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Geometrical Properties of the Gaussian Laser Beam

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ABSTRACT — This paper is a review of the Gaussian Laser beam propagation properties from the point of view of the Geometrical optics. It is meant to be tutorial in nature and useful in scope. And emphasis is placed on formulations, derivations and examples which lead to basic understanding and on results which bear practical significance.

I. Introduction

The field distribution of a fundamental mode gas laser beam is Gaussian. The equations to express the properties of Gaussian laser beam propagating in free space and along the axis on an optical system, in general, differ from the familiar first order equations of geometrical optics. In this paper we will discuss how the Gaussian beams of gas laser are transformed on their passage through free space (Sec.II), thin lens (Sec.III), spherical refracting lens surface (Sec.IV), spherical reflecting mirror surface (Sec.V), mode match (Sec.VI), and the transformation of the Gaussian beam in lens or lens-like systems (Sec.VII). And most of the above discussions are illustrated with numerical examples.

II. Propagation of Gaussian Laser Beam in a Homogeneous Medium [1-3]

In general the Gaussian spherical wave which propagates along the +Z direction can be written in the compressed form

(2-1)
$$\tilde{u}(x,y)=\sqrt{\frac{2}{\pi}}\,\frac{1}{W}\,\exp\left(-jk\frac{x^2+y^2}{2\tilde{q}}\right)$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{1}{W}\,\exp\left(-j\frac{\pi}{\lambda}\,\frac{x^2+y^2}{\tilde{q}}\right)$$

where $\tilde{q}$ is implicity defined as a complex radius of curvature, given by

(2-2) $$\frac{1}{\tilde{q}} = \frac{1}{R} - j\frac{\lambda}{\pi W^2}$$

and W is commonly called the spot size of this Gaussian distribution. This spherical wave with Gaussian transvers amplitude variation across the wave front is shown in Fig. 1.

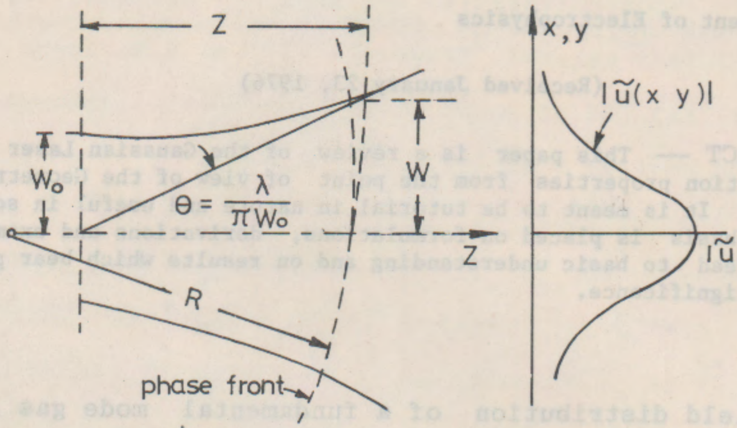

Fig. 1 Contour of a Gaussian laser beam.

Let us assume a Gaussian plane wave at an input Z=0.

(2-3) $$\tilde{u}_o(x_o,y_o) = \sqrt{\frac{2}{\pi}}\,\frac{1}{W_o}\exp\left(-j\frac{k}{2}\,\frac{x_o^2+y_o^2}{\tilde{q}_o}\right)$$

where

(2-4) $$\tilde{q}_o = j\frac{\pi W_o^2}{\lambda}$$

the output at any later is

(2-5) $$\tilde{u}(x,y,z) = \sqrt{\frac{2}{\pi}}\,\frac{1}{W(Z)}\exp\{-j(KZ-\psi(Z))\}\exp\left[-j\frac{k}{2}\,\frac{x^2+y^2}{q(Z)}\right]$$

with the following identifications:

(2-6) $$\tilde{q}(Z) = \tilde{q}_o + Z = Z + j\frac{\pi W_o^2}{\lambda}$$

(2-7) $$\frac{1}{\tilde{q}(Z)} = \frac{1}{R(Z)} - j\frac{\lambda}{\pi W^2(Z)} = \frac{1}{Z + j\dfrac{\pi W_o^2}{\lambda}}$$

(2-8) $$R(Z)=Z+\left(\frac{\pi W_o^2}{\lambda}\right)^2\frac{1}{Z}$$

(2-9) $$W(Z)=W_o\sqrt{1+\left(\frac{\lambda Z}{\pi W_o^2}\right)^2}$$

(2-10) $$\psi(Z)=\tan^{-1}\frac{\lambda Z}{\pi W_o^2}$$

This input plane, where the beam wavefront is planar and the spot size is W_o, we will thereforth refer to as the waist.

