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滿足 Alg T＝｛T｝′之線性變換

On Contractions Satisfying Alg T= {T}

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ABSTRACT — For a bounded linear operator T on a Hilbert space let {T}', {T}" and Alg T denote the commutant, the double commutant and the weakly closed algebra generated by T and 1, respectively. Assume that T is a completely non-unitary contraction with a scalar-valued characteristic function $\psi(\lambda)$. In this note we show that the condition that $|\psi(e^{it})|=1$ on a set of positive Lebesgue measure implies that Alg T={T}'. Moreover, if $\psi(\lambda)$ is assumed to be outer, then these two conditions are equivalent. Our main result generalizes the well-known fact that the compressions of the shift satisfy Alg T={T}'.

For an arbitrary operator T, it is easily seen that Alg T$\subseteq${T}" $\subseteq${T}' holds. Now we assume that T is the compression of the shift on the space $H^2\ominus\psi H^2$, that is, T is defined by

$$(Tf)(\lambda)= P[\lambda f(\lambda)] \text{ for } |\lambda|<1 \text{ and } f\in H^2\ominus\psi H^2 ,$$

where ψ is a scalar-valued inner function and P denotes the (orthogonal) projection onto the space $H^2\ominus\psi H^2$. Then it was shown by Sarason [2] that Alg T={T}'. (In fact, he showed more than this. He proved that every operator in {T}' is of the form u(T), for some $u \in H^\infty$.) Note that the compression of the shift is a completely non-unitary (c.n.u.) contraction whose characteristic function ψ is scalar-valued and satisfies $|\psi(e^{it})|=1$ a.e. In this note a sufficient condition that a c.n.u. contraction having a scalar-valued characteristic function satisfy Alg T={T}' is given. Indeed we show that for such contractions the condition that $|\psi(e^{it})|=1$ on a set of positive Lebesgue measure implies that Alg T={T}' holds. Hence Sarason's result follows as a special case. We also show that if ψ is an outer function then the condition is also necessary and both are equivalent to the condition that every invariant subspace for T is hyperinvariant.

Note that for an arbitrary operator whether the latter condition is equivalent to Alg T={T}' is still unknown.

We should also point out that Sz-Nagy and Foiaş [5] proved the following result:

A c.n.u. contraction T with the scalar-valued characteristic function ψ satisfies $\{T\}''=\{T\}'$ if and only if $\psi(\lambda)\not\equiv 0$.

Hence in a sense our results strengthen theirs.

In the proofs of our theorems we will extensively use the functional model for c.n.u. contractions. The readers are referred to reference [4] for the basic definitions and terminology. Throughout this note results from [4] will be used without specific mentioning.

Our main result is the following:

THEOREM 1. Let T be a c.n.u. contraction having a scalar-valued characteristic function ψ. Assume $|\psi(e^{it})|=1$ on a set of positive Lebesque measure. Then T satisfies Alg T={T}'

Consider the functional model of T, that is, consider T acting on the space $H\equiv[H^2\oplus\overline{\Delta L^2}]\ominus\{\psi W\oplus\Delta W: W\in H^2\}$ by

$$T(f\oplus g)=P(e^{it}f\oplus e^{it}g) \quad \text{for } f\oplus g\in H,$$

where $\Delta=(1-|\psi|^2)^{\frac{1}{2}}$ and P denotes the (orthogonal) projection onto H. Let S be an operator in {T}'. Then S is of the form

$$S=P\begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

where $A\in H^\infty$ and $B,C\in L^\infty$ satisfy $B\psi+C\Delta=\Delta A$ a.e. (cf. [5]). For an arbitrary operator T on H, let Lat T denote the lattice of subspaces invariant under T, and $T^{(n)}$ denote the operator $\underbrace{T\oplus\ldots\oplus T}_{n}$ acting on $\underbrace{H\oplus\ldots\oplus H}_{n}$. To show that $S\in$Alg T, it suffices to show that Lat $T^{(n)}\subseteq$ Lat $S^{(n)}$ for any $n\geq 1$ (cf. [1], Theorem 7.1). Note that the characteristic function of $T^{(n)}$ is given by the n by n matrix-valued function

$$\Phi=\begin{pmatrix} \psi & & 0 \\ & \ddots & \\ 0 & & \psi \end{pmatrix}.$$

Let $K\in$ Lat $T^{(n)}$ and $\Phi=\Phi_2\Phi_1$ be the corresponding regular factorization. We first show the following

LEMMA 2. Φ_1 and Φ_2 are n by n matrix-valued functions.

