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Some Convergence Theorem for Dependent Generalized Gaussian Random Variables

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ABSTRACT — Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables let a_{nk} be real numbers. The convergence of $\sum_{k=1}^{\infty} a_{nk} X_k$ has been studied by Chow for the case where $\{X_k\}_{k=1}^{\infty}$ is an independent sequence. The present paper extends some of Chow's results to the case of dependent sequence $\{X_k\}_{k=1}^{\infty}$.

Definition. A random variable X is a generalized Gaussian random variable if and only if there exists a nonnegative real number α such that for each real number t,

$$E(e^{tX}) \leq e^{\alpha^2 t^2/2} \tag{1}$$

The minimum of those α's satisfying (1) will be denoted by $\tau(X)$. It follows from the definition that if X is a generalized Gaussian random variable, so is aX for all real number a.

If $X_1, X_2, \ldots, X_n$ are generalized Gaussian random variables, then by the Cauchy-Bunyakovsky-Schwartz (C.B.S.) inequality, $X = X_1 + \ldots + X_n$ is a generalized Gaussian random variable with

$$E(e^{tX}) \leq \exp[2^{n-1}(\alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2)t^2/2] \tag{2}$$

where the α_i's satisfy (1).

LEMMA 1. If X is a generalized Gaussian random variable satisfying (1), then for each $\varepsilon > 0$,

$$P(|X|>\varepsilon)\leq 2\exp(-\varepsilon^2/2\alpha^2) \qquad (3)$$

PROOF: This lemma has been proved by Chow [1].

Lemma 1 is essential in the proofs of most of the theorems.

Let $\{a_{nk}\}$ be an infinite matrix of real numbers, and $\{X_n\}$ a sequence of m-dependent generalized Gaussian random variables. The convergence of $\Sigma_{k=1}^{\infty} a_{nk}X_k$ as $n\to\infty$ has been discussed by Chow [1] for the case of independence. The following theorems are extension of Chow's theorems to the case of m-dependence. Some parts of the proofs are similar to those found in [1]. They are included for completeness.

Definition. A sequence of random variables X_n is said to be m-dependent if X_p and X_q are independent whenever $|p-q|\geq m$.

THEOREM 1. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of m-dependent generalized Gaussian random variables with $\tau(X_k)\leq\alpha<\infty$ for all k, and let $A_n=\Sigma_{k=1}^{\infty} a_{nk}^2<\infty$. Let $T_n=\Sigma_{k=1}^{\infty} a_{nk}X_k$. Then

(a) T_n exists a.s. for each n;

(b) For each $\varepsilon>0$, $-\infty<t<\infty$,

$$E[\exp(tT_n)]\leq \exp[2^{m-1}\alpha^2 A_n t^2] \qquad (4)$$

$$P[|T_n|>\varepsilon]\leq 2\exp[-\varepsilon^2/(2^{m-1}\alpha^2 A_n)] \qquad (5)$$

PROOF: (a) Fix n; write a_k as a_{nk}. Let $n_j=2mj$, $S_p=\Sigma_{k=1}^{p} a_k X_k$ and $S_{pq}=S_p-S_q$. For fixed p and q (p>q) there exist integers s and r such that $n_r\leq q<p\leq n_s$, $n_r\leq q<n_{r+1}$, $n_{s-1}<p\leq n_s$.

$$S_{pq}=\sum_{i=r}^{s-1}\sum_{\substack{2mi<k\leq 2(i+1)m\\ q<k\leq p}} a_k X_k=\sum_{i=r}^{s-1}\Big[\sum_{\substack{2mi<k\leq m(2i+1)\\ q<k\leq p}} a_k X_k+$$

$$\sum_{\substack{m(2i+1)+1\leq k\leq 2m(i+1)\\ q<k\leq p}} a_k X_k\Big]$$

$$= \sum_{i=r}^{s-1} \sum_{\substack{2mi<k\leq m(2i+1)\\ q<k\leq p}} a_k X_k + \sum_{i=r}^{s-1} \sum_{\substack{m(2i+1)+1\leq k\leq 2m(i+1)\\ q<k\leq p}} a_k X_k = S'_{pq} + S''_{pq} .$$

Since the convergence set of S_p is

$$C = \bigcap_{n=1}^{\infty} \bigcup_{q=1}^{\infty} \bigcap_{p=q+1}^{\infty} \{|S_p - S_q| \leq 1/n\} ,$$

and $\{|S_p - S_q| > 1/n\} \subseteq \{|S'_{pq}| > 1/2n\} \cup \{|S''_{pq}| > 1/2n\}$,

S_p converges a.s. if $\lim_{n\to\infty} \lim_{q\to\infty} P[\bigcup_{p=q+1} \{|S'_{pq}| > 1/2n\}] = 0$ and $\lim_{n\to\infty} \lim_{q\to\infty} P[\bigcup_{p=q+1} \{|S''_{pq}| > 1/2n\}] = 0.$

Now for $r+1 \leq j \leq s-1$, let

$$Y_j = a_{n_j+1} X_{n_j+1} + \ldots + a_{n_{j+1}-m} X_{n_{j+1}-m}$$

$$Z_j = a_{n_{j+1}-m+1} X_{n_{j+1}-m+1} + \ldots + a_{n_{j+1}} X_{n_{j+1}}$$

Let

$$
\begin{aligned}
Y_r &= a_{q+1} X_{q+1} + \ldots + a_{n_r+m} X_{n_r+m} && \text{if } n_r \leq q \; n_r+m \\
&= 0 && \text{if } n_r+m \leq q < n_{r+1} \\
Y_s &= a_{n_{s-1}+1} X_{n_{s-1}} + \ldots + a_p X_p && \text{if } n_{s-1} < p \leq n_{s-1}+m \\
&= a_{n_{s-1}+1} X_{n_{s-1}+1} + \ldots + a_{n_{s-1}+m} X_{n_{s-1}+m} && \text{if } n_{s-1}+m < p \leq n_s \\
Z_r &= a_{q+1} X_{q+1} + \ldots + a_{n_{r+1}} X_{n_{r+1}} && \text{if } n_r+m \leq q < n_{r+1} \\
&= a_{n_r+m+1} X_{n_r+m+1} + \ldots + a_{n_{r+1}} X_{n_{r+1}} && \text{if } n_r \leq q < n_r+m \\
Z_s &= a_{n_s-m+1} X_{n_s-m+1} + \ldots + a_p X_p && \text{if } n_s-m < p \leq n_s \\
&= 0 && \text{if } n_{s-1} < p \leq n_s-m
\end{aligned}
$$