This results some very important consequences:

1. By definition, the beam profile radius, W, has a minimum at Z=0.
2. The profile radius increases symmetrically on either side of the origin.
3. The phase front of curvature, R, is infinite (plane phase front) at Z=0.
4. R decreases to a minimum of $R_{min}=-2\pi W_o 2/\lambda$ at $Z=-\pi W_o 2/\lambda$, and then monotically increases.
5. In the farfield ($Z>>\frac{\pi W_o^2}{\lambda}$), the phase front radius becomes $R(Z)\approx Z$. And the wave has a spherical wavefront with its center of curvature located essentially at the waist $W(Z)\approx\frac{\lambda Z}{\pi W_o^2}$.

Thus a beam starting as a gaussian plane wave at a waist will remain in the form of a gaussian spherical wave with complex curvature $\tilde{q}(Z)$ at all later plane Z as shown in Fig. 2.

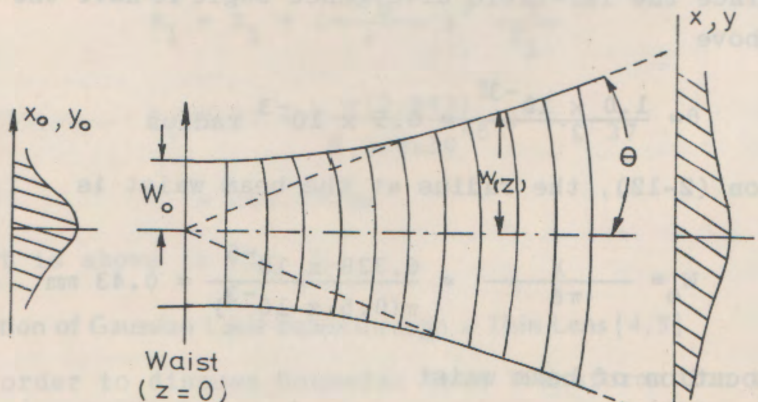

Fig. 2 Outward propagation of a Gaussian laser beam from a waist located at Z=0.

at large distance from the waist of the beam diverges linearly with distance, at a constant cone angle, i.e. in the far field, the beam profile asymptotically approches a straight line through the origin at an angle where

$$\theta = \lim_{Z\to\infty} \left(\frac{W(Z)}{Z}\right) = \lim_{Z\to\infty} \frac{W_o}{Z}\left[1+\left(\frac{\lambda Z}{\pi W_o^2}\right)^2\right]^{1/2} \tag{2-11}$$

$$\theta \cong \frac{\lambda}{W_o} \quad \text{(far-field approximation)} \tag{2-12}$$

This result shows that as the phase front transverses from Z=0 to large Z, its center of curvature transverses from to the origin Z=0. This nonstationary center of curvature and the Gaussian amplitude distribution are the physical differences which cause the classical geometrical lens equation to fail when applied to the near field.

Example 1.

Given: A. The waist radius of the beam, W_o.

B. The location of the waist, Z_1.

C. The value of Z at which the far-field assumptions are valid.

D. The radius of curvature, R_1, of the beam at the exist mirror.

Solution:

A. Beam waist

Since the far-field divergence angle is half the divergence and given above

$$\theta = \frac{1.0 \times 10^{-3}}{2} = 0.5 \times 10^{-3} \text{ radius}$$

From equation (2-12), the radius at the beam waist is

$$W_o = \frac{\lambda}{\pi\theta} = \frac{6.328 \times 10^{-5}}{\pi(0.5 \times 10^{-3})} = 0.43 \text{ mm}$$

