sigma b_I c_I^3 a_{IJ} c_J=1/12$ by (2.7),(2.1).

37. $[\phi^5]=\Sigma b_I c_I^5=1/6$ by (2.1).

38. $[_6\phi]_6=\Sigma b_I a_{IJ} a_{JK} a_{KL} a_{LM} a_{MN} c_N=1/5240$

$[_6\phi]_6=\Sigma b_J a_{JK} a_{KL} a_{LM} a_{MN} c_N-\Sigma b_J c_J a_{JK} a_{KL} a_{LM} a_{MN} c_N$ by (2.6)

$\Sigma b_J a_{JK} a_{KL} a_{LM} a_{MN} c_N=1/720$ by 18

$\Sigma b_J c_J a_{JK} a_{KL} a_{LM} a_{MN} c_N=1/840$ by (2.7),(2.8),(2.10),and $K\geq 5$, (2.9),(2.2).

39. $[_5\phi^2]_5=\Sigma b_I a_{IJ} a_{JK} a_{KL} a_{LM} c_M^2=1/2520$ by 38, $[_6\phi]_6=1/2[_5\phi^2]_5$.

40. $[_4[\phi]\phi]_4=\Sigma b_I a_{IJ} a_{JK} a_{KL} c_L a_{LM} c_M=1/1680$ by(2.7), $(1/6)[_6\phi]_6=\frac{1}{2}[_4[\phi]\phi]_4$.

41. $[_4\phi^3]_4=\Sigma b_I a_{IJ} a_{JK} a_{KL} c_L^3=1/840$ by 40.

42. $[_3[_2\phi]_2\phi]_3=\Sigma b_I a_{IJ} a_{JK} c_K a_{KL} a_{LM} c_M=1/1260$ by (2.7),(2.8),41.

43. $[_3[\phi^2]\phi]_3=\Sigma b_I a_{IJ} a_{JK} c_K a_{KL} c_L^2=1/630$ by (2.6),(2.14),(2.13),(2.8),42.

44. $[_3[\phi]^2]_3=\Sigma b_I a_{IJ} a_{JK} a_{KL} c_L a_{KM} c_M=1/840$ by (2.7),(2.6),(2.1),(2.2).

45. $[_3[\phi]\phi^2]_3=\Sigma b_I a_{IJ} a_{JK} c_K^2 a_{KL} c_L=1/420$ by (2.7),(2.6),(2.1),(2.2).

46. $[_3\phi^4]_3=\Sigma b_I a_{IJ} a_{JK} c_K^4=1/210$ by (2.6),(2.1),(2.2).

47. $[_2[_3\phi]_3\phi]_2=\Sigma b_I a_{IJ} c_J a_{JK} a_{KL} a_{LM} c_M=1/1008$ by (2.7),(2.8),(2.6)(2.14),(2.9).

48. $[_2[_2\phi^2]_2\phi]_2=\Sigma b_I a_{IJ} c_J a_{JK} a_{KL} c_L^2=1/504$ by (2.6),(2.14),(2.13),(2.8),47.

49. $[_2[[\phi]\phi]\phi]_2=\Sigma b_I a_{IJ} c_J a_{JK} c_K a_{KL} c_L=1/336$ by (2.7),(2.14),48.

50. $[_2[\phi^3]\phi]_2=\Sigma b_I a_{IJ} c_J a_{JK} c_K^3=1/168$ by (2.7),(2.9),(2.1).

51. $[_2[_2\phi]_2[\phi]]_2=\Sigma b_I a_{IJ} a_{JK} c_K a_{JL} a_{LM} c_M=1/504$ by (2.7),(2.8),(2.6)(2.1).

52. $[_2[\phi^2][\phi]]_2=\Sigma b_I a_{IJ} a_{JK} c_K a_{JL} c_L^2=1/252$ by (2.7),(2.13),(2.14),(2.6).

53. $[_2[_2\phi]_2\phi^2]_2=\Sigma b_I a_{IJ} c_J^2 a_{JK} a_{KL} c_L=1/252$ by 52.

54. $[_2[\phi^2]\phi^2]_2=\Sigma b_I a_{IJ} c_J^2 a_{JK} c_K^2=1/126$ by 53.

55. $[_2[\phi]^2\phi]_2=\Sigma b_I a_{IJ} c_J a_{JK} c_K a_{JL} c_L=1/168$ by 52.

56. $[_2[\phi]\phi^3]_2=\Sigma b_I a_{IJ} c_J^3 a_{JK} c_K=1/84$ by 53.

57. $[_2\phi^5]_2=\Sigma b_I a_{IJ} c_J^5=1/42$ by 54.

58. $[[_4\phi]_4\phi]=\Sigma b_I a_{IJ} a_{IK} a_{JL} a_{LM} a_{MN} c_N=1/840$ by (2.7),(2.8),(2.10),(2.9),(2.2).

59. $[[_3\phi^2]_3\phi]=\Sigma b_I c_I a_{IJ} a_{JK} a_{KL} c_L^2=1/420$ by (2.14),(2.10),(2.13),(2.8),(2.9),(2.2).

60. $[[_2[\phi]\phi]_2\phi]=\Sigma b_I c_I a_{IJ} a_{JK} c_K a_{KL} c_L=1/280$ by 59.

61. $[[_2\phi^3]_2\phi]=\Sigma b_I c_I a_{IJ} a_{JK} c_K^3=1/140$ by 60.

62. $[[[_2\phi]_2\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J a_{JK} a_{KL} c_L=1/210$ by (2.7),(2.8),(2.2).

63. $[[[\phi^2]\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J a_{JK} c_K^2=1/105$ by (2.14),(2.13),(2.10),(2.8),62.

64. $[[[\phi]^2]\phi]=\Sigma b_I c_I a_{IJ} a_{JK} c_K a_{JL} c_L=1/140$ by (2.7),(2.2).

65. $[[[\phi]\phi^2]\phi]=\Sigma b_I c_I a_{IJ} c_J^2 a_{JK} c_K=1/70$ by 64.

66. $[[\phi^4]\phi]=\Sigma b_I c_I a_{IJ} c_J^4=1/35$ by (2.2).

67. $[[{}_3\phi]_3[\phi]]=\Sigma b_I a_{IJ} a_{IK} c_K a_{JL} a_{LM} c_M=1/336$ by (2.7),(2.8),(2.14),(2.9),(2.1).

68. $[[{}_2\phi^2]_2[\phi]]=\Sigma b_I a_{IJ} c_J a_{IK} a_{KL} c_L^2=1/168$ by (2.7),(2.14),(2.13),(2.8),(2.10),(2.9).

69. $[[[\phi]\phi][\phi]]=\Sigma b_I a_{IJ} c_J a_{IK} c_K a_{KL} c_L=1/112$ by (2.7),68.

70. $[[\phi^3][\phi]]=\Sigma b_I a_{IJ} c_J a_{IK} c_K^3=1/56$ by 69.

