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解析函數中某一特殊族之零點分佈情形
On the Zeros of a Specific Class of Analytic Functions

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ABSTRACT — This paper deals with a specific class S' of entire functions and finds sufficient conditions on the Maclaurin coefficients of an entire function $f(z) \varepsilon S'$, so that there exists a parabolic region H, free of zeros of f(z) and all its partial sums, in the right-half plane symmetric about the nonegative real axis. Moreover, we study the asymptotic behavior of the width function G_f of $f(z) \varepsilon S'$ and solve a conjecture an G_f partially.

I. Introduction

Maclaurin series exists for entire function. Let S' be the class of entire, non-constant functions $f(z)=\sum_{k=0}^{\infty} a_k z^k$ with $a_o>0; a_k \geq 0$ for $k=1,2,..$; $a_m=0$ for $m \geq 1$ implies $a_{m+j}=0$ for $j=1,2,...$; and $\frac{a_{k-1}}{a_k} \geq C(\frac{k}{2})$ for all $k\varepsilon E$ where E is the set of positive integers k for which $a_{k-1}>0$ and $a_k>0$ hold simultaneously. We shall deal particularly with this specific class S' of entire functions and show that if $f(z) \varepsilon S'$ then there exists a parabolic region which is free of zeros of f(z) and all its partial sums. Moreover, we shall study the width function G_f of $f(z) \varepsilon S'$ and prove that $f(z) \varepsilon S'$ implies $G_f(t)=\theta(t^{1-\frac{\rho}{2}+\varepsilon})$ as $t\rightarrow\infty$, where $\rho \equiv \overline{\lim_{r\rightarrow\infty}} \frac{\ell n(\ell n M(r;f))}{\ell n\ r}$ is the order of f, with M(r;f) being the maximum modulus function of f, and ε is an arbitrary positive number. We further prove that if $f(z) \varepsilon S'$ and f(z) of order ρ is of perfectly regular growth*, then $G_f(t) \sim kt^{1-\frac{\rho}{2}}$ as $t \rightarrow\infty$, where k is a positive constant. This particular result partially solves a conjecture on G_f.

II. Parabolic Region Free of Zeros

We first prove the following theorems:

THEOREM 2.1 Let $f(z)=\sum_{k=0}^{\infty} a_k z^k$ be an entire function with $a_k > 0$ for all $k \geq 0$. If there exists a constant $C>0$ such that $\frac{a_{k-1}}{a_k} \geq C\binom{k}{2}$ for all $k \in E$, where $E=\{k \in Z^+ : a_{k-1}>0 \text{ and } a_k>0\}$ and $\binom{k}{2}$ is the binomial coefficient, then $\mathrm{Re} f(z)=\mathrm{Re} \sum_{k=0}^{\infty} a_k z^k$ and all its partial sums have no zeros in $\{(x,y): y^2 \leq 6cx \text{ and } y \neq 0\}$ where $z=x+iy$.

PROOF. Without loss of generality, it is sufficient to prove for $f_n(z)=f_n(x+iy)=\sum_{k=0}^{n} a_k z^k$, the n^{th} partial sum of $f(z)$. By expanding,

$$\text{(2.1)} \qquad \mathrm{Re}[f_n(x+iy)]=\sum_{k=0}^{n} a_k \sum_{j=0}^{[\frac{k}{2}]} \binom{k}{2j} x^{k-2j} (-1)^j y^{2j}$$

$$=\sum_{\ell=0}^{n} \sum_{j=0}^{\min(\ell;n-\ell)} (-1)^j \binom{\ell+j}{2j} a_{\ell+j} x^{\ell-j} y^{2j}$$

$$\equiv \sum_{\ell=0}^{n} T_{\ell,n}(x,y)$$

where

$$T_{\ell,n}(x,y) \equiv \sum_{j=0}^{\min(\ell;n-\ell)} (-1)^j \binom{\ell+j}{2j} a_{\ell+j} x^{\ell-j} y^{2j} .$$

If $\sigma_{\ell,n} \equiv \min(\ell;n-\ell)>0$, the point now is to pair successive terms of $T_{\ell,n}(x,y)$, i.e.,

$$T_{\ell,n}(x,y)= \sum_{p=0}^{[\frac{\sigma_{\ell,n}}{2}]} \{ \binom{\ell+2p}{4p} a_{\ell+2p} x^{\ell-2p} y^{4p}$$

$$- \binom{\ell+2p+1}{4p+2} a_{\ell+2p+1} x^{\ell-2p-1} y^{4p+2} \}$$

+ at most a single nonnegative term.