B. Location of beam waist

At the exist mirror the beam radius is

$$W_1 = \frac{1.00}{2} = 0.50 \text{ mm}$$

From equation (2-9), the value of Z at exit mirror (Z_1) is found

$$W_1 = W_o\sqrt{1 + \left(\frac{\lambda Z_1}{\pi W_o^2}\right)^2}$$

$$Z_1 = \frac{\pi W_o^2}{\lambda}\sqrt{\left(\frac{W_1}{W_o}\right)^2 - 1}$$

$$= \frac{\pi(0.043)^2}{6.328\times10^{-5}}\sqrt{\left(\frac{0.050}{0.043}\right)^2 - 1}$$

$$= 37 \text{ cm}$$

C. The condition of far-field approximation is valid

$$Z >> \frac{\pi W_o^2}{\lambda}$$

$$Z >> \frac{\pi(0.043)^2}{6\text{-}328\times10^{-5}} = 91.66 \text{ cm}$$

Thus, at approximately ten times the above value, or Z=916.6 cm one can safely assume that far field conditions exist.

D. Radius of curvature of beam at exit mirror

In B we found the distance between the exit mirror and the beam waist Z_1=37 cm.

The radius of curvature can be get from equation (2-8)

$$R_1 = Z_1 + \left(\frac{\pi W_o^2}{\lambda}\right)^2 \frac{1}{Z_1}$$

$$R_1 = 37 + \left(\frac{\pi(0.043)^2}{6.328\times10^{-5}}\right)^2 \cdot \frac{1}{37}$$

$$= 264.07 \text{ cm}$$

The result is shown in Fig. 3.

III. Propagation of Gaussian Laser Beam through a Thin Lens [4,5]

In order to discuss Gaussian beam transformation through a thin lens it is covenient to solve the relationships among the parameters in equations (2-8) and (2-9) of a Gaussian beam propagating along Z-axis simultaneously to the forms as

(3-1)
$$W_o = \frac{W(Z)}{\sqrt{1+(\frac{\pi W(Z)^2}{\lambda Z})^2}}$$

(3-2)
$$Z = \frac{R(Z)}{1+(\frac{\lambda R(Z)}{\pi W(Z)^2})^2}$$

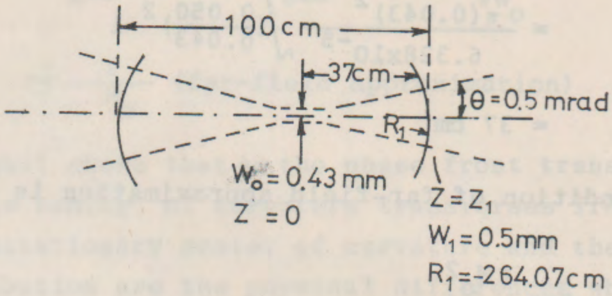

Fig. 3 Example 1 geometry.

Since we now interest the transformation of a Gaussian beam going through a thin lens of focal length f as shown in Fig.4. A laser beamlaser beam with its minimum beam

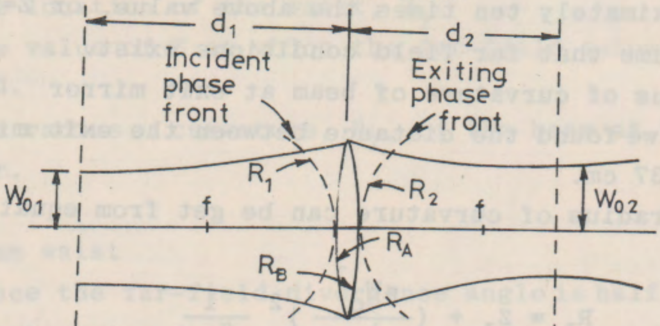

Fig. 4 Gaussian laser beam transformed by a thin lens.

radius W_o1 located at d_1 is travelling from left to right through the thin lens with radius of curvature, R_A, R_B. The phase front radius of curvature incident on the lens surface is R_1, the lens will transform incident phase from radius of curvature, R_1, to an emerging value of R_2. The emerge beam has a minimum beam radius W_{02} located at d_2 to the right of the lens. Because the beam radius remains the same in passing through the thin lens, we obtain from (2-9)