71. $[[{}_3\phi]_3\phi^2]=\Sigma b_I a_{IJ_1} a_{IJ_2} a_{IK} a_{KL} a_{LM} c_M=1/168$ by (2.7),(2.8),70.

72. $[[{}_2\phi^2]_2\phi^2]=\Sigma b_I c_I^2 a_{IJ} a_{JK} c_K^2=1/84$ by (2.14),(2.13),(2.8),71.

73. $[[[\phi]\phi]\phi^2]=\Sigma b_I c_I^2 a_{IJ} c_J a_{JK} c_K=1/56$ by (2.7),72.

74. $[[\phi^3]\phi^2]=\Sigma b_I c_I^2 a_{IJ} c_J^3=1/28$ by 73.

75. $[[{}_2\phi]_2{}^2]=\Sigma b_I a_{IJ} a_{IK} a_{JL} c_L a_{KM} c_M=1/252$ by (2.7),(2.14),(2.8),(2.1).

76. $[[{}_2\phi]_2[\phi^2]]=\Sigma b_I a_{IJ} c_J^2 a_{IK} a_{KL} c_L=1/126$ by (2.7),75.

77. $[[\phi^2]^2]=\Sigma b_I a_{IJ} c_J^2 a_{IK} c_K^2=1/63$ by 76.

78. $[[{}_2\phi]_2[\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J a_{IK} a_{KL} c_L=1/84$ by (2.7),(2.14),(2.8),(2.1).

79. $[[\phi^2][\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J a_{IK} c_K^2=1/42$ by (2.7),78.

80. $[[{}_2\phi]_2\phi^3]=\Sigma b_I c_I^3 a_{IJ} a_{JK} c_K=1/42$ by (2.7),79.

81. $[[\phi^2]\phi^3]=\Sigma b_I c_I^3 a_{IJ} c_J^2=1/21$ by 80.

82. $[[\phi]^3]=\Sigma b_I a_{IJ} c_J a_{IK} c_K a_{IL} c_L=1/56$ by (2.7),(2.1).

83. $[[\phi]^2\phi^2]=\Sigma b_I c_I^2 a_{IJ} c_J a_{IK} c_K=1/28$.

84. $[[\phi]\phi^4]=\Sigma b_I c_I^4 a_{IJ} c_J=1/14$.

85. $[\phi^6]=\Sigma b_I c_I^6=1/7$.

86. $[{}_7\phi]_7=\Sigma b_I a_{IJ} a_{JK} a_{KL} a_{LM} a_{MN} a_{NO} c_O=1/40320$ by (2.7),(2.11),(2.9)(2.1),(2.2),(2.3).

87. $[{}_6\phi^2]_6=\Sigma b_I a_{IJ} a_{JK} a_{KL} a_{LM} a_{MN} c_N^2=1/20160$ by (2.6),(2.14),(2.8)(2.13),(2.9).

88. $[{}_5[\phi]\phi]_5=\Sigma b_I a_{IJ} a_{JK} a_{KL} a_{LM} c_M a_{MN} c_N=1/13440$ by (2.7),86

89. $[{}_5\phi^3]_5=\Sigma b_I a_{IJ} a_{JK} a_{KL} a_{LM} c_M^3=1/6720$ by 88.

90. $[_4[_2\phi]_2\phi]_4 = \Sigma b_I a_{IJ} a_{JK} a_{KL} c_L a_{LM} a_{MN} c_N = 1/10080$ by (2.3),42,(2.3).

91. $[_4[\phi^2]\phi]_4 = \Sigma b_I a_{IJ} a_{JK} a_{KL} c_L a_{LM} c_M^2 = 1/5040$ by (2.6),43,(2.10),(2.13).

92. $[_4[\phi]^2]_4 = \Sigma b_I a_{IJ} a_{JK} a_{KL} a_{LM} c_M a_{LN} c_N = 1/6720$ by 90.

93. $[_4[\phi]\phi^2]_4 = \Sigma b_I a_{IJ} a_{JK} a_{KL} c_L^2 a_{LM} c_M = 1/3360$ by 92.

94. $[_4\phi^4]_4 = \Sigma b_I a_{IJ} a_{JK} a_{KL} c_L^4 = 1/1680$ by 93.

95. $[_3[_3\phi]_3\phi]_3 = \Sigma b_I a_{IJ} a_{JK} c_K a_{KL} a_{LM} a_{MN} c_N = 1/8064$ by (2.7),(2.6),(2.8) (2.14),(2.10), (2.9),(2.1),(2.4).

96. $[_3[_2\phi^2]_2\phi]_3 = \Sigma b_I a_{IJ} a_{JK} c_K a_{KL} a_{LM} c_M^2 = 1/4032$ by 95.

97. $[_3[[\phi]\phi]\phi]_3 = \Sigma b_I a_{IJ} a_{JK} c_K a_{KL} c_L a_{LM} c_M = 1/2688$ by (2.7),95.

98. $[_3[\phi^3]\phi]_3 = \Sigma b_I a_{IJ} a_{JK} c_K a_{KL} c_L^3 = 1/1344$ by 97,(2.10),(2.13),(2.8), (2.3).

99. $[_3[_2\phi]_2[\phi]]_3 = \Sigma b_I a_{IJ} a_{JK} a_{KL} c_L a_{KM} a_{MN} c_N = 1/4032$ by (2.7),(2.8), (2.6),(2.1), (2.4).

100. $[_3[\phi^2][\phi]]_3 = \Sigma b_I a_{IJ} a_{JK} a_{KL} c_L a_{KM} c_M^2 = 1/2016$ by (2.7),(2.6),(2.14), (2.13),(2.8),(2.1),

101. $[_3[_2\phi]_2\phi^2]_3 = \Sigma b_I a_{IJ} a_{JK} c_K^2 a_{KL} a_{LM} c_M = 1/2016$ by (2.7),100.

102. $[_3[\phi^2]\phi^2]_3 = \Sigma b_I a_{IJ} a_{JK} c_K^2 a_{KL} c_L^2 = 1/1008$ by 101.

103. $[_3[\phi]^2\phi]_3 = \Sigma b_I a_{IJ} a_{JK} c_K a_{KL} c_L a_{KM} c_M = 1/1344$ by (2.7),99.

104. $[_3[\phi]\phi^3]_3 = \Sigma b_I a_{IJ} a_{JK} c_K^3 a_{KL} c_L = 1/672$ by (2.7),103.

105. $[_3\phi^5]_3 = \Sigma b_I a_{IJ} a_{JK} c_K^5 = 1/336$ by 104.

106. $[_2[_4\phi]_4\phi]_2 = \Sigma b_I a_{IJ} c_J a_{JK} a_{KL} a_{LM} a_{MN} c_N = 1/6720$ by (2.1),(2.8),(2.6) 40,(2.14),(2.10) (2,9),(2.5).