PROOF: Assume that Φ_1 and Φ_2 are, respectively, m by n and n by m matrix-valued functions. Consider

$$\delta\ (e^{it}) = \dim\ \overline{\Delta\ (e^{it})\ \mathbb{C}^n}$$

$$\delta_1\ (e^{it}) = \dim\ \overline{\Delta_1\ (e^{it})\ \mathbb{C}^n}$$

and

$$\delta_2\ (e^{it}) = \dim\ \overline{\Delta_2\ (e^{it})\ \mathbb{C}^m}\ ,$$

where $\mathbb{C}$ denotes the complex plane, $\Delta(e^{it})=(1-\Phi(e^{it})^*\Phi(e^{it}))^{\frac{1}{2}}$ and $\Delta_j(e^{it})=(1-\Phi_j(e^{it})^*\Phi_j(e^{it}))^{\frac{1}{2}}$, $j=1,2$. That $\Phi=\Phi_2\Phi_1$ is a regular factorization is equivalent to the condition that

$$\delta(e^{it})=\delta_1(e^{it})+\delta_2(e^{it})\quad \text{a.e.} \qquad (1)$$

Since $|\psi(e^{it})|=1$ on a set of positive Lebesque measure, say, α, it follows that $\Delta(e^{it})=0$ on α. Hence $\delta(e^{it})=0$ on α. If $m>n$ then $\Phi_2(e^{it})$ cannot be isometric from $\mathbb{C}^m$ to $\mathbb{C}^n$. Thus $\delta_2(e^{it})>0$ a.e., which contradicts (1). Hence we must have $m\leq n$. Now we want to show that actually $m=n$. Assume the contrary, that is $m<n$. Let $\Phi_1=(\xi_{ij})$ and $\Phi_2=(\psi_{ij})$. Carrying out the matrix multiplication in $\Phi=\Phi_2\Phi_1$, we obtain n^2 equations

$$\sum_{k=1}^{m} \psi_{ik}\ \xi_{kj} = \begin{cases} \psi & \text{, if } i=j \\ 0 & \text{, otherwise .} \end{cases} \qquad (2)$$

Note that the determinant of any m by m submatrix of Φ_1 is the zero function. For otherwise using the Cramer's rule we can show that $\psi_{i1}=\ldots=\psi_{im}=0$ for some $i=1,\ldots,n$, which, by (2), implies $\psi=0$, contradicting our assumption that $|\psi(e^{it})|=1$ on a set of positive Lebesque measure. (A submatrix of Φ_1 is obtained by crossing out some rows and columns from Φ_1.) On the other hand, consider the m by m matrix-valued functions

$$\Phi' = \begin{pmatrix} \psi & & 0 \\ & \ddots & \\ 0 & & \psi \end{pmatrix}, \qquad \Phi_1' = \begin{pmatrix} \xi_{11} & \cdots & \xi_{1m} \\ \xi_{m1} & \cdots & \xi_{mm} \end{pmatrix}$$

and $$\Phi_2' = \begin{pmatrix} \psi_{11} & \cdots & \psi_{1m} \\ \psi_{m1} & \cdots & \psi_{mm} \end{pmatrix}$$

Since $\Phi' = \Phi_2'\Phi_1'$, we have $\psi^m = \det \Phi' = (\det \Phi_2')(\det \Phi_1') = 0$. This implies $\psi = 0$, again a contradiction. Thus we must have $m=n$, as asserted.