Then $Y_r, Y_{r+1}, \ldots, Y_s$ are independent, and so are $Z_r, Z_{r+1}, \ldots, Z_s$

$$S'_{pq}=Y_r+Y_{r+1}+\dots+Y_s$$

$$S''_{pq}=Z_r+Z_{r+1}+\dots+Z_s$$

For each $r<j<s$, Y_j is generalized Gaussian with

$$E[\exp(tY_j)]\le \exp[2^{m-2}\alpha^2t^2(a^2_{n_j+1}+a^2_{n_j+2}+\dots+a^2_{n_{j+1}-m})]$$

For $j=r$,

$$E[\exp(tY_j)]\le\exp[2^{m-2}\alpha^2t^2(a^2_{q+1}+\dots+a^2_{n_r+m})]$$

For $j=s$,

$$E[\exp(tY_j)]\le\exp[2^{m-2}\alpha^2t^2(a^2_{n_{s-1}+1}+\dots+a^2_p)].$$

Hence

$$E[\exp(tS'_{pq})]\le \exp[2^{m-2}\alpha^2t^2\sum_{j=r}^{s}A_j]$$

where $A_r=a^2_{q+1}+a^2_{q+2}+\dots+a^2_{n_r+m}$,

$$A_s=a^2_{n_{s-1}+1}+a^2_{n_{s-1}+2}+\dots+a^2_p,$$

and $A_j=a^2_{n_j+1}+a^2_{n_j+2}+\dots+a_{n_{j+1}-m}$ for $r+1\le j\le s-1$

Therefore by Lemma 1,

$$P[|S'_{pq}|>\varepsilon]\le 2\exp[-\varepsilon^2/2^m\alpha^2\sum_{j=r}^{s}A_j]\le 2\exp[-\varepsilon^2/2^m\alpha^2\sum_{j=0}^{p}a_j^2]$$

which tends to 0, as $p,q\to\infty$, since $\sum_{k=1}^{\infty}a^2_{nk}<\infty$ for all n. Similarly $P[|S''_{pq}|>\varepsilon]\to 0$, as $p,q\to\infty$. Hence S_p converges a.s. as $p\to\infty$.

(b) Since

$$\int e^{tS_p}dp=\int e^{t(S'_p+S''_p)}dp\le\exp(c_1^2+c_2^2)t^2$$

where $c_1^2=2^{m-1}\alpha^2\sum_{j=0}^{p}(a^2_{n_j+1}+\dots+a^2_{n_{j+1}-m})$

$$c_2^2=2^{m-1}\alpha^2\sum_{j=0}^{p}(a^2_{n_{j+1}-m+1}+\dots+a^2_{n_{j+1}})$$

we have $\int e^{tS_p} dP \le \exp [2^{m-1} \alpha^2 \sum_{j=1}^{2m(p+1)} a_j^2 t^2]$

and $\int e^{tT_n} dP \le \exp[2^{m-1}\alpha^2 A_n t^2]$.

(5) is a consequence of (4).

THEOREM 2. Let $\{X_{nk}\}_{k=1}$, n=1,2,... be sequences of m-dependent (in k) generalized Gaussian random variables such that for each n, $\sup_k \tau(X_{nk}) \le \alpha < \infty$, (we may assume $\alpha > 0$). Let $T_n = \Sigma_{k=1}^{\infty} a_{nk} X_{nk}$, let $A_n = \Sigma_{k=1}^{\infty} a_{nk}^2 < \infty$ for all n. If for every $\beta > 0$,

$$\Sigma_{n=1}^{\infty} \exp(-\beta/A_n) < \infty, \quad (6)$$

then the series T_n converges a.s. for each n and $\lim_n T_n = 0$ a.s..

PROOF: By Theorem 1, T_n exists a.s. for each n, and by (5)

$$\Sigma_{n=1}^{\infty} P [|T_n| > \varepsilon] \le 2 \Sigma_{n=1}^{\infty} \exp[-\varepsilon^2 / 2^{m-1} \alpha^2 A_n]$$

Hence $\lim_n T_n = 0$ a.s..

THEOREM 3. Let $\{X_{nk}\}_{k=1}$, n=1,2,... be sequences of m-dependent (in k) random variables such that $Y_{nk} = X_{nk} - E(X_{nk})$ are generalized Gaussian with $\sup_k \tau(X_{nk}) \le \alpha < \infty$, $(\alpha > 0)$, for each n. Let $C_n = \Sigma_{k=1}^{\infty} a_{nk} E(X_{nk})$ converge for each n, and $\lim_n C_n = C$ finite. If (6) holds, in particular if $A_n = o(1/\log n)$, then the series $T_n = \Sigma_{k=1}^{\infty} a_{nk} X_{nk}$ converges a.s. for each n and $\lim_n T_n = C$ a.s..

PROOF: Let $S_n = \Sigma_{k=1}^{\infty} a_{nk} Y_{nk}$. If (6) holds, then by Theorem 2, the series S_n converges a.s. for each n and $\lim_n S_n = 0$. Hence T_n converges a.s. for each n and $\lim T_n = C$ a.s.. If $A_n = o(1/\log n)$, then (6) holds and the theorem follows.