If $\frac{a_{k-1}}{a_k} \geq C\binom{k}{2}$ for all $k \in E$ and $k \geq 2$, then since

$$\binom{k}{2} \geq \frac{\binom{k}{2(m+1)}}{\binom{k-1}{2m}} \quad \text{with } k \geq 2(m+1)>0$$

we have

$$a_{k-1}\binom{k-1}{2m} \geq a_k\binom{k}{2(m+1)}, \text{ where } k \geq 2(m+1) > 0.$$

Now since all $z=x+iy$ that are under consideration satisfy the inequality $y^2 \leq Cx$, we then have

$$a_{k-1}\binom{k-1}{2m} x \geq a_k \binom{k}{2(m+1)} y^2 \quad , \text{ where } k \geq 2(m+1) > 0.$$

This in turn implies that

$$T_{\ell,n}(x,y) \geq 0,$$

with strict inequality for at least one ℓ . Hence $\mathrm{Re}[f_n(x+iy)]$ and consequently the limit function $\mathrm{Re}f(z)$ are positive. Thus the first part of this theorem follows.

Similarly,

$$I_m[f_n(x+iy)] = \sum_{k=1}^{n} a_k \sum_{j=0}^{[\frac{k-1}{2}]} \binom{k}{2j+1} x^{k-2j-1} (-1)^j y^{2j+1}$$

$$= \sum_{\ell=1}^{n} \sum_{j=0}^{\min(\ell;n-\ell)} (-1)^j \binom{\ell+j}{2j+1} a_{\ell+j} x^{\ell-j-1} y^{2j+1}$$

$$\equiv \sum_{\ell=1}^{n} \tilde{T}_{\ell,n}(x,y)$$

where

$$\tilde{T}_{\ell,n}(x,y) \equiv \sum_{j=0}^{\min(\ell;n-\ell)} (-1)^j \binom{\ell+j}{2j+1} a_{\ell+j} x^{\ell-j-1} y^{2j+1} \quad .$$

If $\sigma_{\ell,n} \equiv \min(\ell;n-\ell) > 0$, we again pair successive terms of $\tilde{T}_{\ell,n}(x,y)$, i.e.,

$$\tilde{T}_{\ell,n}(x,y) = \sum_{p=0}^{[\frac{\sigma_{\ell,n}}{2}]} \left\{ \binom{\ell+2p}{4p+1} a_{\ell+2p} x^{\ell-2p-1} y^{4p+1} \right.$$

$$\left. - \binom{\ell+2p+1}{4p+3} a_{\ell+2p+1} x^{\ell-2p-2} y^{4p+3} \right\}$$

$$+ \text{ at most a single nonnegative term.}$$

With $y>0$, a similar argument as in the first part of this theorem yields that there exists at least one ℓ for which $\tilde{T}_{\ell,n}(x,y)$ is positive. Hence $\mathrm{Im}[f_n(x+iy)]$ and consequently $I_m f(z)$ are both positive, which proves the second part of this theorem.

An easy consequence of Theorem 2.1 is the following:

COROLLARY 2.1.1. With the hypothesis as in THEOREM 2.1, $f(z)=\sum_{k=0}^{\infty} a_k z^k$ and all $f_n(z)=\sum_{k=0}^{n} a_k z^k$ have no zeros in $T=\{(x,y):y^2\leq 6cx\}$.

As an example, consider $\cosh(\sqrt{z})=\sum_{k=0}^{\infty}\frac{z^k}{(2k)!}$ where $a_k=\frac{1}{(2k)!}$ for all $k\geq 0$. Since

$$\frac{a_{k-1}}{a_k}=(2k)(2k-1)\geq 4\cdot\left(\frac{k\cdot(k-1)}{2}\right)\cdot 2=8\binom{k}{2},$$

we have that C=8. Thus by COROLLARY 2.1.1, $\cosh\sqrt{z}$ has no zeros in the region $\{(x,y):y^2\leq 48x\}$.