Proof of Theorem 1. To complete the proof we have to show that $K \epsilon \operatorname{Lat} S^{(n)}$. Note that in the functional model,

$$K = \{\Phi_2 u \oplus Z^{-1}(\Delta_2 u \oplus \nu) : u \epsilon H^2(\mathbb{C}^n), \nu \ \epsilon\ \overline{\Delta_1 L^2(\mathbb{C}^n)}\}$$

$$\ominus\{\Phi w \oplus \Delta w : w \epsilon H^2(\mathbb{C}^n)\},$$

where Z denotes the unitary operator from $\overline{\Delta L^2(\mathbb{C}^n)}$ to $\overline{\Delta_2 L^2(\mathbb{C}^n)} \oplus \overline{\Delta_1 L^2(\mathbb{C}^n)}$ defined by

$$Z(\Delta\nu) = \Delta_2\ \Phi_1 \nu \oplus \Delta_1 \nu\ , \qquad \nu \epsilon L^2(\mathbb{C}^n).$$

Let $\Phi_2 u \oplus t$ be an element in K, where $u = (u_i) \epsilon H^2(\mathbb{C}^n)$ and $t = (t_i) \epsilon \overline{\Delta L^2(\mathbb{C}^n)}$ satisfying $Zt = \Delta_2\ u \oplus v$, for some $v = (v_i) \epsilon \overline{\Delta_1 L^2(\mathbb{C}^n)}$. We want to show that $S^{(n)}(\Phi_2 u \oplus t) \epsilon K$. Note that $\Phi_2 u \oplus t = \sum_{i=1}^{n} \oplus\ (\sum_{j=1}^{n} \psi_{ij} u_j \oplus t_i)$ and $\sum_{j=1}^{n} \psi_{ij} u_j \oplus t_i \epsilon [H^2 \oplus \Delta L^2] \ominus \{\psi W \oplus W : W \epsilon H^2\}$. Hence

$$S^{(n)}(\Phi_2 u \oplus t) = \sum_{i=1}^{n} \oplus\ [P\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}\begin{pmatrix} \sum_{j=1}^{n} \psi_{ij}\ u_j \\ t_i \end{pmatrix}]$$

$$= \sum_{i=1}^{n} \oplus \left[P \begin{pmatrix} A \sum_{j=1}^{n} \psi_{ij} u_j \\ B \sum_{j=1}^{n} \psi_{ij} u_j + C t_i \end{pmatrix} \right]$$

$$= \sum_{i=1}^{n} \oplus \begin{pmatrix} A \sum_{j=1}^{n} \psi_{ij} u_j - \psi w_i \\ B \sum_{j=1}^{n} \psi_{ij} u_j + C t_i - \Delta w_i \end{pmatrix} \qquad (3)$$

for some $W_i \varepsilon H^2$, $i=1,2,...,n$. Solving the following n equations for $\psi_{i1},...,\psi_{in}$

$$\sum_{k=1}^{n} \psi_{ik}\, \xi_{kj} = \begin{cases} \psi & \text{, if } j=i \\ 0 & \text{, otherwise} \end{cases} \qquad j=1,2,..,n,$$

by means of the Cramer's rule, we obtain

$$(\det\Phi_1)\psi_{ik} = \psi\eta_{ik} \qquad i,k=1,2....n,$$

where η_{ik} is the determinant, multiplied by $(-1)^{i+k}$, of the matrix obtained from Φ_1 by deleting its i-th row and k-th column. It follows that

$$(\det\Phi_1) B \sum_{j=1}^{n} \psi_{1j} u_j = B \sum_{j=1}^{n} (\det\Phi_1)\psi_{ij} u_j = B \sum_{j=1}^{n} \psi\eta_{ij} u_j$$

$$= \sum_{j=1}^{n} \Delta(A-C)\eta_{ij} u_j = \Delta(A-C) \sum_{j=1}^{n} \eta_{ij} u_j.$$

Note that $\sum_{j=1}^{n} \oplus [(\det \Phi_1) B \sum_{j=1}^{n} \psi_{ij} u_j]$ is an element of $\overline{\Delta L^2(C^n)}$. Thus we have

$$Z(\sum_{i=1}^{n} \oplus [(\det\Phi_1) B \sum_{j=1}^{n} \psi_{ij} u_j]) = Z(\sum_{i=1}^{n} \oplus [\Delta(A-C) \sum_{j=1}^{n} \eta_{ij} u_j])$$

$$= \Delta_2 \Phi_1 (A-C) \begin{pmatrix} \sum_{j=1}^{n} \eta_{1j} u_j \\ \vdots \\ \sum_{j=1}^{n} \eta_{nj} u_j \end{pmatrix} \oplus \Delta_1 (A-C) \begin{pmatrix} \sum_{j=1}^{n} \eta_{1j} u_j \\ \vdots \\ \sum_{j=1}^{n} \eta_{nj} u_j \end{pmatrix}$$