107. $[_2[_3\phi^2]_3\phi]_2 = \Sigma b_I a_{IJ} c_J a_{JK} a_{KL} a_{LM} c_M^2 = 1/3360$ by 106.

108. $[_2[_2[\phi]\phi]_2\phi]_2 = \Sigma b_I a_{IJ} c_J a_{JK} a_{KL} c_L a_{LM} c_M = 1/2240$ by (2.7),106.

109. $[_2[_2\phi^3]_2\phi]_2 = \Sigma b_I a_{IJ} c_J a_{JK} a_{KL} c_L^3 = 1/1120$ by 108.

110. $[_2[[_2\phi]_2\phi]\phi]_2 = \Sigma b_I a_{IJ} c_J a_{JK} c_K a_{KL} a_{LM} c_M = 1/1680$ by (2.7),(2.8), (2.6),(2.2).

111. $[_2[[\phi^2]\phi]\phi]_2 = \Sigma b_I a_{IJ} c_J a_{JK} c_K a_{KL} c_L^2 = 1/840$ by 110.

112. $[_2[[\phi]^2]\phi]_2=\Sigma b_I a_{IJ} c_J a_{JK} a_{KL} c_L a_{KM} c_M=1/1120$ by 111.

113. $[_2[[\phi]\phi^2]\phi]_2=\Sigma b_I a_{IJ} c_J a_{JK} c_K^2 a_{KL} c_L=1/560$ by 112.

114. $[_2[\phi^4]\phi]_2=\Sigma b_I a_{IJ} c_J a_{JK} c_K^4=1/280$ by 113.

115. $[_2[_3\phi]_3[\phi]]_2=\Sigma b_I a_{IJ} a_{JK} c_K a_{JL} a_{LM} a_{MN} c_N=1/2688$ by (2.7),(2.8),(2.6),(2.14),(2.9),(2.1).

116. $[_2[_2\phi^2]_2[\phi]]_2=\Sigma b_I a_{IJ} a_{JK} c_K a_{JL} a_{LM} c_M^2=1/1344$ by 115.

117. $[_2[[\phi]\phi][\phi]]_2=\Sigma b_I a_{IJ} a_{JK} c_K a_{JL} c_L a_{LM} c_M=1/896$ by 11,(2.7).

118. $[_2[\phi^3][\phi]]_2=\Sigma b_I a_{IJ} a_{JK} c_K a_{JL} c_L^3=1/448$ by 117,(2.7).

119. $[_2[_3\phi]_3\phi^2]_2=\Sigma b_I a_{IJ} c_J^2 a_{JK} a_{KL} a_{LM} c_M=1/1344$ by (2.7),(2.8),118.

120. $[_2[_2\phi^2]_2\phi^2]_2=\Sigma b_I a_{IJ} c_J^2 a_{JK} a_{KL} c_L^2=1/672$ by 119.

121. $[_2[\phi]\phi]\phi^2]_2 =\Sigma b_I a_{IJ} c_J^2 a_{JK} c_K a_{KL} c_L=1/448$ by (2.7),120.

122. $[_2[\phi^3]\phi^2]_2=\Sigma b_I a_{IJ} c_J^2 a_{JK} c_K^3=1/224$ by 121.

123. $[_2[_2\phi]_2{}^2]_2=\Sigma b_I a_{IJ} a_{JK} a_{KL} c_L a_{JM} a_{MN} c_N=1/2016$ by (2.7),(2.8),(2.6),(2.1).

124. $[_2[_2\phi]_2[\phi^2]]_2=\Sigma b_I a_{IJ} a_{JK} c_K^2 a_{JL} a_{LM} c_M=1/1008$ by 123.

125. $[_2[\phi^2]^2]_2=\Sigma b_I a_{IJ} a_{JK} c_K^2 a_{JL} c_L^2=1/504$ by 124.

126. $[_2[_2\phi]_2[\phi]\phi]_2=\Sigma b_I a_{IJ} c_J a_{JK} c_K a_{JL} a_{LM} c_M=1/672$ by (2.7),(2.8),123.

127. $[_2[\phi^2][\phi]\phi]_2=\Sigma b_I a_{IJ} c_J a_{JK} c_K a_{JL} c_L^2=1/336$ by (2.7),124.

128. $[_2[_2\phi]_2\phi^3]_2=\Sigma b_I a_{IJ} c_J^3 a_{JK} a_{KL} c_L=1/336$ by 126,(2.7),(2.8).

129. $[_2[\phi^2]\phi^3]_2=\Sigma b_I a_{IJ} c_J^3 a_{JK} c_K^2=1/168$ by 127.

130. $[_2[\phi]^3]_2=\Sigma b_I a_{IJ} a_{JK} c_K a_{JL} c_L a_{JM} c_M=1/448$ by 128,(2.6).

131. $[_2[\phi]^2\phi^2]_2=\Sigma b_I a_{IJ} c_J^2 a_{JK} c_K a_{JL} c_L=1/224$ by 130.

132. $[_2[\phi]\phi^4]_2=\Sigma b_I a_{IJ} c_J^4 a_{JK} c_K=1/112$ by 131.

133. $[_2\phi^6]_2=\Sigma b_I a_{IJ} c_J^6=1/56$ by 132.

134 $[[_5\phi]_5\phi]=\Sigma b_I c_I a_{IJ} a_{JK} a_{KL} a_{LM} a_{MN} c_N=1/5760$ by (2.7),(2.8),(2.11),(2.9),(2.3).

135. $[[_4\phi^2]_4\phi]=\Sigma b_I c_I a_{IJ} a_{JK} a_{KL} a_{LM} c_M^2=1/2880$ by 134.

136. $[[_3[\phi]\phi]_3\phi]=\Sigma b_I c_I a_{IJ} a_{JK} a_{KL} c_L a_{LM} c_M=1/1920$ by 135,(2.6).

137. $[[_3\phi^3]_3\phi]=\Sigma b_I c_I a_{IJ} a_{JK} a_{KL} c_L^3=1/960$ by 136.

138. $[[_2[_2\phi]_2\phi]_2\phi]=\Sigma b_I c_I a_{IJ} a_{JK} c_K a_{KL} a_{LM} c_M=1/1440$ by (2.7),(2.8), (2.3).

139. $[[_2[\phi^2]\phi]_2\phi]=\Sigma b_I c_I a_{IJ} a_{JK} c_K a_{KL} c_L^2=1/720$ by 138.

140. $[[_2[\phi]^2]_2\phi]=\Sigma b_I c_I a_{IJ} a_{JK} a_{KL} c_L a_{KM} c_M=1/960$ by 139,(2.6).

141. $[[_2[\phi]\phi^2]_2\phi]=\Sigma b_I c_I a_{IJ} a_{JK} c_K^2 a_{KL} c_L=1/480$ by 140.

142. $[[_2\phi^4]_2\phi]=\Sigma b_I c_I a_{IJ} a_{JK} c_K^4=1/240$ by 141.