(3-3)
$$W_{01}\sqrt{1+(\frac{\lambda d_1}{\pi W_{01}^2})^2} = W_{02}\sqrt{1+(\frac{\lambda d_2}{\pi W_{02}^2})^2}$$

The thin lens formula states that the change of the phase front curvature may be approximated by the reciprocal of the focal length. Thus using (2-6), one obtain

$$\frac{1}{d_1[1+(\frac{\pi W_{01}^2}{\lambda d_1})^2]}+\frac{1}{d_2[1+(\frac{\pi W_{02}^2}{\lambda d_2})^2]}=\frac{1}{f} \tag{3-4}$$

Let the phase=front radius of curvature incident on the lens is

$$R_1=-d_1\,[1+(\frac{\pi W_{01}^2}{\lambda d_1})^2] \tag{3-5}$$

and the size of the beam at the lens is

$$W_\ell^2=W_{01}^2\,[1+(\frac{\lambda d_1}{\pi W_{01}^2})^2] \tag{3-6}$$

Then the location and size of the new waist in image space can be get from equation (3-3) and (3-4)

$$d_2=\frac{-R_2}{1+(\frac{\lambda R_2}{\pi W_\ell^2})^2} \tag{3-7}$$

$$W_{02}^2=W_\ell^2\,[\frac{1}{1+(\frac{\pi W_\ell^2}{\lambda R_2})^2}] \tag{3-8}$$

where $R_2=(\frac{1}{R_1}+\frac{1}{f})^{-1}=\frac{R_1 f}{R_1+f}$

straight forward algebra will lead to the followings

Solution for (3-3) and (3-4)

$$\frac{d_2}{f}-1=\frac{\frac{d_1}{f}-1}{(\frac{d_1}{f}-1)^2+(\frac{\pi W_{01}^2}{\lambda f})^2} \tag{3-9}$$

$$\frac{W_{02}^2}{W_{01}^2}=\frac{1}{(\frac{d_1}{f})^2+(\frac{\pi W_{01}^2}{\lambda f})^2} \tag{3-10}$$

Equations (3-9) and (3-10) describe the basic relationships of the waist location and size at the input to those at the output and thus contain all essential information concerning the transformation caused by thin lenses by assigning the output waist size and location from one lens as the input the parameters for the subsequent lens.

For the special cases as below, the equations (3-9) and (3-10) can be reduced as:

(1) when $d_1=0$ and $f=\infty$ (no lens)

$$W_2=W_0\left[\left(1-\frac{d_2}{f}\right)^2+\frac{1}{\left(\frac{\pi W_0^2}{\lambda d_2}\right)^2}\right]^{1/2} \tag{3-11}$$

(2) $d_1=0$ (waist at lens)

$$W_2=W_0\left[1+\left(\frac{\lambda d_2}{\pi W_0^2}\right)^2\right] \tag{3-12}$$

$$W_{02}=\frac{\lambda f}{\pi W_0}\left[1+\frac{1}{\left(\frac{\pi W_0^2}{\lambda f}\right)^2}\right]^{-1} \tag{3-13}$$

(waist radius for $d_1=0$)

$$d_2=f\left[1+\frac{1}{\left(\frac{\pi W_0^2}{\lambda f}\right)^2}\right]^{-1} \quad \text{(waist location for } d_1=0) \tag{3-14}$$

Example 2

A He-Ne laser beam is transmitted from a thin lens. The waist location of the incident laser beam d_1 is 40 cm, wait radius W_{01} is 0.4 mm. The parameters of the thin lens are $R_A=10$ cm, $R_B=-50$ cm, n=1.5. Find the location and size of the waist in emerging beam.