$$= \Delta_2 (A-C) \begin{pmatrix} \sum_{k=1}^{n} \xi_{1k} \sum_{j=1}^{n} \eta_{kj} u_j \\ \vdots \\ \sum_{k=1}^{n} \xi_{nk} \sum_{j=1}^{n} \eta_{kj} u_j \end{pmatrix} \oplus \Delta_1 (A-C) \begin{pmatrix} \sum_{j=1}^{n} \eta_{1j} u_j \\ \vdots \\ \sum_{j=1}^{n} \eta_{nj} u_j \end{pmatrix}$$

$$= \Delta_2 (A-C) \begin{pmatrix} (\det\Phi_1) u_1 \\ \vdots \\ (\det\Phi_1) u_n \end{pmatrix} \oplus \Delta_1 (A-C) \begin{pmatrix} \sum_{j=1}^{n} \eta_{1j} u_j \\ \vdots \\ \sum_{j=1}^{n} \eta_{nj} u_j \end{pmatrix}$$

$$=\Delta_2(A-C)(\det\Phi_1)u \oplus \Delta_1(A-C)\begin{pmatrix}\sum_{j=1}^{n}\eta_{1j}u_j\\ \vdots\\ \sum_{j=1}^{n}\eta_{nj}u_j\end{pmatrix}. \qquad (4)$$

On the other hand,

$$Z(\sum_{i=1}^{n}\oplus[(\det\Phi_1)B\sum_{j=1}^{n}\psi_{ij}u_j])$$

$$=(\det\Phi_1)\ Z(\sum_{i=1}^{n}\oplus(B\sum_{j=1}^{n}\psi_{ij}u_j))$$

$$=(\det\Phi_1)(X\oplus Y), \qquad (5)$$

say, for some element $X\oplus Y$ in $\overline{\Delta_2 L^2(\mathbb{C}^n)}\oplus\overline{\Delta_1 L^2(\mathbb{C}^n)}$. Equating the first components in (4) and (5) we obtain

$$\Delta_2(A-C)(\det\Phi_1)u=(\det\Phi_1)X . \qquad (6)$$

Since $\det\Phi=\psi^n\not\equiv 0$ and $\det\Phi=(\det\Phi_2)(\det\Phi_1)$, we have $\det\Phi_1\not\equiv 0$. Hence $\det\Phi_1\neq 0$ a.e. by the F. and M. Riesz Theorem. Thus (6) yields that

$$\Delta_2(A-C)u=X ,$$

and

$$Z(\sum_{i=1}^{n}\oplus(B\sum_{j=1}^{n}\psi_{ij}u_j+Ct_i))=Z(\sum_{i=1}^{n}\oplus(B\sum_{j=1}^{n}\psi_{ij}u_j))$$

$$+Z(\sum_{i=1}^{n}\oplus Ct_i)=[(\Delta_2(A-C)u)\oplus Y]+C(\Delta_2 u\oplus v)$$

$$=\Delta_2 Au\oplus(Y+Cv),$$

Hence (3) can be written as

$S^{(n)}(\Phi_2 u\oplus t)=\{\Phi_2 Au\oplus Z^{-1}[\Delta_2 Au\oplus(Y+Cv)]\}\ominus(\Phi_1 W\oplus\Delta W)$, where $W=(W_i)\in H^2(\mathbb{C}^n)$. This shows that $S^{(n)}(\Phi_2 u\oplus t)\in K$ as asserted and completes the proof.

COROLLARY 3. If T is the compression of the shift on the space $H^2\ominus\psi H^2$, then T satisfies Alg $T=\{T\}'$.

If ψ is an outer function then more can be said.

THEOREM 4. Let T be a c.n.u. contraction having an outer scalar-valued characteristic function ψ. Then the following are equivalent to each other:

(1) $|\psi(e^{it})|=1$ on a set of positive Lebesque measure;

(2) Alg $T=\{T\}'$;

(3) every invariant subspace for T is hyperinvariant, that is, invariant under every operator in $\{T\}'$.

The proof of this theorem follows immediately from the following lemma.

LEMMA 5. Let T be a c.n.u. contraction having an outer scalar-valued characteristic function ψ. Then T has the same lattice of hyperinvariant subspaces as the bilateral shift on $L^2(E)$, where $E=\{e^{it}: |\psi(e^{it})|<1\}$.