143. $[[[_3\phi]_3\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J a_{JK} a_{KL} a_{LM} c_M=1/1152$ by (2.7),(2.8),(2.10).

144. $[[[_2\phi^2]_2\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J a_{JK} a_{KL} c_L^2=1/576$ by 143.

145. $[[[[\phi]\phi]\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J a_{JK} c_K a_{KL} c_L=1/384$ by 143.

146. $[[[\phi^3]\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J a_{JK} c_K^3=1/192$ by 145.

147. $[[[_2\phi]_2[\phi]]\phi]=\Sigma b_I c_I a_{IJ} a_{JK} c_K a_{JL} a_{LM} c_M=1/576$ by (2.7),(2.8), (2.4).

148. $[[[\phi^2][\phi]]\phi]=\Sigma b_I c_I a_{IJ} a_{JK} c_K a_{JL} c_L^2=1/288$ by 147.

149. $[[[_2\phi]_2\phi^2]\phi]=\Sigma b_I c_I a_{IJ} c_J^2 a_{JK} a_{KL} c_L=1/288$ by (2.7),148.

150. $[[[\phi^2]\phi^2]\phi]=\Sigma b_I c_I a_{IJ} c_J^2 a_{JK} c_K^2=1/144$ by (2.7),149.

151. $[[[\phi]^2\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J a_{JK} c_K a_{JL} c_L=1/192$ by (2.7),(2.4).

152. $[[[\phi]\phi^3]\phi]=\Sigma b_I c_I a_{IJ} c_J^3 a_{JK} c_K=1/96$ by (2.7),(2.4).

153. $[[\phi^5]\phi]=\Sigma b_I c_I a_{IJ} c_J^5=1/48$ by (2.4).

154. $[[_4\phi]_4[\phi]]=\Sigma b_I a_{IJ} c_J a_{IK} a_{KL} a_{LM} a_{MN} c_N=1/1920$ by (2.7),(2.8),(2.10) (2.9).

155. $[[_3\phi^2]_3[\phi]]=\Sigma b_I a_{IJ} c_J a_{IK} a_{KL} a_{LM} c_M^2=1/960$ by 154.

156. $[[_2[\phi]\phi]_2[\phi]]=\Sigma b_I a_{IJ} c_J a_{IK} a_{KL} c_L a_{LM} c_M=1/640$ by 154.

157. $[[_2\phi^3]_2[\phi]]=\Sigma b_I a_{IJ} c_J a_{IK} a_{KL} c_L^3=1/320$ by 155.

158. $[[[_2\phi]_2\phi][\phi]]=\Sigma b_I a_{IJ} c_J a_{IK} c_K a_{KL} a_{LM} c_M=1/480$ by (2.7),(2.8),(2.5).

159. $[[[\phi^2]\phi][\phi]]=\Sigma b_I a_{IJ} c_J a_{IK} c_K a_{KL} c_L^2=1/240$ by 158.

160. $[[[\phi]^2][\phi]]=\Sigma b_I a_{IJ} c_J a_{IK} a_{KL} c_L a_{KM} c_M=1/320$ by (2.7),(2.5).

161. $[[[\phi]\phi^2][\phi]]=\Sigma b_I a_{IJ} c_J a_{IK} c_K^2 a_{KL} c_L=1/160$ by (2.7),(2.5).

162. $[[\phi^4][\phi]]=\Sigma b_I a_{IJ} c_J a_{IK} c_K^4=1/80$ by 161.

163. $[[_4\phi]_4\phi^2]=\Sigma b_I c_I^2 a_{IJ} a_{JK} a_{KL} a_{LM} c_M=1/960$ by (2.7),(2.8),154.

164. $[[_3\phi^2]_3\phi^2]=\Sigma b_I c_I^2 a_{IJ} a_{JK} a_{KL} c_L^2=1/480$ by 163.

165. $[[_2[\phi]\phi]_2\phi^2]=\Sigma b_I c_I^2 a_{IJ} a_{JK} c_K a_{KL} c_L=1/320$ by (2.7),163.

166. $[[_2\phi^3]_2\phi^2]=\Sigma b_I c_I^2 a_{IJ} a_{JK} c_K^3=1/160$ by 165.

167. $[[[_2\phi]_2\phi]\phi^2]=\Sigma b_I c_I^2 a_{IJ} c_J a_{JK} a_{KL} c_L=1/240$ by (2.7),(2.8),(2.5).

168. $[[[\phi^2]\phi]\phi^2]=\Sigma b_I c_I^2 a_{IJ} c_J a_{JK} c_K^2=1/120$ by 167.

169. $[[[\phi]^2]\phi^2]=\Sigma b_I c_I^2 a_{IJ} a_{JK} c_K a_{JL} c_L=1/160$ by (2.7),(2.5).

170. $[[[\phi]\phi^2]\phi^2]=\Sigma b_I c_I^2 a_{IJ} c_J^2 a_{JK} c_K=1/80$ by (2.7),(2.5).

171. $[[\phi^4]\phi^2]=\Sigma b_I c_I^2 a_{IJ} c_J^4=1/40$ by 170.

172. $[[_3\phi]_3[_2\phi]_2]=\Sigma b_I a_{IJ} a_{JK} c_K a_{IL} a_{LM} a_{MN} c_N=1/1152$ by (2.14),(2.7) (2.8),(2.9).

173. $[[_3\phi]_3[\phi^2]]=\Sigma b_I a_{IJ} c_J^2 a_{IK} a_{KL} a_{LM} c_M=1/576$ by 172.

174. $[[_2\phi^2]_2[_2\phi]_2]=\Sigma b_I a_{IJ} a_{JK} c_K a_{IL} a_{LM} c_M^2=1/576$ by (2.14),(2.13),(2.7),(2.8),(2.9).

175. $[[_2\phi^2]_2[\phi^2]]=\Sigma b_I a_{IJ} c_J^2 a_{IK} a_{KL} c_L^2=1/288$ by 174.

176. $[[[\phi]\phi][_2\phi]_2]=\Sigma b_I a_{IJ} c_J a_{JK} c_K a_{IL} a_{LM} c_M=1/384$ by (2.14).

177. $[[[\phi]\phi][\phi^2]]=\Sigma b_I a_{IJ} c_J^2 a_{IK} c_K a_{KL} c_L=1/192$ by 176.

178. $[[\phi^3][_2\phi]_2]=\Sigma b_I a_{IJ} c_J^3 a_{IK} a_{KL} c_L=1/192$ by 177.

179. $[[\phi^3][\phi^2]]=\Sigma b_I a_{IJ} c_J^3 a_{IK} c_K^2=1/96$ by 178.

180. $[[_3\phi]_3[\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J a_{IK} a_{KL} a_{LM} c_M=1/384$ by (2.14),(2.7),(2.8),(2.9).

181. $[[_2\phi^2]_2[\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J a_{IK} a_{KL} c_L^2=1/192$ by (2.14),(2.13),(2.8),(2.9),(2.1).