COROLLARY 3.1. Let $\{X_n\}$ be a sequence of m-dependent random variables such that $Y_n = X_n - E(X_n)$ are generalized Gaussian with $\tau(Y_n) \le \alpha < \infty$ for all n. Let $S_n = X_1 + X_2 + \ldots + X_n$. If for some $\beta > 0$,

$$\lim E(S_n)/n^{\frac{1}{2}}(\log n)^{\frac{1}{2}(1+\beta)}=0$$

then $\lim S_n/n^{\frac{1}{2}}(\log n)^{\frac{1}{2}(1+\beta)}=0$ a.s..

COROLLARY 3.2. Let $\{X_n\}$ be a sequence of m-dependent random variables such that $Y_n=X_n-E(X_n)$ are generalized Gaussian with $\tau(Y_n)\leq\alpha<\infty$ for all n. Let

$$T_n=\Sigma_{j=1}^{n} j^{-1}X_j/\Sigma_{j=1}^{n} j^{-1} .$$

If $\lim E(X_n)=0$, then $\lim T_n=0$ a.s.. In particular, X_n is summable (C,γ) to 0 for every $\gamma>0$.

LEMMA 2. Let X_n be a sequence of m-dependent identically distributed random variables with mean μ. If $E(|X_1|)<\infty$, then $\Sigma_{k=1}^{n} X_k/n\to\mu$ a.s. .

PROOF: Let $n_j=2mj$,

$$Y_j=X_{n_j+1}+X_{n_j+2}+\cdots+X_{n_{j+1}-m} ,$$

$$Z_j=X_{n_{j+1}-m+1}+\cdots+X_{n_{j+1}} .$$

Then $\{Y_j\}$ and $\{Z_j\}$ are sequences of independent random variables, , and $E(Y_j)=E(Z_j)=m\mu$. Now for any given integer n, there exist integers k and r such that $0\leq r<2m$, and $n=n_k+r$. Let

$$S_n=X_1+X_2+\cdots+X_n=X_1+\cdots+X_{n_k}+X_{n_k+1}+\cdots+X_{n_k+r}$$

$$=\Sigma_{j=0}^{k-1} Y_j+\Sigma_{j=0}^{k-1} Z_j+X_{n_k+1}+\cdots+X_{n_k+r}=S'_n+S''_n+R_n .$$

Then

$$S_n/n=(S'_n+S''_n+R_n)/n=(n_k/n)(S'_n/n_k)+(n_k/n)(S''_n/n_k)+R_n/n.$$

Since $E(|X_1|)<\infty$, $\lim R_n/n=0$ a.s.. Hence the lemma follows by applying the strong law of large numbers.

THEOREM 4. Let $\{X_n\}$ be a sequence of m-dependent, identically distributed random variables such that $E(X_1^2)<\infty$. Then $E(X_1)=0$ if and only if for every array $\{a_{nk}\}$ of real numbers such that $\Sigma_{k=1}^{n} a_{nk}^2 \to 1$,

$$\lim_n \Sigma_{k=1}^{n} a_{nk}X_k/n^{\frac{1}{2}}=0 \text{ a.s. .} \quad (7)$$

PROOF: Assume that $E(X_1)=0$. Let $X_k'=X_k$ if $|X_k| \le k^{\frac{1}{4}}$, and $X_k'=0$ if $|X_k|>k^{\frac{1}{4}}$. Then $X_k'-E(X_k')$ are m-dependent generalized Gaussian random variables. By Theorem 2, $\lim_n \Sigma_{k=1}^{n} a_{nk}[X_k'-E(X_k')]/n^{\frac{1}{2}}=0$ a.s. . As in [1], $\lim_n \Sigma_{k=1}^{n} a_{nk}E(X_k')/n^{\frac{1}{2}}=0$. Hence $\Sigma_{k=1}^{n} a_{nk}X_k' \to 0$ a.s. .

Let $\varepsilon>0$ be given. Since $E(X_1^2)<\infty$, there is an N such that

$$\int_{|X_1|>N} X_1^2 dP<\varepsilon \quad .$$

Let $Y_k=X_k$ if $|X_k| \ge N$, and $Y_k=0$ if $|X_k|<N$. By Lemma 2, $\Sigma_{k=1}^{n} Y_k^2/n \to E(Y_1^2)$ a.s. . Now let $X_k''=X_k-X_k'$. Then

$$(n^{-\frac{1}{2}}\Sigma_{k=1}^{n} a_{nk}X_k'')^2 \le (1+o(1))\Sigma_{k=1}^{n} (X_k'')^2/n \le (1+o(1))\Sigma_{k=1}^{n}(Y_k)^2/n+o(1) \to$$

$$E(Y_1^2)<\varepsilon \quad \text{a.s.}$$

Therefore $\lim_n \Sigma_{k=1}^{n} a_{nk}X_k''=0$ a.s. and hence (7) holds.

For the second part of Theorem 4, assume that for every array $\{a_{nk}\}$ such that $\lim_n \Sigma_{k=1}^{n} a_{nk}^2=1$, $\lim_n \Sigma_{k=1}^{n} a_{nk}X_k/n^{\frac{1}{2}}=0$ a.s.. In particular, let $a_{nk}=1/n^{\frac{1}{2}}$ if $1 \le k \le n$, and $a_{nk}=0$ if $k>n$. Then $\lim_n (X_1+X_2+\cdots+X_n)/n=0$ a.s. Hence $E(X_1)=0$ by Lemma 2.

LEMMA 3. Let $\{X_n\}$ and $\{U_n\}$ be two sequences of random variables, and $\{a_n\}$ a sequence of real numbers. Let $Q(t)=\Sigma_{k=1}^{n} a_k X_k \cos(kt+U_k)$, and $M=\max_t |Q(t)|$. Then for $k \ge 0$,

$$[M \ge K] \subset \bigcup_{k=1}^{16n^2} [|Q(k/(2m^2))| \ge K/2] \quad (8)$$

PROOF: See Lemma 4 [1] .