In fact, Sickler [3] showed that in this case T has the same lattice of invariant subspaces as the bilateral shift on $L^2(E)$. Our proof will be based on his descriptions of this lattice of T.

PROOF: The invariant subspaces of T are given by

$$K_S=\{f \oplus g\epsilon H: -\Delta f+\psi g\epsilon SH^2\} \quad K_F=\{f \oplus g\epsilon H: -\Delta f+\psi g\epsilon L^2(F)\}$$

where S runs over all unimodular functions and F over all measurable subsets of the unit circle (cf. [3], Theorem 5.1). In light of Sickler's result (Theorem 6.1 in [3]) it suffices to show that each K_F is hyperinvariant and each K_S is not. Since the operators in $\{T\}'$ are of the form

$$S=P\begin{pmatrix} A & 0 \\ B & C \end{pmatrix},$$

where $A\epsilon H^\infty$ and $B,C\epsilon L^\infty$ satisfy $B\psi+C\Delta=\Delta A$ a.e., we have

$$S\begin{pmatrix} f \\ g \end{pmatrix} = P\begin{pmatrix} A & 0 \\ B & C \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = P\begin{pmatrix} Af \\ Bf+Cg \end{pmatrix} = \begin{pmatrix} Af-\psi w \\ Bf+Cg-\Delta w \end{pmatrix}$$

for some $w\epsilon H^2$. . Note that

$$\begin{aligned} -\Delta(Af-\psi w)+\psi(Bf+Cg-\Delta w) &= -\Delta Af+\psi Bf+\psi Cg \\ &= -(B\psi+C\Delta)f+\psi Bf+\psi Cg=C(-\Delta f+\psi g). \end{aligned}$$

Hence $f\oplus g\epsilon K_F$ implies that $S(f\oplus g)\epsilon K_F$, which proves our first assertion.

To prove the second assertion we may assume that $|\psi(e^{it})|< 1$ a.e., for otherwise all of the invariant subspaces are of the form K_F (cf. [3], Theorem 7.2). Assume that K_S is hyperinvariant for some unimodular function S. From the calculation above we conclude that for any operator S in $\{T\}'$ $-\Delta f+\psi g\epsilon SH^2$ implies that $C(-\Delta f+\psi g)\epsilon SH^2$. Since the set $\{-\Delta f+\psi g: f\oplus g\epsilon K_S\}$ is dense in SH^2 (cf. the proof of Theorem 6.1 in [3]), the multiplication by C defines an operator on SH^2. Thus $C\epsilon H^\infty$. Let $\alpha_n=\{e^{it}:\frac{1}{n}\leq|\psi(e^{it})|\leq 1-\frac{1}{n}\}$ for $n=2,3,\ldots,$. Then $\bigcup_n\alpha_n$=the unit circle. Consider $A=0$, $B_n=\chi_{\alpha n}$ and $C_n=\frac{-\psi\chi_{\alpha n}}{\Delta}$. It is easily seen unit $A\epsilon H^\infty$ and $B_n,C_n\epsilon L^\infty$ satisfy $B_n\psi+C_n\Delta=\Delta A$ a.e. Such choices of A, B_n and C_n give rise to an operator S_n in $\{T\}'$ for each n. Thus $C_n\epsilon H^\infty$ for all n. Note that $\alpha_n{}^c$, the complement of α_n with respect to the unit circle, has Lebesque measure zero, for at least one $n\geq 2$, for otherwise $C_n=-\frac{\psi\chi_{\alpha n}}{\Delta}=0$ on $\alpha_n{}^c$ implies, by the F and M. Riesz Theorem and the fact that $\psi\not\equiv 0$, that α_n has Lebesque measure zero for all n, which is absurd. Thus $\frac{1}{n}\leq|\psi|\leq 1-\frac{1}{n}$ a.e. for some $n\geq 2$, and $C=\frac{-\psi}{\Delta}$ a.e. is an element of H^∞. It follows that Δ is an element of H^∞. Since Δ is real-valued, it can be easily shown that ψ is a constant function, the trivial case which we can exclude in the beginning. This completes the proof.

Proof of Theorem 4. The implication (3)$\Rightarrow$(1) is essentially given in the proof of Lemma 5.

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