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Matrix Inversion by Decomposition

馬爾傑 Er-Chieh Ma
Department of Applied Mathematics

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ABSTRACT — This paper starts with the inversion of a matrix which differs in one row, one column or both, from a matrix having a known inverse. The results lead to a method of matrix inversion by decomposition for any non-singular matrix.

I. Introduction

In dealing with linear programming problems, new bases are formed during computational procedures by changing an element or a vector in the old bases. The entire basis may even assume different values in parametric programming. The inversion of a new basis using informations available in the old one is essential for effective computations.

Sherman and Morrison [1] considered the inversion of matrix due to a change of one element. In a method of "tearing" by Kron [2], the inverse of a nonsingular matrix with a special form H=A+BDC is given by

$$H^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1} CA^{-1} ,$$

where A and D are non-singular, B and C are rectangular matrices. A similar result, called inverting modified matrices, was obtained by Woodbury [3], Bartlett [4] and described by Householder [5].

II. Inversion of a Matrix for Changing a Row And/Or a Column

Let the ℓ^{th} row of an nxn nonsingular matrix A, $A^{(\ell)}=(a_{\ell j})$, be changed to $A^{*(\ell)}=(a^*_{\ell j})$, the new matrix, assumed to be nonsingular, can be written as

$$A^* = A + R = (I + RA^{-1})A . \qquad (1)$$

where $R=(\delta_{i\ell} r_{ij})$, $r_{\ell j}=a^*_{\ell j}-a_{\ell j}$ and $\delta_{i\ell}$ is the Kronecker delta. Assume $A^{-1}=(A^{\dagger}_{ij})$, then

$$I+RA^{-1}=(\delta_{ij}+\delta_{i\ell}\Sigma_{s=1}^{n} r_{\ell s}a^{\dagger}_{sj}),$$

and

$$(I+RA^{-1})^{-1}=(\delta_{ij}+\beta\delta_{i\ell}\Sigma_{s=1}^{n} r_{\ell s}a^{\dagger}_{sj})$$
$$=I+\beta RA^{-1}$$

where $\beta=-(1+\alpha)^{-1}$ and $\alpha=\Sigma_{s=1}^{n} r_{\ell s} a^{+}_{s\ell}$. The inverse of A^* can be expressed in a product from in terms of A^{-1} and R

$$(A^*)^{-1}=A^{-1}(I+\beta RA^{-1})=(I+\beta A^{-1}R)A^{-1} \quad (2)$$

This leads to the following theorem:

THEOREM 1: Let $A^{-1}=(a_{ij})$ $R=(\delta_{i\ell}r_{ij})$ $r_{\ell j}=a^*_{\ell j}-a_{\ell j}$

$\alpha=\Sigma_{s=1}^{n} r_{\ell s} a^{\dagger}_{s\ell}$ and $\beta=-(1+\alpha)^{-1}$, the inverse of $A^*=A+R$ is given by

$$(A^*)^{-1}= A^{-1}(I+\beta RA^{-1}) = (I+\beta A^{-1}R)A^{-1}.$$

Since A is nonsingular in (1), A* is singular if and only if $(I+RA^{-1})$ is singular. Since $|I+RA^{-1}|=1+\alpha$, $(I+RA^{-1})$ is singular if and only if $1+\alpha=0$. This leads to the second theorem:

THEOREM 2. A* is singular if and only if $(I+RA^{-1})$ is singular, therefore if and only if $1+\alpha=0$.

If the k^{th} column A_k of A is changed to A^*_k, A* may be decomposed as

$$A^*= A + C = A(I + A^{-1}C), \quad (3)$$

where $C=(\delta_{kj} c_{ij})$ and $c_{ik}=a^*_{ik}-a_{ik}$. A similar expression of $(A^*)^{-1}$ may be given by

$$(A^*)^{-1}=A^{-1}(I+\mu CA^{-1})=(I+\mu A^{-1}C)A^{-1}, \quad (4)$$

where $\mu=-(1+\tau)^{-1}$ and $\tau=\Sigma_{s=1}^{n} a^{+}_{ks} c_{sk}$. A^* is singular if and only if $1+\tau=0$.

We notice that equations (2) and (4) have similar expressions. In case that the ℓ^{th} row of A is changed.

$$I + \beta RA^{-1} = (\delta_{ij}+\beta\delta_{i\ell}\Sigma_{s=1}^{n} r_{\ell s} a^{\dagger}_{sj})$$

and

$$I + \beta A^{-1}R = (\delta_{ij}+\beta a^{\dagger}_{i\ell}r_{\ell j})$$

the expression $(A^*)^{-1}=A^{-1}(I+\beta RA^{-1})$ can be computed more effectively. If the k^{th} column of A is changed, the expression $(A^*)^{-1}=(I+\mu A^{-1}C)A^{-1}$ will be preferred.

In case that the ℓ^{th} row and the k^{th} column of A are changed simulaneously, then

$$A^* = A + R\ C,$$

where $r_{\ell k}+c_{\ell k}=a^*_{\ell k}-a_{\ell k}$. We may assign arbitrarily $r_{\ell k}=0$ if $1+ \neq 0$ or $c_{\ell k}=0$ if $1+\tau\neq 0$. The inverse of A^* can be computed by (2) and (4) successively in either order.

III. Matrix Inverion by Decomposition

Let A be an nxn nonsingular matrix. It may be decomposed as

$$A=B_o+\Sigma_{s=1}^{n} R_s \tag{5}$$

$$A=B_o+\Sigma_{s=1}^{n} C_s \tag{6}$$

or combination of (5) and (6),

where $B_o=(\delta_{ij}a_{ij})$, $R_s=((\delta_{is}-\delta_{is}\delta_{sj})a_{ij})$ and $C_s=((\delta_{sj}-\delta_{is}\delta_{sj})a_{ij})$.

Further, let $B_i=B_{i-1}+R_i$ or $B_i=B_{i-1} + C_i$, i=1,2,...n, then

$$B_i^{-1}=B_{i-1}^{-1}(I+\beta_{i-1}R_iB_{i-1}^{-1}) \tag{7}$$

or

$$B_i^{-1}=(I+\mu_{i-1}B_{i-1}^{-1}C_i)B_{i-1}^{-1}, \tag{8}$$

i=1,2,...n. The inverse of A is given by B_n^{-1}.

Equations (7) or (8) are valid if and only if $1+\tau_{i-1}\neq 0$ or $1+\alpha_{i-1}\neq 0$ for i=1, 2, ... n. Assume that A is decomposed according to (5) or (6), these conditions are satisfied if all the principal submatrices obtained in succession from the top left corner are non-singular. In case one of the submatrices is singular, a change of order of decomposition in (5) or (6) usually will take care of the situation.

If some $a_{hh}=0$ in A, we may assign $a_{hh}=1$ in B_o and let $R_h=(\delta_{ih}a_{ij}-\delta_{ih}\delta_{hj})$ or $C_h=(\delta_{hj}a_{ij}-\delta_{ih}\delta_{hj})$in (5) or (6).

The number of products including multiplication and divisions required in computing $B_i^{-1}=B_{i-1}^{-1}(I+\beta_{i-1}R_iB_{i-1}^{-1})$ are listed in the following table.

Terms	No. of Products
α_{i-1}	i-1
β_{i-1}	1
$R_iB_{i-1}^{-1}$	(i-1)(n-1)+i
$\beta R_iB_{i-1}^{-1}$	n
B_i^{-1}	(i-1)[2(n-1)x1]+ n

For an nxn matrix, the total number of products required in computing its inverse is $n(3n^2+n+4)/2$.

References

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