THEOREM 5. (Extension of Salem-Zygmund-Chow's Theorem). Let $\{U_n\}$ be a sequence of independent random variables, and $\{X_n\}$ an independent sequence of m-dependent generalized Gaussian random variables with $\sup_n \tau(X_n) \leq \alpha < \infty$. Let $\{a_n\}$ be a sequence of real numbers such that for some $\beta > 0$,

$$\sum_{n=1}^{\infty} a_n^2 (\log n)^{1+\beta} < \infty \tag{9}$$

Then for each t, $\sum_{n=1}^{\infty} a_n X_n \cos(nt+U_n)$ converges a.s. to a stochastic process f(t) which is a.s. sample continuous.

PROOF: Let $n_j = 2mj$

$$Y_j(t) = a_{n_j+1} X_{n_j+1} \cos((n_j+1)t+U_{n_j+1}) + \cdots$$

$$+ a_{n_{j+1}-m} X_{n_{j+1}-m} \cos((n_{j+1}-m)t+U_{n_{j+1}-m}),$$

$$Z_j(t) = a_{n_{j+1}-m+1} X_{n_{j+1}-m+1} \cos((n_{j+1}-m+1)t+U_{n_{j+1}-m+1}) + \cdots$$

$$+ a_{n_{j+1}} X_{n_{j+1}} \cos((n_{j+1})t+U_{n_{j+1}}).$$

Since $E\{\exp[tX_n \cos(ns+U_n)]\} = E\{E[\exp(tX_n \cos(ns+U_n)) | U_n]\}$

$$\leq \exp(\alpha^2 t^2/2),$$

$\{X_n \cos(nt+U_n)\}$ is a sequence of m-dependent generalized Gaussian random variables for each t. By Theorem 1, the series $\sum_{n=1}^{\infty} a_n X_n \cos(nt+U_n)$ converges a.s. for each t to a stochastic process f(t).

Now

$$E[\exp[tY_j(s)]] \leq \exp[\alpha^2 t^2 2^{m-2} (a_{n_j+1}^2 + a_{n_j+2}^2 + \cdots + a_{n_{j+1}-m}^2)]$$

$$E[\exp[tZ_j(s)]] \leq \exp[\alpha^2 t^2 2^{m-2} (a_{n_{j+1}-m+1}^2 + \cdots + a_{n_{j+1}}^2)]$$

Hence for each t, $Y_j(t)$ and $Z_j(t)$ are sequence of independent generalized Gaussian random variables.

Let $k(i)=2^{2^i}$,

$$Q_i(t)=Y_{k(i)+1}(t)+Y_{k(i)+2}(t)+\ldots+Y_{k(i+1)}(t),$$

$$R_i(t)=Z_{k(i)+1}(t)+Z_{k(i)+2}(t)+\ldots+Z_{k(i+1)}(t).$$

Then

$$E[\exp[tQ_i(s)]] \leq \exp[\alpha^2 t^2 2^{m-2}(b_{k(i)+1}+\cdots+b_{k(i+1)})],$$

$$E[\exp[tR_i(s)]] \leq \exp[\alpha^2 t^2 2^{m-2}(c_{k(i)+1}+\cdots+c_{k(i+1)})],$$

where $b_j=a^2_{n_j+1}+a^2_{n_j+2}+\cdots+a^2_{n_{j+1}-m}$,

$$c_j=a^2_{n_{j+1}-m+1}+a^2_{n_{j+1}-m+2}+\cdots+a^2_{n_{j+1}}.$$

Let $S_i(t)=R_i(t)+Q_i(t)$. By the C.B.S. inequality,

$$E[\exp[tS_i(s)]] \leq \exp[\alpha^2 t^2 2^{m-1}[A_{k(i)+1}+A_{k(i)+2}+\cdots+A_{k(i+1)}]] \quad (10)$$

where $A_j=b_j+c_j$. Let

$$B_i=A_{k(i)+1}+A_{k(i)+2}+\cdots+A_{k(i+1)},$$

$$C_j=a^2_{n_j+1}[\log(n_j+1)]^{1+\beta}+\cdots+a^2_{n_{j+1}}(\log n_{j+1})^{1+\beta}$$

$$D_i=C_{k(i)+1}+C_{k(i)+2}+\cdots+C_{k(i+1)}.$$

Then $A_j \leq C_j/[\log(n_j+1)]^{1+\beta} \leq C_j/(\log n_j)^{1+\beta}$

and $B_i \leq D_i/[\log(2m2^{2^i})]^{1+\beta} \leq D_i/[2^i \log 2]^{1+\beta}$.

Let $t_j=j/4mk(i+2)$. By (10), as in Lemma 1,

$$P[S_i(t_j) > i^{-2}] \leq 2\exp[-(2^i \log 2)^{1+\beta}/(\alpha^2 2^{m-1} D_i i^4)] \qquad (11)$$

Let $||S_i|| = \max_t |S_i(t)|$. By Lemma 3,

$$P[||S_i|| > 2i^{-2}] \leq \sum_{j=1}^{32mk(i+2)} P[|S_i(t_j)| > i^{-2}], \text{ whence}$$

$$P[||S_i|| \ 2i^{-2}] \leq 64mk(i+2)\exp[-2^{i(1+\beta)}(\log 2)^{1+\beta}/(\alpha^2 2^{m-1} D_i i^4)]$$

The right-hand side equals

$$64m\cdot\exp[2^{i+2}\log 2 - \frac{2^{i(1+\beta)}(\log 2)^{1+\beta}}{\alpha^2 2^{m-1} D_i i^4}]$$

Here the part in brackets is

$$2^i \log 2[2^2 - 2^{i\beta}(\log 2)^{\beta}/(\alpha^2 2^{m-1} D_i i^4)] \leq 2^{i(1+\beta/2)} \text{ for large } i.$$

So $\sum_{i=1}^{\infty} P[||S_i|| > 2i^{-2}] < \infty$ and by Borel-Cantelli lemma, $\sum_{i=1}^{\infty} ||S_i||$ converges a.s.. Consequently $\sum_{i=1}^{\infty} S_i(t)$ converges a.s. and uniformly to a stochastic process $g(t)$ which is a.s. sample continuous, since $S_i(t)$ are continuous in t. Since for each t, $\sum_{i=1}^{\infty} S_i(t) = f(t)$ a.s., we have $P[g(t)=f(t)]=1$ for each t.