182. $[[[\phi]\phi][\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J a_{IK} c_K a_{KL} c_L=1/128$ by (2.14),(2.7),(2.9)(2.1).

183. $[[\phi^3][\phi]\phi]=\Sigma b_I c_I a_{IJ} c_J a_{IK} c_K^3=1/64$ by (2.14),(2.7),(2.9),(2.1).

184. $[[_3\phi]_3\phi^3]=\Sigma b_I c_I^3 a_{IJ} a_{JK} a_{KL} c_L=1/192$ by (2.14),(2.7),(2.8),(2.9)(2.1).

185. $[[_2\phi^2]_2\phi^3]=\Sigma b_I c_I^3 a_{IJ} a_{JK} c_K^2=1/96$ by (2.14),(2.13),(2.8),(2.9),(2.1).

186. $[[[\phi]\phi]\phi^3]=\Sigma b_I c_I^3 a_{IJ} c_J a_{JK} c_K=1/64$ by (2.7),(2.14),(2.9),(2.1).

187. $[[\phi^3]\phi^3]=\Sigma b_I c_I^3 a_{IJ} c_J^3=1/32$ by (2.14),(2.9),(2.1).

188. $[[_2\phi]_2^2\phi]=\Sigma b_I c_I a_{IJ} a_{JK} c_K a_{IL} a_{LM} c_M=1/288$ by (2.7),(2.8),(2.1).

189. $[[_2\phi]_2[\phi^2]\phi]=\Sigma b_I c_I a_{IJ} c_J^2 a_{IK} a_{KL} c_L=1/144$ by (2.7),(2.8),(2.14)(2.1).

190. $[[\phi^2]^2\phi]=\Sigma b_I c_I a_{IJ} c_J^2 a_{IK} c_K^2=1/72$ by (2.14),(2.8),(2.1).

191. $[[_2\phi]_2[\phi]^2]=\Sigma b_I a_{IJ} c_J a_{IK} c_K a_{IL} a_{LM} c_M=1/192$ by (2.7),(2.8),(2.1).

192. $[[\phi^2][\phi]^2]=\Sigma b_I a_{IJ} c_J a_{IK} c_K a_{IL} c_L^2=1/96$ by 191.

193. $[[_2\phi]_2[\phi]\phi^2]=\Sigma b_I c_I^2 a_{IJ} c_J a_{IK} a_{KL} c_L=1/96$ by (2.7),(2.8),(2.1).

194. $[[\phi^2][\phi]\phi^2]=\Sigma b_I c_I^2 a_{IJ} c_J a_{IK} c_K^2=1/48$ by (2.14),(2.7),(2.8),(2.1).

195. $[[_2\phi]_2\phi^4]=\Sigma b_I c_I^4 a_{IJ} a_{JK} c_K=1/48$ by (2.7),(2.8),(2.1).

196. $[[\phi^2]\phi^4]=\Sigma b_I c_I^4 a_{IJ} c_J^2=1/24$ by (2.14),(2.8),(2.1).

197. $[[\phi]^3\phi]=\Sigma b_I c_I a_{IJ} c_J a_{IK} c_K a_{IL} c_L=1/64$ by (2.7).

198. $[[\phi]^2\phi^3]=\Sigma b_I c_I^3 a_{IJ} c_J a_{IK} c_K=1/32$ by (2.7).

199. $[[\phi]\phi^5]=\Sigma b_I c_I^5 a_{IJ} c_J=1/16$ by (2.7).

200. $[\phi^7]=\Sigma b_I c_I^7=1/8$.

then the proof of theorem 1 is completed.

THEOREM 2: The reduced system of $6\nu+4$ equations possesses only $6\nu-2$ independent equations. Hence it appears reasonable to try $\nu=11$.

PROOF: There are six redundant equations in (2.6). First by (2.14), (2.6) becomes

$$\sum_{I=6}^{\nu} b_I a_{IJ} = b_J(1-c_J) \qquad J=1,6,7,\ \ldots,\nu$$

$$=0 \qquad J=2,3,4,5$$

for the cases: J=2 and 3.

$$\sum_{I=6}^{\nu} b_I a_{I2} = \sum_{I=6}^{\nu} b_I a_{I3} = 0 \quad \text{holds by (2.12),(2.13)}$$

Next, for r=0,1,2,3 by (2.14)

$$\text{(2.15)} \qquad \sum_{J=1}^{\nu} c_J^r \{ \sum_{I=1}^{\nu} b_I a_{IJ} - b_J(1-c_J) \}$$

$$= \sum_{I=6}^{\nu} b_I (\sum_{J=1}^{\nu} a_{IJ} c_J^r) - \sum_{J=1}^{\nu} b_J c_J^r + \sum_{J=1}^{\nu} b_J c_J^{r+1} \quad \text{by (2.11)}$$

$$= \sum_{I=1}^{\nu} (1/(r+1)) b_I c_I^{r+1} - (1/(r+1)) + 1/(r+2) \quad \text{by (2.1),(2.8),(2.9),(2.1)}$$

$$=(1/(r+1))-(1/(r+2))-(1/(r+1))+1/(r+2)=0$$

Since $\nu \geq 0$ (see [2] and [3]) hence (2.1) implies there are at least 6 of $c_4, c_5, \ldots, c_\nu$ are nonzero and any two of them are unequal (i.e., $\nu \geq 6$ [2])

∴ (2.15) implies 4 of the equations

$$\sum_{I=1}^{\nu} b_I a_{IJ} = b_J(1-c_J) \qquad J \geq 4$$

hold. Therefore in the reduced system (2.6) contains only $\nu-6$ independent equations.

THEOREM 3: When $\nu=11$, the reduced system has a solution.

In the following Lemmas, we let $h_0=1$, $h_1=c_6+c_7+c_8$, $h_2=c_6c_7+c_6c_8+c_7c_8$ and $h_3=c_6c_7c_8$, assume any two of $c_6, c_7, c_8, c_9, c_{10}$ and $c_{11}=1$ are unequal (from [2] and [3], this is reasonable).