Solution:

Use the optic equation $\frac{1}{f}=(n-1)\left(\frac{1}{R_A}-\frac{1}{R_B}\right)$ find the focal length of the lens

$$\frac{1}{f}=(1.5-1)\left(\frac{1}{10}-\frac{1}{50}\right)$$

$f = 16.667$ cm

A. In equation (3-9)

$$\frac{d_2}{16.667} - 1 = \frac{\frac{40}{16.667} - 1}{\left(\frac{40}{16.667} - 1\right)^2 + \left[\frac{\pi (4\times10^{-2})^2}{(6.328\times10^{-5})(16.667)}\right]^2}$$

we get

$d_2 = 17.62$ cm

B. In equation (3-10)

$$\frac{W_{02}}{(0.4)^2} = \frac{1}{\left(\frac{40}{16.667} - 1\right)^2 + \left(\frac{\pi (4\times10^{-2})^2}{(6.328\times10^{-5})(16.667)}\right)^2}$$

we get

$W_{02} = 0.008$ mm

IV. Propagation of Gaussian Laser Beam through a Spherical Refracting Lens Surface [6,7]

The parameters to be used for the general refracting surface are shown in Fig. 5. The laser beam is travelling from

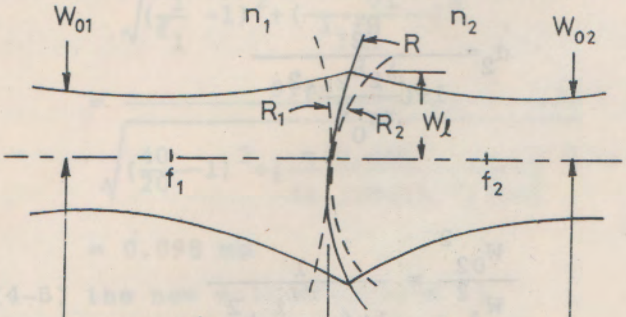

Fig. 5 Spherical refracting surface parameters

left to right. The input beam has a waist, W_{01}, at distance d_1 to the left of the surface. After passing the refracting surface with radius of curvature, R, will become another beam with its phase front radius of curvature R_2, and its waist size of W_{02} at a distance d_2 to the right of the surface. The refractive index in the indicent side and the image side are n_1 and n_2 respectly.

Use the equations (3-5) and (3-6) for the thin lens case to express the incident phase front radius R_1 and the beam size W_L at refracting surface. From the principles of geometry optic we know that the beam size is constant across the surface, but the wavelength changes according to

(4-1) $$n_1\lambda_1=n_2\lambda_2$$

And spherical surface transforms the incident wavefront radius to an emerging radius by

(4-2) $$\frac{n_2}{R_2}-\frac{n_1}{R_1}=\frac{n_2}{f_2}=\frac{n_1}{f_1}$$

with the equations (3-5), (3-6), (4-1) and (4-2), the image beam radius R, beam size W_L, new wavelength, new waist W_{02} and new waist location may be calculated from free space propagation equations (3-1) and (3-2).

$$\lambda_2=\frac{n_1}{n_2}\lambda_1$$

$$R_2=\frac{n_2}{n_1}\left(\frac{1}{f_1}+\frac{1}{R_1}\right)^{-1}$$

$$d_2=-\frac{R_2}{1+\left(\frac{\lambda_2 R_2}{\pi W_0^{\ 2}}\right)^2}$$

and

$$\frac{W_{02}^{\ 2}}{W_\ell^{\ 2}}=\frac{1}{1+\left(\frac{\pi W_\ell}{\lambda_2 R_2}\right)^2}$$

From the above equations, the final results are:

(4-7) $$\frac{d_2}{f_2}-1=\frac{(d_1-f_1)}{\left(\frac{d_1}{f_1}-1\right)^2+\left(\frac{\pi W_{01}^{\ 2}}{\lambda_1 f_1}\right)}$$

and

$$\frac{W_{02}^2}{W_{01}^2} = \frac{1}{(\frac{d_1}{f_1}-1)^2+(\frac{\pi W_{02}^2}{\lambda_1 f_1})^2}$$

Equations (4-7) and (4-8) are nearly identical to thin lens equations (3-9) and (3-10)

Example 3.

The radii of curvature of the surfaces of a thick lens are +10 and -50cm. The index is 1.50. Compute the waist size and the waist position of the image of a incident He-Ne laser beam whose wasit radius W_{01} is 0.4 mm, waist location d_1 is 40 cm.

Surface 1 Calculation. Fig. 6.