LEMMA 4. If X is a generalized Gaussian random variable, then

$$E[e^{t|X|}] \leq 2e^{\tau^2(x)t^2/2} \qquad (12)$$

PROOF:
$$E[e^{t|X|}] = \int_{\{X<0\}} e^{t(-X)}dP + \int_{\{X>0\}} e^{tX}dP$$

$$\leq \int e^{t(-X)}dP + \int e^{tX}dP \leq 2e^{\tau^2(X)t^2/2}$$

THEOREM 6. Let $\{X_n\}_{n=1}$ be a sequence of m-dependent Gaussian random variables with $E(X_n)=0$ for $n=1,2,\ldots$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} 2^n a_n^2 < \infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous, uniformly bounded functions defined on a closed interval T. Then the stochastic process $\{Y(t), t\in T\}$ definded by $Y(t)=\Sigma_{k=1}^{\infty} a_k f_k(t) X_k$ is a Gaussian process.

PROOF: Since

$$E[\exp(tf_n(s)X_n \leq \exp(\alpha^2 t^2 M^2/2)]$$

$\{f_n(t)X_n\}_{n=1}^{\infty}$ is a sequence of m-dependent generalized Gaussian random variables. Hence, by Theorem 1, the series $\Sigma_{n=1}^{\infty} a_n f_n(t) X_n$ converges a.s. for each t. Let $n_j=2mj$,

$$Y_j(t)=a_{n_j+1} f_{n_j+1}(t) X_{n_j+1} + \cdots + a_{n_{j+1}-m} f_{n_{j+1}-m}(t) X_{n_{j+1}-m}.$$

Let $Q_j=\max|Y_j(t)|$. Then

$$Q_j \leq \Sigma_{k=n_j+1}^{n_{j+1}-m} \max|a_k f_k(t) X_k| \leq \Sigma_{k=n_j+1}^{n_{j+1}-m} M|a_k X_k|$$

By Lemma 4, (12), and the C.B.S. inequality, for t>0,

$$E[\exp(tQ_j)] \leq 2^{\Sigma_{k=1}^{m}(1/2)^k} \cdot \exp[t^2\alpha^2 M^2 2^{m-2}(a^2_{n_{j+1}} + \cdots + a^2_{n_{j+1}-m})]$$

$$\leq 2\exp[t^2\alpha^2 M^2 2^{m-2}(a^2_{n_{j+1}} + \cdots + a^2_{n_{j+1}-m})].$$

Let $S_{pq}=Q_p+Q_{p+1}+\cdots+Q_q$. By the C.B.S. inequality, for t>0,

$$E[\exp(tS_{pq})] \leq 2\exp[t^2\alpha^2 M^2 2^{m-2} \Sigma_{j=p}^{q} (2^{n_{j+1}} a^2_{n_{j+1}} + \cdots + 2^{n_{j+1}-m} a^2_{n_{j+1}-m})].$$

As in Lemma 1,

$$P(|S_{pq}|>\varepsilon) \leq 2\exp[-\varepsilon^2\alpha^2 M^2 2^{m-1} \Sigma_{j=p}^{q} (2^{n_{j+1}} a^2_{n_{j+1}} + \cdots + 2^{n_{j+1}-m} a^2_{n_{j+1}-m})].$$

Since $\Sigma_{n=1}^{\infty} 2^n a_n^2 < \infty$, $P(|S_{pq}|>\infty) \to 0$, as $p,q\to\infty$. Therefore $\Sigma_{j=1}^{\infty} Q_j$ converges in probability and also a.s., since the sequence $\{Q_j\}_{j=1}^{\infty}$ is independent. Let

$$Z_j(t)=a_{n_{j+1}-m+1} f_{n_{j+1}-m+1}(t) X_{n_{j+1}-m+1} + \cdots + a_{n_{j+1}} f_{n_{j+1}}(t) X_{n_{j+1}}$$

and $R_j=\max|Z_j(t)|$. Then it can be shown in the same manner that

$\Sigma_{j=1}^{\infty} R_j$ converges a.s. . Hence $\Sigma_{n=1}^{\infty} a_n f_n(t) X_n$ converges a.s., and uniformly in t. Since $\{f_n\}_{n=1}$ is continuous, $Y(t)=\Sigma_{n=1}^{\infty} a_n f_n(t) X_n$ is a.s. sample continuous.

It is obvious that any finite number of members in $\{Y(t), t\varepsilon T\}$ have joint normal distribution. So, $\{Y(t), t\varepsilon T\}$ is a Gaussian process.

THEOREM 7. (Extension of Hsu-Robbins-Chow's theorem). Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of m-dependent identically distributed random variables with $E(X_1)=0$. Let $\{a_{nk}\}$ be an infinite matrix of real numbers such that $a_{nk}=0$ if $k>n$, and $|a_{nk}|\le KA_n$ for some $0<K<\infty$, where $A_n = \Sigma_{k=1}^{n} a_{nk}^2$. If for some $0<\alpha\le 1$, $A_n\le Kn^{-\alpha}$ and $E(|X_1|^{2/\alpha})\le K$, then

$$\lim_n \Sigma_{k=1}^{n} a_{nk} X_k = 0 \quad \text{a.s.}$$

PROOF: The proof follows the same line as in Chow's proof for the case of independence, except some minor changes.

