The Journal of National Chiao Tung University
Vol. I, April 1976
pp.211-216

帕德表中 $\{R_{n-3,n}(z)\}_{n=3}^{\infty}$ 序列之討論

On the Sequence $\{R_{n-3,n}(z)\}_{n=3}^{\infty}$ in the Padé Table

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(Received January 15, 1976)

ABSTRACT — The basic aim of this paper is to show that a specific sequence $\{R_{n-3,n(z)}\}_{n=3}^{\infty}$ in the Padé table is bounded along the imaginary axis. Such a result can be applied to verify that $\{R_{n-3,n}(z)\}_{n=3}^{\infty}$ converges uniformly to e^{-z} on the sector $S_\delta \equiv \{z=re^{i\theta}: |\theta| \leq \frac{\pi}{2} - \delta\}$

I. Introduction

In the Padé table, it is already known that $||R_{n-1,n}(iy)||_{L_\infty(-\infty,+\infty)}$ and $||R_{n-2,n}(iy)||_{L_\infty(-\infty,+\infty)}$ are both bounded by 1. We shall consider, in particular, the sequence $\{R_{n-3,n}(z)\}_{n=3}^{\infty}$ in this table. Although $||R_{n-3,n}(iy)||_{L_\infty(-\infty,\infty)}$ is not bounded by 1, we shall verify that $\lim\limits_{n\to\infty} ||R_{n-3,n}(iy)||_{L_\infty(-\infty,+\infty)} = 1$. Applying a similar argument as that for the uniform convergence of $\{R_{n-1,n}(z)\}_{n=1}^{\infty}$ and $\{R_{n-2,n}(z)\}_{n=2}^{\infty}$ to e^{-z}, we also show that $\{R_{n-3,n}(z)\}_{n=3}^{\infty}$ converges uniformly to e^{-z} on the sector $S_\delta \equiv \{z=re^{i\theta}: |\theta| \leq \frac{\pi}{2} - \delta\}$.

II. Results and Discussion

Let π_m denote the set of all complex polynomials in the variable z with degree at most m, and let $\pi_{\nu,n}$ denote the set of all complex rational functions $R_{\nu,n}(z)$ of the form $R_{\nu,n}(z) = \frac{Q_{\nu,n}(z)}{P_{\nu,n}(z)}$, where $Q_{\nu,n}(z) \in \pi_\nu$, $P_{\nu,n} \in \pi_n$ and $P_{\nu,n}(0) = 1$. For any function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, analytic in a neighborhood of z=0, and for any nonnegative integers ν and n, the (ν,n)-th Pade' approximant to f(z) is defined as the element $R_{\nu,n} \in \pi_{\nu,n}$ for which

$$f(z)-R_{\nu,n}(z)=\mathcal{O}(|z|^m)$$

as $|z|\to 0$ is valid for the largest integer m. It is known that for $f(z)=e^{-z}$, its (ν,n)-th Padé approximant $R_{\nu,n}(z)\frac{Q_{\nu,n}(z)}{P_{\nu,n}(z)}$ is given explicitly by [1]

$$Q_{\nu,n}(z)\equiv\sum_{k=0}^{\nu}\frac{(\nu+n-k)!\nu!(-z)^k}{(\nu+n)!k!(\nu-k)!}$$

and

$$P_{\nu,n}(z)\equiv\sum_{k=0}^{n}\frac{(\nu+n-k)!n!z^k}{(\nu+n)!k!(n-k)!}$$

In [1] , we have the results that $R_{n-1,n}(z)$ and $R_{n-2,n}(z)$ are analytic for Re $z\le 0$ and bounded there by unity. Moreover, it is also shown [1] that $R_{n-3,n}(iy)$ is not bounded by unity over the interval $-\sqrt{n^2-2n}<y<\sqrt{n^2-2n}$ for $n\ge 3$.

To show that $\lim_{n\to\infty}||R_{n-3,n}(iy)||_{L_\infty(-\infty,+\infty)}=1$, we state the following lemmas which can be easily established using simple calculations.

LEMMA 2.1 For any $n\ge 3$,

$$|R_{n-3,n}(iy)|^2=1-(y^2-n^2+2n)\left[\frac{(n-3)!y^{n-1}}{(2n-3)!}\right]^2/|P_{n-3,n}(iy)|^2 .$$

LEMMA 2.2 For $|y|>\sqrt{n^2-2n}$ and $n\ge 3$, $|R_{n-3,n}(iy)|<1$.