The focal lengths of the lens from equation (4-2) are

$$f_1=\frac{n_1 R}{n_2-n_1} = \frac{10}{1.5-1}=20 \text{ cm}$$

$$f_2=\frac{n_2}{n_1} f_1=1.5(20)= \Delta 30\text{cm}$$

From equation (4-6) find the new waist size

$$W_{02}=\frac{W_{01}}{\sqrt{(\frac{d_1}{f_1}-1)^2+(\frac{\pi W_{01}^2}{\lambda_1 f_1})^2}}$$

$$= \frac{0.04}{\sqrt{(\frac{40}{20}-1)^2+[\frac{\pi(0.04)^2}{(6.328\text{x}10^{-5})(20)}]^2}}$$

$$= 0.098 \text{ mm}$$

From equation (4-5) the new waist location is

$$d_2=f_2[1+\frac{d_1-f_1}{(\frac{d_1}{f_1}-1)^2+(\frac{\pi W_{01}^2}{\lambda_1 f_1})^2}]$$

$$= 30[1+\frac{40-20}{(\frac{40}{20}-1)^2+(\frac{\pi(0.04)^2}{(6.328\text{x}10^{-5})(20)})^2}]$$

$$= 65.79 \text{ cm}$$

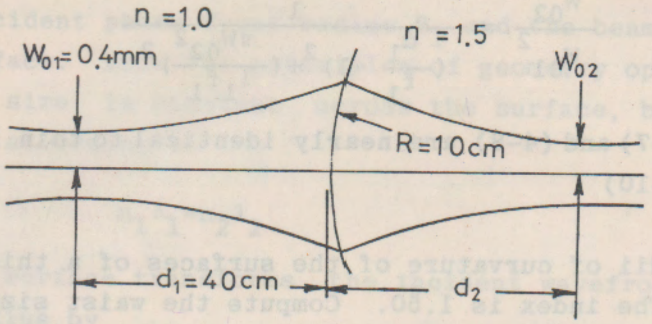

Fig. 6 Surface 1 Calculation.

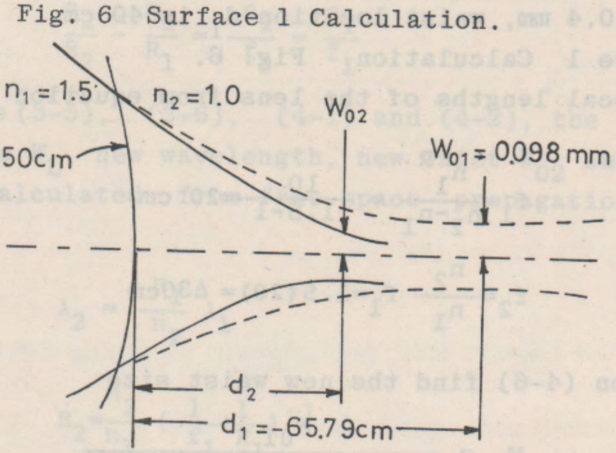

Fig. 7 Surface 2 Calculation.

The input to surface 2 of the lens is the output from surface 1. Thus, for the second surface

$$d_1 = -65.79 \text{ cm}$$

and

$$W_{01} = 0.098 \text{ mm}$$

$$f_1 = \frac{-50}{1-1.5} = 150 \text{ cm}$$

$$f_2 = \frac{1}{1.50}(150) = 100 \text{ cm}$$

Since the incident beam is in a medium of refractive index 1.5. Thus

$$\lambda_1 = \frac{n_2}{n_1}\lambda_2 = \frac{6.328\text{x}10^{-5}}{1.5} = 4.219\text{x}10^{-5} \text{ cm}$$