Let N be a fixed positive integer. Define, for $0<\beta<\alpha$,

$$X_k'=X_k I(X_k\le n^{\beta}) \qquad \text{if } a_{nk}\ge 0$$

$$=X_k I(X_k\ge -n^{\beta}) \qquad \text{if } a_{nk}<0$$

$$X_k''=X_k I(X_k\ge \varepsilon n^{\alpha}/(NK^2)) \qquad \text{if } a_{nk}\ge 0$$

$$=X_k I(X_k\le -\varepsilon n^{\alpha}/(NK^2)) \qquad \text{if } a_{nk}<0$$

where I(A) is the indecator function of A.

Let $X_k'''=X_k-X_k'-X_k''$, $T_n'=\Sigma_{k=1}^{\infty} X_k' a_{nk}$, $T_k''=\Sigma_{k=1}^{\infty} a_{nk} X_k''$, and $T_n'''=\Sigma_{k=1}^{\infty} a_{nk} X_k'''$.

We may assume that $K\ge 1$, and $A_n>0$ for each n. Let $0<t\le 1/(kn^{\beta})$. Then, as it has been proved by Chow [1] ,

$$E[\exp(ta_{nk}X_k'/A_n)] \le \exp(t^2 a_{nk}^2 K/A_n^2, \qquad (13)$$

Now let Y_j and Z_j be defined as in the proof of Theorem 1 with X_k replaced by X_k'. Then $\{Y_j\}$ and $\{Z_j\}$ are sequences of independent random variables. For $0<t\leq 1/(Kn^{\beta})$, by the C.B.S. inequality and (13), we have

$$E[\exp(tY_j/A_n)]\leq \exp[2^{m-1}t^2(a^2_{n_j+1}+\cdots+a^2_{n_{j+1}-m})/A_n^2]$$

Let $S_p=\Sigma_{j=1}^{p} Y_j$. Then

$$E(tS_p/A_n)\leq \exp(2^{m-1}t^2K/A_n)$$

Hence $p(S_p\geq\varepsilon)=P(tS_p/A_n\geq t\varepsilon A_n^{-1})\leq \exp[-t(\varepsilon-tK)/A_n]$.

Let $t=n^{-\beta}K^{-1}$. Then for large n,

$$P(S_p\geq\varepsilon)\leq \exp[-\varepsilon n^{\alpha-\beta}/(2K)]$$

Hence $\Sigma_{p=1}^{\infty} P(S_p\geq\varepsilon)<\infty$. Let $R_p=\Sigma_{j=1}^{p} Z_j$, then similarly $\Sigma_{p=1}^{\infty}P(R_p\geq\varepsilon)<\infty$. Therefore $\Sigma_{n=1}^{\infty} P(T_n'\geq\varepsilon)<\infty$. Now $P(T_n''\geq\varepsilon)\leq nP(|X_1|\geq\varepsilon n^{\alpha}/(NK^2))$ $=nP\,[(NK^2\,|X_1|/\varepsilon)^{\alpha-1}\geq n]$. Since $E(|X_1|^{2/\alpha})<\infty$, $\Sigma_{n=1}P(T_n''\geq\varepsilon)<\infty$.

By the definition of X_k, we have $a_{nk}X'''_k\leq\varepsilon/N$. So $X'''_k>\varepsilon$ implies that there are at least N of the $X_1, X_2,\ldots,X_n$ which are not zero. For large n and small N, we have

$$P(T_n'''\geq\varepsilon)\binom{n}{N}P^N(|X_1|>n^{\beta})\leq\binom{n}{N}P^N(|X_1|^{\alpha-1}>n^{\beta/\alpha})$$

By Tchebichev's inequality,

$$P(T_n>\varepsilon)\leq\binom{n}{N}(Kn^{-2\beta/\alpha})^N\leq Mn^{(1-2\beta/\alpha)N}$$

where M is independent of n. Let $\beta=2\alpha/3$ and N=6. Then for all large n, $P(T_n'''\geq\varepsilon)\leq Mn^{-2}$. Hence

$$\Sigma_{n=1}P(T_n'''\geq\varepsilon)<\infty .$$

Therefore, we have $\Sigma_{n=1}P(T_n\geq\varepsilon)<\infty$. By symmetry, we can show that $\Sigma_{n=1}P(T_n\,-\varepsilon)$. Thus the theorem is proved.

Some of the results above can be extended to the case of total dependence.

THEOREM 8. Let $\{X_k\}$ be a sequence of generalized Gaussian random variables with $\tau(X_k)=\alpha<\infty$ for all k, and let $T_n=\Sigma_{k=1}^{\infty} a_{nk}X_k$ If $A_n=\Sigma_{k=1}^{\infty} 2^k a_{nk}^2<\infty$, then T_n converges a.s. for each n, and for each $\varepsilon>0$, $-\infty<t<\infty$,

$$E[\exp(tT_n)] \leq \exp(t^2\alpha^2 A_n/2) \quad (14)$$

$$P(|T_n|>\varepsilon)\leq 2\exp[-\varepsilon^2/(2\alpha^2 A_n)] \quad (15)$$

PROOF: Let $S_{pq}^n = \Sigma_{k=p}^{q} a_{nk}X_k$. Then by (2),

$$E[\exp(ts_{pq}^n)] \leq \exp\ [t^2\alpha^2(2a_{np}^2+2^2a_{np+1}+\cdots+2^{q-p-1}a_{pq}^2)/2]$$

$$\leq \exp\ (t^2\alpha^2\ \Sigma_{k=p}^{q}\ 2^k a_{nk}^2/2) \quad (16)$$

As in Lemma 1,

$$P(|S_{pq}^n|>\varepsilon\ ) \leq 2\exp[-\varepsilon^2/(2\alpha^2\Sigma_{k=p}^{q}\ 2^k a_{nk}^2)].$$

Since $\Sigma_{k=1}^{\infty}\ 2^k a_{nk}^2<\infty$, $P(|S_{pq}^n|>\varepsilon\ )\to 0$, as $p,q\to\infty$. Hence the series $T_n =\Sigma_{k=1}^{\infty} a_{nk}X_k$ converges in probability.