LEMMA 2.3 If

$$(2.1)\quad \sup_{|y|\le n-1}\left\{(y^2-n^2+2n)\left[\frac{(n-3)!y^{n-1}}{(2n-3)!}\right]^2/|P_{n-3,n}(iy)|^2\right\}\to 0$$

as $n\to\infty$, then

$$\lim_{n\to\infty}||R_{n-3,n}(iy)||_{L_\infty(-\infty,+\infty)}=1 .$$

Now, since $P_{n-3,n}(iy)$ can be written in the form

$$P_{n-3,n}(iy)=K\prod_{j=1}^{n}(iy-\lambda_j) ,$$

where $K=\frac{(n-3)!}{(2n-3)!}$ and λ_j is a zero of $P_{n-3,n}(z)$, we can rewrite the quantity in (2.1) as:

$$(y^2-n^2+2n)\cdot y^{2n-2} / \left|\prod_{j=1}^{n}(iy-\lambda_j)\right|^2$$

LEMMA 2.4 $P_{n-3,n}(z)$ has no zeros in the region $G=\{z=x+it: t^2 \leq 4(n-2)(n+x-2),\ x>-(n-2)\}$.

PROOF. This is just a special case of the Parabola theorem in [3] with $\nu=n-3$.

LEMMA 2.5 For any two conjugate zeros λ_j and $\bar{\lambda}_j$ of $P_{n-3,n}(z)$, there exist two conjugate points z_j and $\bar{z}_j$ on the parabola $t^2=4(n-2)(x+n-2)$ such that

$$(2.2)\qquad |iy-\lambda_j|\cdot|iy-\bar{\lambda}_j| \geq |iy-z_j|\cdot|iy-\bar{z}_j|,$$

for all y with $|y| \leq n-1$ and $n \geq 3$.

PROOF. We consider first the case when $|\mathrm{Im}\lambda_j|>2(n-2)$. Without loss of generality, we may assume $\mathrm{Im}\,\lambda_j>0$. We claim that the choice $z_j \equiv 2i(n-2)$ satisfies (2.2) in this case. By LEMMA 2.4, we know that the zeros of $P_{n-3,n}(z)$ have to lie outside the parabola $t^2=4(n-2)\cdot\{x+n-2\}$. From Figure 1, it is evident in this case that for any real y with $|y| \leq n-1$,

$$|iy-\lambda_j|>|iy-2i(n-2)| \text{ and } |iy-\bar{\lambda}_j|>|iy+2i(n-2)|,$$

whence

$$|iy-\lambda_j|\cdot|iy-\bar{\lambda}_j|>|iy-z_j|\cdot|iy-z_j|\cdot|iy-\bar{z}_j|.$$

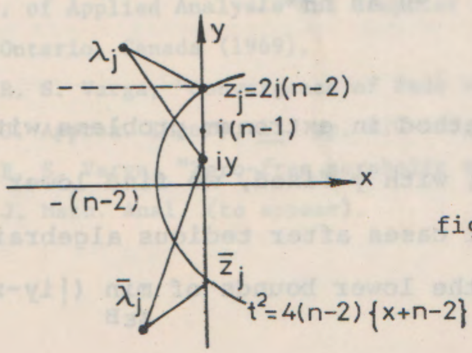

fig. 1

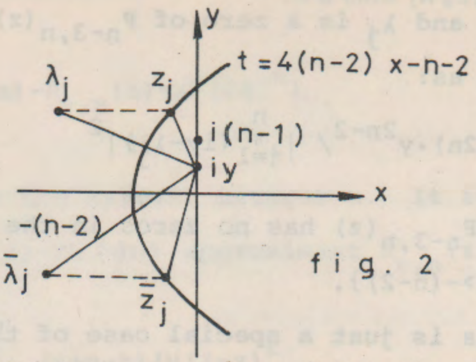

fig. 2

Next, consider the case when $|\text{Im}\,\lambda_j| \le 2(n-2)$. Again, without loss of generality, we may assume that $\text{Im}\,\lambda_j \ge 0$. Define z_j to be the point on the parabola $t^2=4(n-2)\{x+n-2\}$ such that $\text{Im}\,z_j=\text{Im}\,\lambda_j$, as shown in Figure 2. In like manner, one sees from Figure 2 that for any real y with $|y| \le n-1$,