Restricting our consideration to polynomials $f(z)=\sum_{k=0}^{n} a_k z^k$ and following the same argument as in THEOREM 2.1 with a slightly different way of pairing, we obtain the following result.

THEOREM 2.2. If f(z) is a polynomial of degree n with $a_k>0$, then f(z) has no zeros in $\{(x,y):|y|\leq \frac{C}{n}x\}$ where C can be any constant less than or equal to $\frac{n}{\sqrt{\binom{n}{2}}}$.

III. Asymptotic Behavior of $G_f(t)$ as $t\to+\infty$

To continue our discussion, we need to define a few terminologies. Let $f(z)=\sum_{k=0}^{\infty} a_k z^k$ be an entire function and U be the sequence of the moduli of the terms in the power series representation for f(z) with $|z|=r$, i.e.,

$$U(r)=\{|b_k|\cdot r^k\}_{k=0}^{\infty}\ .$$

Since f(z) is entire, the sequence tends to zero for all finite non-negative real values of r. We define the maximum term m(r) for a given value of $r\geq 0$ to be the greatest of the terms of the sequence U(r) and set

$$N(r)\equiv \max\{k:\ |b_k|\cdot r^k=m(r)\}\ .$$

Furthermore, let $F(u) \equiv \sum_{n=1}^{\infty} e^{H(n)} u^n$, where u is a positive real variable, $H(x) \in C^1[1,+\infty)$, with H increasing on $[1,+\infty)$, and with $H'(x)$ decreasing on $[1,+\infty)$. We assume that the radius of convergence of F is unity and that $H(n)/n \to 0$ as $n \to \infty$. The values of r are called ordinary values of r with respect to F(u) if we can find two numbers k and ℓ corresponding to r for which f(z) with $|z|=r$ and $kF(r/\ell)$ have the same maximum term and rank, and f(z) is dominated by $kF(r/\ell)$. Other values of r are called exceptional values of r with respect to F(u). It is known (cf. Valiron [3, p.95]) that the exceptional values of r in [0,R] is a set consisting of a finite number of intervals, i.e., exceptional intervals. It has also been proved [4, ch.5] that if $\tau(R)$ is the total length of exceptional intervals of an entire function f between 0 and R>0, then $\lim_{R\to\infty} \frac{\tau(R)}{R} = 0$.

A few definitions in [1] are also required for our discussion. We define $M(r;f)$, the maximum modulus function for f(z), an entire function of the complex variable $z=re^{i\theta}$, by $M(r;f) \equiv \max\{|f(re^{i\theta})|: 0 \le \theta \le 2\pi\}$. If f is a non-constant entire function, then $\rho \equiv \overline{\lim_{r\to\infty}} \frac{\ell n(\ell n\, M(r;f))}{\ell n\, r}$ is called the order of f. If $\rho<+\infty$, then f is said to be of finite order. An entire function of finite order ρ satisfying $\lim_{r\to\infty} \frac{\ell n(\ell n\, M(r;f))}{\ell n\, r} = \rho$ is called an entire function of regular growth. It is known [3, p.44-46] that a necessary and sufficient condition for an entire function $f(z)=\sum_{k=0}^{\infty} a_k z^k$ of finite order $\rho > 0$ to be of regular growth is that the coefficients a_k should satisfy the inequality $|a_n|^{\frac{1}{n}} < n^{-\frac{1}{\rho+\varepsilon}}$, ε being arbitrarily small and positive, for all sufficiently large values of n, and that there should be an infinite sequence $N=\{n_p\}_{p=1}^{\infty}$ of positive integers such that $\lim_{p\to\infty} \frac{\ell n\, n_{p+1}}{\ell n\, n_p} = 1$ for which $|a_{n_p}| > n_p^{\frac{-1}{\rho-\varepsilon_p}}$, where $\varepsilon_p \to 0$ as $p \to +\infty$. If we let N be the set consisting of all even, or odd positive integers or both, we say that an entire function f(z) is of reqular growth * (R.G.*) providing that f(z) is of order $\rho=0$ or f (z), of order $\rho>0$, is of regular growth with respect to the set N. Entire functions of finite order $\rho>0$ satisfying $0<\beta = \lim_{r\to\infty} \frac{\ln M(r;f)}{r^\rho} <+\infty$ are called functions of perfectly regular growth. If f(z) of order $\rho>0$ is of perfectly regular growth with the set N consisting of all even or odd positive integers, or both, then $f(z)\varepsilon$ P.R.G. * (perfectly regular growth *).