Since the sequence $S_{nm}=\Sigma_{k=1}^{m} a_{nk}X_k$ converges in probability in m, there is a subsequence $\{S_{nm_k}\}$ that converges a.s.. In the following we may assume that $S_{nm}\to 0$ in probability for each n, and hence $S_{nm_k}\to 0$ a.s. . Let

$$D_{nk}= \max_{m_k+1\leq p<m_{k+1}} |S_{nm_k} - S_{np}|$$

$$B_{nk}=|a_{nm_k+1}\ X_{nm_k+1}| + \cdots+|a_{nm_{k+1}-1}X_{nm_{k+1}-1}|\ .$$

Then $D_{nk} \leq B_{nk}$. Since $E[\exp(t|a_{nm_k+j}X_{nm_k+j}|)] \leq 2\exp(\alpha^2t^2a^2_{nm_k+j}/2)$, $j=1,2,..,m_{k+1}-m_k-1$, by the C.B.S. inequality,

$$E[\exp(tB_{nk})] \le 2^{\Sigma_{k=1}^{\infty} 1/2^k} \cdot \exp(\alpha^2 t^2 \Sigma_{j=m_k+1}^{m_{k+1}} 2^{j-m_k} a_{nj}^2)$$

$$\le 2\exp(\alpha^2 t^2 \Sigma_{j=m_k+1}^{m_{k+1}} 2^{j-m_k} a_{nj}^2)$$

Consequently, for $t>0$,

$$E[\exp(tD_{nk})] \le 2\exp(\alpha^2 t^2 \Sigma_{j=m_k+1}^{m_{k+1}} 2^{j-m_k} a_{nj}^2).$$

By (16), as in Lemma 1,

$$P(|D_{nk}|\varepsilon) \le 2\exp[-\varepsilon^2/(\alpha^2 \Sigma_{j=m_k+1}^{m_{k+1}} 2^{j-m_k} a_{nj}^2)]$$

For k large, m_k is large and

$$k\alpha^2 \Sigma_{j=m_k+1}^{m_{k+1}} 2^{j-m_k+1} a_{nj}^2 \le (k\Sigma_{j=m_k+1}^{m_{k+1}} 2^j a_{nj}^2)/2^{m_{k-1}} \le \Sigma_{j=m_k+1}^{m_{k+1}} 2^j a_{nj}^2$$

which is small since the series $\Sigma_{j=1}^{\infty} 2^j a_{nj}^2$ converges. Hence

$$\exp[-\varepsilon^2/k\alpha^2 \Sigma_{j=m_k+1}^{m_{k+1}} 2^{j-m_k+1} a_{nj}^2] \to 0$$

and by the root test, $\Sigma_{k=1}^{\infty} 2\exp[-\varepsilon^2\alpha^2/\Sigma_{j=m_k+1}^{m_{k+1}} 2^{j-m_k+1} a_{nj}^2]$ is convergent. By the Borel-Cantelli lemma, $D_{nk} \to 0$ a.s. for each n. Since $|S_{np}| \le |S_{nm_k}| + D_{nk}$ for $m_k+1 \le p < m_{k+1}$, $S_{np} \to 0$ a.s. for each n.

The first inequality of the theorem follows from (16) and Fatou's lemma as in Theorem 1. As in Lemma 1, we obtain the second inequality from the first.

THEOREM 9. Let $\{X_{nk}\}_{k=1}^{\infty}$ $n=1,2,\ldots$ be sequences of generalized Gaussian random variables such that for each n, $\sup_k \tau(X_{nk}) \le \alpha < \infty$. Let $A_n = \Sigma_{k=1}^{\infty} 2^k a_{nk}^2$, $T_n = \Sigma_{k=1}^{\infty} a_{nk} X_{nk}$. If for every $\beta>0$,

$$\Sigma_{n=1}^{\infty} e^{-\beta/A_n} < \infty$$

Then the series T_n converges a.s. for each n, and $\lim_{n\to\infty} T_n=0$ a.s. .

PROOF: By Theorem 8, T_n converges a.s. for each n, and by the second inequality of Theorem 8,

$$\Sigma_{n=1}^{\infty} P(|T_n|>\varepsilon) \leq 2\Sigma_{n=1}^{\infty} e^{-\varepsilon^2/(2\alpha^2 A_n)} \tag{17}$$

Hence $\lim_{n\to\infty} T_n = 0$ a.s. by the Borel-Cantelli lemma.

THEOREM 10. Let $\{X_{nk}\}_{k=1}$, n=1,2,.. , be sequences of random variables such that $Y_{nk}=X_{nk}-E(X_{nk})$ are generalized Gaussian with $\sup_k \tau(X_{nk})\leq\alpha<\infty$ for each n. Let $C_n=\Sigma_{k=1}^{\infty} a_{nk}E(X_{nk})$ converges for each n and $\lim_{n\to\infty} C_n=C$. If (17) holds, in particular if $A_n=o(1/\log n)$ then the series $T_n=\Sigma_{k=1}^{\infty} a_{nk}X_{nk}$ converges a.s. for each n and $\lim_{n\to\infty} T_n =C$ a.s. .

PROOF: The proof is like that of Theorem 3.