$$|iy-\lambda_j|>|iy-z_j| \text{ and } |iy-\bar{\lambda}_j|>|iy-\bar{z}_j| ,$$

whence

$$|iy-\lambda_j|\cdot|iy-\bar{\lambda}_j|>|iy-z_j|\cdot|iy-\bar{z}_j| .$$

LEMMA 2.6 If $\{\lambda_j\}_{j=1}^{n}$ are the zeros of $P_{n-3,n}(z)$ and $|y| \le n-1$, then

$$\left|\prod_{j=1}^{n}(iy-\lambda_j)\right|^2 \ge \min_{z\in B}\,(|iy-z|\cdot|iy-\bar{z}|)^n ,$$

where $B=\{z=x+it:t^2=4(n-2)(x+n-2)\}$.

PROOF. This follows simply from LEMMA 2.5.

The point now is to show that

$$\frac{y^2-n^2+2n}{\left|\prod_{j=1}^{n}(iy-\lambda_j)\right|^2} \to 0 \text{ as } n\to\infty .$$

Applying Lagrange's method in extremum problems with side conditions to $|iy-z|\cdot|iy-\bar{z}|$ with y fixed, we find lower bounds of $\min_{z\in B}($ $|iy-z|\cdot|iy-\bar{z}|)^n$ in all cases after tedious algebraic manipulations. A comparison between the lower bounds of $\min_{z\in B}(|iy-z|\cdot|iy-\bar{z}|)^n$ and

y^2-n^2+2n gives,

$$\frac{y^2-n^2+2n}{\min\limits_{z\varepsilon B}\ (|iy-z|\cdot|iy-\bar{z}|)^n} \to 0 \text{ as } n\to\infty.$$

By the virtue of LEMMA 2.6, we have that

$$\frac{y^2-n^2+2n}{|\prod\limits_{j=1}^{n}(iy-\lambda_j)|^2} \to 0 \text{ as } n\to\infty$$

and consequently the following theorem is proved.

THEOREM 2.1. $\lim\limits_{n\to\infty} ||R_{n-3,n}(iy)||_{L_\infty(-\infty,+\infty)}=1$, where $R_{n-3,n}(z)$ is the Padé approximation of e^{-z} with $z=x+iy$.

With this result, we may use a similar argument as in [2-, THEOREM 4.3] which yields the following theorem.

THEOREM 2.2 Given any δ with $0\leq\delta<\frac{\pi}{2}$, the sequence $\{R_{n-3,n}(z)\}_{n=3}^{\infty}$ converges uniformly to e^{-z} on the sector $S_\delta\equiv\{z=re^{i\theta}:|\theta| \leq \frac{\pi}{2}-\delta\}$.

Numerical results have indication that

$$\lim_{n\to\infty}||R_{n-\nu,n}(iy)||_{L_\infty(-\infty,\ +\infty)} = 1$$

for any fixed positive integer ν . Such a general statement, however, is still left to be proved.

References

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3. E. B. Saff and R. S. Varga, "Zero-free parabolic regions for sequences of polynomials", SIAM J. Math. Anal. (to appear).

y^2-n^2+2n gives.

$$\frac{y^2-n^2+2n}{\min_{z\in B}(|iy-z|\cdot|iy-\bar{z}|)^n} \to 0 \text{ as } n\to\infty.$$

By the virtue of LEMMA 2.6, we have that

$$\frac{y^2-n^2+2n}{\left|\prod_{j=1}^{n}(iy-\lambda_j)\right|^2} \to 0 \text{ as } n\to\infty$$

and consequently the following theorem is proved.

THEOREM 2.1. $\lim_{n\to\infty}\|R_{n-3,n}(iy)\|_{L_\infty(-\infty,+\infty)}=1$, where $R_{n-3,n}(z)$ is the Padé approximation of e^{-z} with $z=x+iy$.

With this result, we may use a similar argument as in [2, THEOREM 4.3] which yields the following theorem.

THEOREM 2.2 Given any δ with $0<\delta<\frac{\pi}{2}$, the sequence $\{R_{n-3,n}(z)\}_{n=3}^{\infty}$ converges *uniformly* to e^{-z} on the sector $S_\delta \equiv \{z=re^{i\theta}: |\theta| \le \frac{\pi}{2}-\delta\}$.

Numerical results have indication that

$$\lim_{n\to\infty}\|R_{n-v,n}(iy)\|_{L_\infty(-\infty,+\infty)} = 1$$

for any fixed positive integer v. Such a general statement, however, is still left to be proved.

References

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