We adopt the notation $A\sim B$ as $r\to\infty$ to mean $\lim_{r\to\infty} A/B=1$. Wiman [5] has obtained that $M(r;f^{(j)})\sim M(r;f)(\frac{N(r)}{r})^j$, for j=1,2,3,.., if $r\to\infty$ through ordinary values common to $z^k f^{(k)}(z)$, where the comparison functions are $u^k F_\alpha^{(k)}(u)$, k=1,2,3,..., with $F_\alpha(u)=\sum_{n=1}^{\infty} e^{n^\alpha} u^n$, $0<\alpha<1$. Thus, if we take r to run through such common ordinary values of $r>0$, then $M(r;f^{(j)})/M(r;f^{(j+1)})\sim \frac{r}{N(r)}$. The result of chapter III in [4] shows that for any entire function f of order $\rho>0$, given any ε with $0<\varepsilon<\rho$, $N(r)>r^{\rho-\varepsilon}$ for all r sufficiently large. Thus, for all common ordinary values of r sufficiently large,

$$r/N(r) < r^{(1-\rho-\varepsilon)} \quad (3.1)$$

We then have

$$\frac{M(r;f^{(j)})}{M(r;f^{(j+1)})} < r^{(1-\rho+\varepsilon)} \quad (3.2)$$

From THEOREM 2.1, we know that $f(z)\varepsilon S'$ has no zeros in the set $\{(x,y):y^2<Cx\}$ where

$$C = \inf_{k\geq 2}\{\frac{a_{k-1}}{a_k\binom{k}{2}}\}.$$

In particular, given any ε_o, with $0<\varepsilon_o<C$, there exists a j such that f(z) has no zeros in $\{(x,y):y^2<(\frac{a_{j-1}}{a_j\binom{j}{2}}-\varepsilon_o)x\}$ which is properly contained in $\{(x,y):y^2<Cx\}$. If we define $A_f(x)\equiv\sqrt{(\frac{a_{j-1}}{a_j\binom{j}{2}}-\varepsilon_o)x}$, then f(z) has no zeros in the region $\{(x,y):y^2<A_f(x)\}$. After a

performance of a horizontal translation: $z=\xi+\beta$ where $\beta\geq 0$,

$$f_n(z)=\sum_{k=0}^{n} a_k z^k = f_n(\xi+\beta)\equiv \tilde{f}_{n,\beta}(\xi)$$

$$=\sum_{k=0}^{n} a_{k,n}(\beta)\,\xi^k$$

where

$$a_{k,n}(\beta)=\begin{cases}\sum_{p=0}^{n-k} a_{k+p}\binom{k+p}{p}\beta^p, & \text{for } k\leq n\\ 0, & \text{for } k>n\end{cases}$$

Let $\tilde{f}(x)=f(x+\beta)$ be a horizontal translation of f. Checking the necessary properties of the involved coefficients, we have that $\tilde{f}\in S'$. Therefore all the previous results hold for $\tilde{f}$, and we have

$$(3.3)\qquad A_{\tilde{f}}(x)=\sqrt{\left(\frac{a_{j-1}(\beta)}{a_j(\beta)\binom{j}{2}}-\varepsilon_o\right)x}\ .$$

If we define $G_f(t)\equiv A_{\tilde{f}}(x)$ where $t=x+\beta$, then G_f has no zeros in $\{(t,y):|y|<G_f(t)\}$. Now since

$$M(\beta;f^{(j)})=f^{(j)}(\beta)=j!\,a_j(\beta)\ ,$$

we have

$$\frac{a_{j-1}(\beta)}{\binom{j}{2}a_j(\beta)}=\frac{M(\beta;\ f^{(j-1)})}{(j-1)\,M(\beta;f^{(j)})}\ .$$