LEMMA 1. $(15h_0-20h_1+28h_2-42h_3)-(c_9+c_{10})(20h_0-28h_1+42h_2-70h_3)+c_9c_{10}(28h_0-42h_1+70h_2-140h_3)=0$

PROOF: From (2.1),(2.14) ∵ $c_1=0$, we have

$$\begin{cases} b_6c_6+b_7c_7+\cdots+b_{10}c_{10}+b_{11}-1/2=0 \\ b_6c_6^2+b_7c_7^2+\cdots+b_{10}c_{10}^2+b_{11}-1/3=0 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ b_6c_6^7+b_7c_7^7+\cdots+b_{10}c_{10}^7+b_{11}-1/8=0 \end{cases}$$

This is equivalent to the determinant

$$\begin{vmatrix} c_6 & c_7 & \cdots & c_{10} & 1 & -1/2 \\ c_6^2 & c_7^2 & \cdots & c_{10}^2 & 1 & -1/3 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ c_6^7 & c_7^7 & & c_{10}^7 & 1 & -1/8 \end{vmatrix}=0$$

By the theory of determinant (esp. the theory of Vandermonde determinant) we get

$$(1/56)-(1/42)\sum_{I=6}^{10} c_I+(1/30)\sum_{I<J} c_Ic_J-(1/20)\sum_{I<J<K} c_Ic_Jc_K$$

$$+(1/12)\sum_{I<J<K<L} c_Ic_Jc_Kc_L-(1/6)c_6c_7c_8c_9c_{10}=0$$

$$(1/56)h_0-(1/42)(h_1+c_9+c_{10})+(1/30)(h_2+(c_9+c_{10})h_1+c_9c_{10})$$

$$-(1/20)(h_3+(c_9c_{10})h_2\div c_9c_{10}h_1)+(1/10)((c_9+c_{10})h_3+c_9c_{10}h_2)$$

$$-(1/6)c_9c_{10}h_3=0$$

LEMMA 2. $(5h_0-8h_1+14h_2-28h_3)-c_9(8h_0-14h_1+28h_2-70h_3)=0$

PROOF: Let $\sigma_1=\sum_{I,J=1}^{11} b_I(1-c_I)a_{IJ}(c_J-c_6)(c_J-c_7)(c_J-c_8)(c_J$

$$-c_9)c_J=\sum_{J=2}^{11}\Delta_J\sum_{I=1}^{11} b_I(1-c_I)a_{IJ} \quad \because c_1=0$$

where $\Delta_J=c_J(c_J-6)(c_J-7)(c_J-8)(c_J-9)$, $J=2,3,\ldots,\nu=11$

From (2.12),(2.13),(2.14), and $c_{11}=1$, we have

$$\sigma_1=[\sum_{I=6}^{11} b_I(1-c_I)a_{I4}]\Delta_4+\sum_{I=6}^{11} b_I(1-c_I)a_{I5}\Delta_5$$

$\therefore \sum_{I=6}^{11} b_I c_I a_{I4} = \sum_{I=6}^{11} b_I c_I a_{I5} = 0, \quad ((2.10),\ r=1)$

$\therefore \sigma_1 = (\sum_{I=6}^{11} b_I a_{I4})\Delta_4 + (\sum_{I=6}^{11} b_I a_{I5})\Delta_5$

$= b_4(1-c_4)\Delta_4 + b_5(1-c_5)\Delta_5 = 0 \quad ((2.14),(2.6)$

In the other hand, multiplying out the factors, we have

$$\sigma_1 = \sum_{I,J=1}^{11} b_I a_{IJ} c_J^5 - (\sum_{I,J=1}^{11} b_I a_{IJ} c_J^4)H_1 + \Sigma b_I a_{IJ} c_J^3)H_2$$

$$-(\sum_{I,J} b_I a_{IJ} c_J^2)H_3 + (\sum_{I,J} b_{IJ} a_{IJ} c_J)H_4 - \sum_{I,J=1}^{11} b_I c_I a_{IJ} c_J^5$$

$$+(\sum_{I,J} b_I c_I a_{IJ} c_J^4)H_1 - (\sum_{I,J} b_I c_I a_{IJ} c_J^3)H_2$$

$$+(\sum_{I,J} b_I c_I a_{IJ} c_J^2)H_3 - (\sum_{I,J} b_I c_I a_{IJ} c_J)H_4$$

where $H_1 = \sum_{i=6}^{9} c_i \quad H_2 = \sum_{\substack{i,j=6\\ i<j}}^{9} c_i c_j \quad H_3 = \sum_{\substack{i,j,k=6\\ i<j<k}}^{9} c_i c_j c_k$

$H_4 = c_6 c_7 c_8 c_9.$

Hence by theorem 1, we have

$$\sigma_1 = (1/336) - (1/210)H_1 + (1/120)H_2 - (1/60)H_3 + (1/24)H_4$$
$$= (1/336)h_0 - (1/120)(h_1 + c_9) + (1/120)(h_2 + c_9 h_1)$$
$$-(1/60)(h_3 + c_9 h_2) + (1/24)c_9 h_3$$

From this, we obtain the Lemma 2.

LEMMA 3. $(9h_0 - 15h_1 + 28h_2 - 63h_3) - (3/2)c_{10}(8h_0 - 14h_1 + 28h_2 - 70h_3)$
$= 0$

PROOF: Let $\sigma_2 = \sum_{I,J=1}^{11} b_I(1-c_I)(c_I - c_{10})a_{IJ}(c_J - c_6)(c_J - c_7)(c_J - c_8)c_J$

$$= \sum_{J=2}^{11} \Delta_J \sum_{I=1}^{11} b_I(1-c_I)(c_I - c_{10})a_{IJ} \quad (c_1 = 0)$$

where, now $\Delta_J = c_J(c_J - c_6)(c_J - c_7)(c_J - c_8)$

From (2.12),(2.13),(2.14), terms of J=2,3,6,7,8,10,11 vanish.

$$\therefore \quad \sigma_2=\sum_{I=6}^{11} b_I(1-c_I)(c_I-c_{10})a_{I4}\Delta_4+\sum_{I=6}^{11} b_I(1-c_I)(c_I-c_{10})a_{I5}\Delta_5$$

$$+\sum_{I=10}^{11} b_I(1-c_I)(c_I-c_{10})a_{I9}\Delta_9$$

$$=\sum_{I=6}^{11} b_I(1-c_I)(c_I-c_{10})a_{I4}\Delta_4+\sum_{I=6}^{11} b_I(1-c_I)(c_I-c_{10})a_{I5}\Delta_5$$

$$\because \quad \sum_{I=6}^{11} b_I c_I^r a_{IJ}=0, \quad r=1,2; \quad J=4,5 \tag{2.10}$$

and

$$\sigma_2=\sum_{I=6}^{11} b_I(c_I-c_{10}-c_I^2+c_{10}c_I)a_{I4}\Delta_4+\sum_{I=6}^{11} b_I(c_I-c_{10}-c_I^2+c_{10}c_I)a_{I5}\Delta_5$$

$$\therefore \quad \sigma_2=-c_{10}[(\sum_{I=6}^{11} b_I a_{I4})\Delta_4+(\sum_{I=6}^{11} b_I a_{I5})\Delta_5]$$

$$=-c_{10}[b_4(1-c_4)\Delta_4+b_5(1-c_5)\Delta_5]=0$$

On the other hand, multiplying out the factors of each terms we have (from [1], pp. 191-193)

$$\sigma_2=((1/180)h_0-(1/168)h_1+(1/90)h_2-(1/40)h_3)-c_{10}((1/210)h_0$$

$$-(1/120)h_1+(1/60)h_2-(1/24)h_3)$$

and obtain the lemma.