COROLLARY 10.1. Let $\{X_n\}$ be a sequence of random variables such that $Y_n=X_n-E(X_n)$ are generalized Gaussian with $\tau(X_n)\leq\alpha<\infty$. for all n. Let $S_n=X_1+X_2+\cdots+X_n$. If for some $\beta>0$,

$$\lim_n E(S_n)/(2^n n^{1/2}(\log n)^{1/2(1+\beta)})=0,$$

then $$\lim_n S_n/[2^n n^{1/2}(\log n)^{1/2(1+\beta)}]=0 \text{ a.s. .}$$

PROOF: Let

$$a_{nk}=\begin{cases} 1/2^n(\log n)^{1/2(1+\beta)}n^{1/2} & \text{if } 1\leq k\leq n \\ 0 & \text{if } k>n. \end{cases}$$

Then $A_n=\Sigma_{k=1}^{\infty} 2^k a_{nk}^2 \leq 1/(\log n)^{1+\beta}=o(1/\log n)$. By Theorem 10,

$$\lim_n S_n/[2^n n^{1/2}(\log n)^{1/2(1+\beta)}]=0 \text{ a.s. .}$$

THEOREM 11. Let $\{U_n\}$ be a sequence of independent random variables, and $\{X_n\}$ a sequence of generalized Gaussian random variables with $\sup \tau(X_p) \leq \alpha < \infty$. Let $\{a_n\}$ be a sequence of real number such that for some $\beta > 0$,

$$\Sigma_{n=1} 2^n a_n^2 (\log n)^{1+\beta} < \infty$$

If $\{U_n\}$ and $\{X_n\}$ are independent, then for each t, the series $\Sigma_{n=1}^{\infty} a_n X_n \cos(nt+U_n)$ converges to a stochastic process $Y(t)$, and furthermore, there exists another process $Z(t)$ such that for each t $P(Y(t)=Z(t))=1$, and $Z(t)$ is a.s. sample continuous.

PROOF: Since the sequences $\{U_n\}$ and $\{X_n\}$ are independent, we have

$$E(\exp[tX_n \cos(ns+U_n)]) = E(E[\exp(tX_n \cos(ns+U_n)) | U_n])$$
$$\leq \exp(t^2\alpha^2/2).$$

Hence, the sequence $\{X_n \cos(nt+U_n)\}$ is generalized Gaussian for each t. By Theorem 10, the series $\Sigma_{n=1}^{\infty} a_n X_n \cos(nt+U_n)$ converges a.s. for each t. It remains to show that the convergence is uniform in t.

Let $n_j = 2^{2j}$, $Q_j(t) = \Sigma_{k=n_j+1}^{n_{j+1}} a_k X_k \cos(kt+U_k)$ and $||Q_j|| = \max_t |Q_j(t)|$. Then by (2),

$$E[\exp(sQ_j(t)] \leq \exp(\alpha^2 t^2 \Sigma_{k=n_j+1}^{n_{j+1}} 2^{k-n_j} a_k^2).$$

Let $B_j = \Sigma_{k=n_j+1}^{n_{j+1}} 2^{k-n_j} a_k^2$, $C_j = \Sigma_{k=n_j+1}^{n_{j+1}} 2^{k-n_j} a_k^2 (\log k)^{1+\beta}$. Since $B_j \leq C_j/(\log n_j)^{1+\beta}$,

$$E[\exp(sQ_j(t)] \leq \exp(\alpha^2 s^2 B_j) \leq \exp[\alpha^2 s^2 C_j/(2^j \log 2)^{1+\beta}] \quad (18)$$

Hence by (18) and Lemma 1,

$$P(|Q_j(t)| > j^{-2}) < 2\exp[-2^{j(1+\beta)-2}(\log 2)^{1+\beta}/(\alpha^2 C_j j^4)]$$

Therefore by Lemma 3, with $t_k = k2n_{j+2}$,

$$P(\|Q_j\| > 2j^{-2}) \leq \Sigma_{k=1}^{16n_{j+2}} P(|Q_j(t)| > j^{-2})$$

$$\leq 32n_{j+2}\exp[-2^{j(1+\beta)-2}(\log 2)^{1+\beta}/(\alpha^2 C_j j^4)]$$

$$= \exp[(2^{j+2}+5)\log 2 - 2^{j(1+\beta)}(\log 2)^{1+\beta}/(4\alpha^2 C_j j^4)]$$

$$= \exp[-2^{j(1+\beta/2)}[2^{j\beta/2}(\log 2)^{1+\beta}/(4\alpha^2 C_j j^4)$$

$$-2^{-j\beta/2+2}\log 2] - 5\log 2].$$

Now $2^{-j\beta/2-2}\log 2$ is small when j is large. Since C_j is small when j is large and $2^{j\beta/2}(\log 2)^{1+\beta}/4\alpha^2 j^4$ is large when j is large, $2^{j\beta/2}(\log 2)^{1+\beta}/(4\alpha^2 C_j j^4)$ is large if j is large. Hence for large j,

$$2^{j\beta/2}(\log 2)^{1+\beta}/(4\alpha^2 C_j j^4) - 2^{-j\beta/2}\log 2 - 5\log 2 \geq 1.$$

Therefore, for all large j,

$$E[(2^{j+2}+5)\log 2 - 2^{j(1+\beta)}(\log 2)^{1+\beta}/(4\alpha^2 C_j j^4)] \leq \exp[-2^{j(1+\beta/2)}]$$

so,

$$\Sigma_{j=1}^{\infty} P(\|Q_j\| > 2j^{-2}) \leq \Sigma_{j=1}^{\infty}\exp[-2^{j(1+\beta/2)}] < \infty$$

By the Borel-Cantelli lemma $\Sigma_{j=1}^{\infty}\|Q_j\|$ converges a.s. . Hence the series $\Sigma_{j=1}^{\infty} Q_j(t)$ converges a.s. and uniformly to $Z(t)$ which is a. s. sample continuous since $Q_j(t)$ is continuous in t for each j. But by Theorem 10, for each t, $\Sigma_{j=1} Q_j(t)$ is convergent a.s. to $Y(t)$, so $P(Y(t)=Z(t))=1$.

References

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