Hence by (3.1) and (3.2) and for β sufficiently large,

$$(3.4)\qquad \frac{a_{j-1}(\beta)}{\binom{j}{2}a_j(\beta)}<K\beta^{1-\rho+\varepsilon}\ ,$$

where K is a positive constant dependent only on j. Combining (3.3) and (3.4),

$$G_f(t)<\sqrt{(K\beta^{1-\rho+\varepsilon}-\varepsilon_o)(x+\beta)}$$

for β sufficiently large, where $t=x+\beta$. Now, if we let $x=\beta$, then for β sufficiently large,

$$G_f(t) < K_1 \cdot t^{1-\frac{\rho}{2}+\varepsilon} ,$$

where K_1 is a positive constant. Thus we have established the following theorem.

THEOREM 3.1. If $f(z) \varepsilon S'$, then

$$G_f(t) = \theta(t^{1-\frac{\rho}{2}+\varepsilon}) \text{ as } t \to \infty$$

where ε is an arbitrary positive number.

Let S denote the set of all entire functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for which

i) $a_o > 0$ and $a_k \geq 0$ for all $k \geq 1$;

ii) if $a_m = 0$, then $a_{m+2j} = 0$ for every $j \geq 1$;

and

iii) if $E \equiv \{k : a_k > 0 \text{ and } a_{k+2} > 0\}$ is non-empty, then

$$\inf_{k \varepsilon E} \{\frac{a_k}{(k+1)(k+2)a_{k+2}}\} > 0 .$$

Note that the set S' defined before is a proper subset of S.

In the special case that $f \varepsilon S'$ is also in P.R.G.*, a sharper bound on the asymptotic behavior of $G_f(t)$ can be obtained. It is known [4, ch.3] that $N(r) \sim B\rho r^{\rho}$, $n \to \infty$, $B > 0$, $\rho > 0$ is both a necessary and sufficient condition for $f(z) \varepsilon$P.R.G.* As in the proof of THEOREM 3.1, we can conclude that

$$G_f(t) \sim Kt^{1-\frac{\rho}{2}} , \text{ as } t \to \infty ,$$

where K is a positive constant. Therefore, we have the following:

THEOREM 3.2. If $f(z) \varepsilon S'$ and $f(z) \varepsilon$P.R.G.* is of order ρ, then $G_f(t) \sim K_2 t^{1-\frac{\rho}{2}}$, as $t \to \infty$, where K_2 is a positive constant.

The above THEOREM solves partially the conjecture in [2] that for elements $f \varepsilon S$ of perfectly regular growth, the associated function G_f satisfies $G_f(t) \sim Kt^{\frac{2-\rho}{2}}$ as $t \to +\infty$.

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$$G_f(t) < K_1 \cdot t^{1-\frac{\rho}{2}+\varepsilon} ,$$

where K_1 is a positive constant. Thus we [illegible] following theorem:

THEOREM 3.1. If $f(z) \in S^*$, then

$$[illegible] \; t^{1-\frac{\rho}{2}+\varepsilon}$$

where ε is an arbitrary positive number.

[illegible] for which

i) $a_0 > 0$ and $a_k \geq 0$ for all $k \geq 1$;

ii) if $a_m = 0$, then $a_{m+2j} = 0$ for every $j \geq 1$;

and

iii) if $E \equiv \{k : a_k > 0$ and $a_{k+2} > 0\}$ is non-empty, then

$$\inf_{k \in E} \left\{ \frac{a_k}{(k+1)(k+2)a_{k+2}} \right\} > 0 .$$

Note that the set S' defined before is a proper subset of S.

In the special case that $f \in S'$ is also in P.R.G.*, a sharper bound on the asymptotic behavior of $G_f(t)$ can be obtained. It is known [4, ch.3] that $N(r) \sim B\rho r^{\rho}$, $r \to \infty$, $B > 0$, $\rho > 0$ is both a necessary and sufficient condition for $f(z) \in$ P.R.G.* As in the proof of THEOREM 3.1, we can conclude that

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The above THEOREM solves partially the conjecture in [2] that for elements $f \in S$ of perfectly regular growth, the associated function G_f satisfies $G_f(t) \sim K t^{\frac{2-\rho}{2}}$ as $t \to \infty$.