LEMMA 4. $3h_0-6h_1+14h_2-42h_3=0$

PROOF: Let $\sigma_3=\sum_{I,J,K=1}^{11} b_I(1-c_I)a_{IJ}a_{JK}(c_K-c_6)(c_K-c_7)(c_K-c_8)c_K$

$$=\sum_{K=2}^{11}\Delta_K\sum_{I,J=1}^{11} b_I(1-c_I)a_{IJ}a_{JK}$$

the terms, vanish when K=1,6,7,8,9,10,11

$$K=2, \quad \sum_{\substack{J=3\\I=6}}^{11} b_I(1-c_I)a_{IJ}a_{JL}=\sum_{\substack{J=4\\I=6}}^{11} b_I(1-c_I)a_{IJ}a_{JL}=0$$

$$k=3, \quad \sum_{\substack{J=4\\I=6}}^{11} b_I(1-c_I)a_{IJ}a_{J3}=[\sum_{I=6}^{11} b_I(1-c_I)a_{I4}]a_{43}$$

$$+\left[\sum_{I=6}^{11} b_I(1-c_I)a_{I5}\right]a_{53}=0 \quad \text{(Pf. of } \sigma_1,)((2.13))$$

$$K=4, \quad \sum_{\substack{J=5\\I=6}}^{11} b_I(1-c_I)a_{IJ}a_{J4} \quad \text{(by (2.11))}$$

$$=\sum_{\substack{J=5\\I=6}}^{11} b_I a_{IJ} a_{J4} = \sum_{J=5}^{11}\left(\sum_{I=6}^{11} b_I a_{IJ}\right)a_{J4}$$

$$=\sum_{J=5}^{11} b_J(1-c_J)a_{J4}=0 \quad \text{(Pf. of lemma 2)}$$

Similarly,

$$K=5, \quad \sum_{\substack{J=6\\I=7}}^{11} b_I(1-c_I)a_{IJ}a_{J5}=0 \quad \text{(by (2.11),(2.6))}$$

On the other hand, similar to the last lemma, we have

$$\sigma_3=(1/1680)h_0-(1/840)h_1+(1/360)h_2-(1/120)h_3=0$$

and, then the proof of the lemma is completed.

LEMMA 5. $(4h_0-7h_1+14h_2-35h_3)\cdot\{(15-21c_{10})-(c_6+c_7)(28-42c_{10})$
$+c_6c_7(63-105c_{10})\}$
$=840\{3-7(c_6+c_7)+21c_6c_7\}\cdot\{b_9(c_9-c_6)(c_9-c_7)(c_9-c_8)c_9(1-$
$-c_9)(c_9-c_{10})\}$

PROOF: Define σ_4,σ_5 and σ_6 as [4] ((4.7),(4.8) & (4.9)) Similar to the proof of the last lemma, we get (4.12),(4.13), and (4.14) of [4].

From (4.12) of [4], we have

$$b_{10}(1-c_{10})a_{10,9}=((1/210)h_0-(1/120)h_1+(1/60)h_2$$
$$-(1/24)h_3)/(c_9-c_6)(c_9-c_7)(c_9-c_8)c_9$$

From (4.14) of [4], we have

$$a_{9,8}(c_8-c_6)(c_8-c_7)c_8=\{(1/168)-(1/120)c_{10}-((1/90)$$
$$-(1/60)c_{10})(c_6+c_7)+((1/40)-(1/24)c_{10})c_6c_7\}/b_9(1-c_9)(c_9$$

$-c_{10})$

hence (4.13) of [4] becomes

$$((1/120)h_0-(1/120)h_1+(1/60)h_2-(1/24)h_3\cdot\{(1/168)$$
$$-(1/120)c_{10}-((1/90)-(1/60)c_{10})(c_6+c_7)+((1/40)$$
$$-(1/24)c_{10})c_6c_7\}$$
$$=\{(1/840)-(1/360)(c_6+c_7)+(1/120)c_6c_7\}b_9(c_9-c_6)(c_9$$
$$-c_7)(c_9-c_8)c_9(1-c_9)(c_9-c_{10}).$$

This proves the lemma.

From these lemmas we can get a solution of b_I, a_{IJ} (Ref. Curtis [4], sec. 5), this completes the proof of theorem 3.

III. Numerical Examples

Now, we use Curtis' data ([4], pp. 274-276) and the computer of Chiao-Tung University to compute some numerical examples.

1. $y^{[4]}=y''(12y^2+8);\quad x_0=0,\quad y_0=y_0''=0,\quad y_0'=1,\quad y_0'''=2$

The true (theoretical) solution is $y=\tan(x)$.

Let $y''=z$, then $z''=z(12y^2+8)$

Let
$$y'=u=f_1(x,y,u,z,v)=u$$
$$u'=y''=z=f_2(x,y,u,z,v)=z$$
$$z'=v-f_3(x,y,u,z,v)=v$$
$$v'=z''=y^{[4]}=f_4(x,y,u,z,v)=z(12y^2+8)$$

Let
$$g_i^{(I)}=f_i(x_0+c_1h,\ y_0+h\sum_{J=1}^{I-1}a_{IJ}g_1^{(J)},\ u_0+h\sum_{J=1}^{I-1}a_{IJ}g_2^{(J)}$$
$$z_0+h\sum_{J=1}^{I-1}a_{IJ}g_3^{(J)},\ v_0+h\sum_{J=1}^{I-1}a_{IJ}g_4^{(J)},$$
$$i=1,2,3,4;\qquad I=1,2,\ ....,11$$

then
$$g_1^{(I)}=u_0+h\sum_{J=1}^{I-1}a_{IJ}g_2^{(J)}$$
$$g_2^{(I)}=z_0+h\sum_{J=1}^{I-1}a_{IJ}g_3^{(J)}$$
$$g_3^{(I)}=v_0+h\sum_{J=1}^{I-1}a_{IJ}g_4^{(J)}$$

$$g_4^{(I)}=(z_0+h\sum_{J=1}^{I-1} a_{IJ}g_3^{(J)})[12(y_0+h\sum_{J=1}^{I-1} a_{IJ}g_1^{(J)})^2+8],$$

and the numerical result

$$y(x_0+h)=y_0+h\sum_{I=1}^{11} b_I g_1^{(I)}$$

using the computer and for some h, we get

h	numerical result y(h)	true number tan(h)	error y(h) - tan(h)
0.10	0.100334671705	0.100334671326	$0.379305753473\times10^{-9}$
0.20	0.202710032724	0.202710034326	$-0.160125759874\times10^{-8}$
0.25	0.255341906169	0.255341917276	$-0.111074419662\times10^{-7}$
0.30	0.309336172371	0.309336245060	$-0.726888043645\times10^{-7}$
0.40	0.422792176416	0.422793217003	$-0.104058745864\times10^{-5}$
0.50	0.546294960439	0.546302489939	$-0.752949921567\times10^{-5}$
0.60	0.684101751215	0.684136822820	$-0.350716044484\times10^{-4}$
0.70	0.842184023131	0.842288367450	$-0.104344319663\times10^{-3}$
0.80	1.02954106466	1.02963852882	$-0.974641592901\times10^{-4}$
0.90	1.26127835689	1.26015816629	$0.112019059693\times10^{-2}$
1.00	1.56707577780	1.557740772188	$0.966805592090\times10^{-2}$

2. To compare the (11,8) method with 4th order method, we use Ralston's 4th order Runge-Kutta formula [6] to solve the last equation.

$$y^{[4]}=y''(12y^2+8), \qquad (x_0,y_0,y_0',y_0'',y_0''')=(0,0,1,0,2)$$

Now $a_{21}=0.4$

$a_{31}=0.29697761$ $a_{32}=0.15875964$

$a_{41}=0.21810040$ $a_{42}=-3.05096516$

$a_{43}=3.83286476$

$b_1=0.17476028$ $b_2=-0.55148066$ $b_3=1.20553560$ $b_4=0.17118478$

we get

h	numerical result y(h)	true result tan(h)	error y(h) - tan(h)
0.10	0.100333332975	0.100334671326	$-0.133835085547\times10^{-5}$
0.20	0.202666666036	0.202710034326	$-0.433682895352\times10^{-4}$
0.30	0.309000001299	0.309336245060	$-0.336243760580\times10^{-3}$

h	numerical result y(h)	true result tan(h)	error y(h) - tan(h)
0.40	0.421333332760	0.422793217003	$-0.145988424353\times 10^{-2}$
0.50	0.541666668923	0.546302489936	$-0.463582101379\times 10^{-2}$
0.60	0.672000005925	0.684136822820	$-0.121368168950\times 10^{-1}$
0.70	0.814333332863	0.842288367450	$-0.279550345875\times 10^{-1}$
0.80	0.970666671019	1.02963852820	$-0.589718578046\times 10^{-1}$
0.90	1.14300001046	1.26015816629	-0.117158155830
1.00	1.33333335138	1.55740772188	-0.224074370497

3. In Curtis' (11,8) method, the coefficients, b_9 and b_{10} are arbitrary. If we let $b_9=26/180$ $b_{10}=23/180$ (instead of $b_9=0.2$, $b_{10}=13/180$ used by Curtis) in example 1 ($y^{[4]}=y''(12y^2+8)$ $x_0=0$, $y_0=y_0''=0$, $y_0'=1$, $y_0'''=2$. We get

h	numerical result y(h)	error y(h) = tan(h)
0.10	0.100334671705	$0.379305753473 \times 10^{-9}$
0.20	0.202710032724	$-0.160125762649 \times 10^{-8}$
0.30	0.309336172371	$-0.726888043645 \times 10^{-7}$
0.40	0.422792176416	$-0.104058745864 \times 10^{-5}$
0.50	0.546294960437	$-0.752949921567 \times 10^{-5}$
0.60	0.684101751215	$-0.350716044485 \times 10^{-4}$
0.70	0.842184023131	$-0.104344319663 \times 10^{-3}$
0.80	1.02954106466	$-0.974641592904 \times 10^{-4}$
0.90	1.26127835689	$0.112019059693 \times 10^{-2}$
1.00	1.56707577780	$0.966805592090 \times 10^{-2}$

We see that the change of the results are very very small.

4. The equation $y^{[4]}=y''(4y^2+8y')$, $(x_0,y_0,y_0',y_0'',y_0''')=(0,0,1,0,2)$ has the same true solution $y=\tan(x)$. Using the same (11,8) method as example 1 and 3 respectively, we get

(1) $b_9=0.2$ $b_{10}=13/180$

h	numerical result y(h)	error y(h)-tan(h)
0.10	0.100334671705	$0.379275638673 \times 10^{-9}$
0.20	0.202710032662	$-0.166419222847 \times 10^{-8}$
0.30	0.309336166731	$-0.783289504924 \times 10^{-7}$

h	numerical result y(h)	error y(h)-tan(h)
0.40	0.422792035816	$-0.118118775647 \times 10^{-5}$
0.50	0.546293204719	$-0.928521726512 \times 10^{-5}$
0.60	0.684087432310	$-0.493905096174 \times 10^{-4}$
0.70	0.842095749938	$-0.192617512274 \times 10^{-3}$
0.80	1.02954106466	$-0.548865095044 \times 10^{-3}$
0.90	1.25923506475	$-0.923101543121 \times 10^{-3}$
1.00	1.55850353099	$0.109580911201 \times 10^{-2}$

(2) $b_9=26/180$ $b_{10}=23/180$

h	numerical result y(h)	error y(h)-tan(h)
0.10	0.100334671705	$0.379275638673 \times 10^{-9}$
0.20	0.202710032662	$-0.166419222847 \times 10^{-8}$
0.30	0.309336166731	$-0.783289504924 \times 10^{-7}$
0.40	0.422792035816	$-0.118118775644 \times 10^{-5}$
0.50	0.546293204719	$-0.928521726512 \times 10^{-5}$
0.60	0.684087432310	$-0.493905096174 \times 10^{-4}$
0.70	0.842095749938	$-0.192617512274 \times 10^{-3}$
0.80	1.02908966373	$-0.548865095044 \times 10^{-3}$
0.90	1.25923506475	$-0.923101543121 \times 10^{-3}$
1.00	1.55850353099	$0.109580911201 \times 10^{-2}$

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h	numerical result y(h)	error y(h)-tan(h)
0.40	0.422792035816	$-0.118118775647 \times 10^{-8}$
0.50	0.5462933204719	$-0.928531728512 \times 10^{-8}$
0.60	0.684087432310	$-0.483605096174 \times 10^{-8}$
0.70	0.842095748938	$-0.192617512274 \times 10^{-8}$
0.80	1.029638966373	$-0.548565095044 \times 10^{-8}$
0.90	1.259823506475	$-0.923101543131 \times 10^{-8}$
1.00	1.558503530099	$0.109568011291 \times 10^{-8}$

(2) $b_9=26/180$, $b_{10}=28/180$

h	numerical result y(h)	error y(h)-tan(h)
0.10	0.1003346471705	$0.379275633867 \times 10^{-9}$
0.20	0.202710032662	$-0.186419922884 \times 10^{-8}$
0.30	0.309336166731	$-0.783289504924 \times 10^{-8}$
0.40	0.422792035816	$-0.118118775644 \times 10^{-8}$
0.50	0.5462933204719	$-0.928531728512 \times 10^{-8}$
0.60	0.684087432310	$-0.483605096174 \times 10^{-8}$
0.70	0.842095748938	$-0.192617512274 \times 10^{-8}$
0.80	1.029638966373	$-0.548565095044 \times 10^{-8}$
0.90	1.258235064475	$-0.923101543131 \times 10^{-8}$
1.00	1.55850353099	$-0.109568011291 \times 10^{-8}$

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