Finally, the waist size and the waist location of the emerging laser beam from the lens are:

$$W_{02}=\frac{W_{01}}{\sqrt{\left(\frac{d_1}{f_1}-1\right)^2+\left(\frac{\pi W_{01}{}^2}{\lambda_1 f_1}\right)^2}}$$

$$=\frac{9.8\times 10^{-3}}{\sqrt{\left(\frac{-65.79}{150}-1\right)^2+\left(\frac{\pi(9.8\times10^{-3})^2}{(4.219\times10^{-5})(150)}\right)^2}}$$

$$= 0.068 \text{ mm}$$

$$d_2=100\left[1+\frac{-65.79-150}{\left(\frac{-65.79}{150}-1\right)^2+\left(\frac{\pi(9.8\times10^{-3})^2}{(4.219\times10^{-5})(150)}\right)^2}\right]$$

$$=30.54 \text{ cm}$$

V. Propagation of Gaussian Laser Beam through a Spherical Reflecting Mirror Surface [7]

The difference between the spherical reflecting mirror surface with the refracting surface is that the emerging beam is travelling from right to left and in the same medium as the incident beam. So that, all the previous refracting equations are applicable if the substitution $n_2=n_1$ is made.

The parameters to be used for the general reflecting mirror surface are shown in Fig. 9, and the equation

(4-2) becomes

$$\frac{1}{R_2}+\frac{1}{R_1}=-\frac{1}{f_1}=\frac{1}{f_2}$$

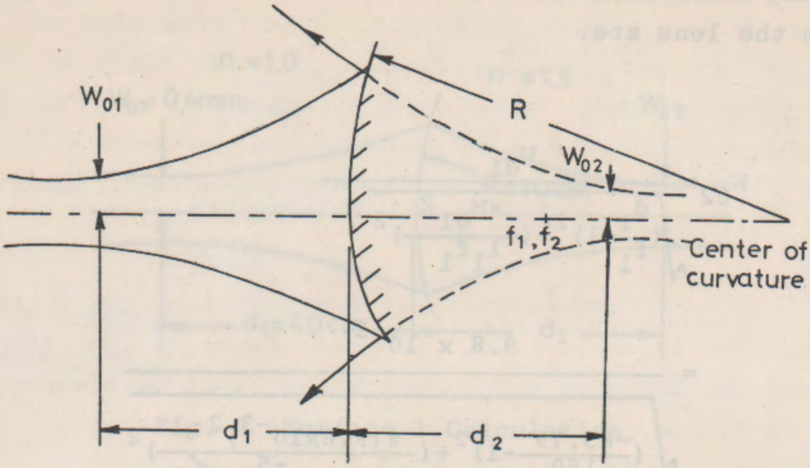

Fig.8 Parameters for spherical reflecting surface

From this equation we get $f_1=-f_2$, since f_1 is positive to the left of the surface and f_2 is positive to the right, so that two focal points are actually at the same physical location as shown in Fig. 9.

Calculations for mirrors are simply made by computing f_1 and f_2 from equation (5-1), and substituting the results in refraction equations (4-7) and (4-8).

In the case of a plane mirror which focal lengths become infinite, from (4-7) and (4-8) equations, the waist and the waist location reduce to

$$d_2=d_1$$

and

$$W_{02} = W_{01}$$

Example 4.

A He-Ne laser beam with a 0.4 mm waist size is focused by a concave spherical mirror whose radius of curvature R is -10 cm, the waist location of beam is 40 cm from the vertex of the mirror.(Fig.9) Find the location and size of the focused laser beam waist:

Solution:

From equation (5-1)

$$f_1=-\frac{R}{2}=\frac{-(-10)}{2}=5\text{ cm}$$

$$f_2=-5\text{ cm}$$

From equation (4-7)

$$d_2 = \left[\frac{(d_1 - f_1)}{(\frac{d_1}{f_1} - 1)^2 + (\frac{\pi W_{01}^2}{\lambda_1 f_1})^2} + 1\right] f_2$$

$$= \left[\frac{(40-5)}{(\frac{40}{5} - 1)^2 + (\frac{\pi (4\times 10^{-2})^2}{6.328\times 10^{-5})(5)})^2} + \right] (-5)$$

$$= -5.58 \text{ cm}$$

From equation (4-8)

$$W_{02} = \frac{W_{01}}{\sqrt{(\frac{d_1}{f_1} - 1)^2 + (\frac{\pi W_{01}^2}{\lambda_1 f_1})^2}}$$

$$= \frac{0.04}{\sqrt{(\frac{40}{5} - 1)^2 + (\frac{\pi (4\times 10^{-2})^2}{6.328\times 10^{-5})(5)})^2}}$$

$$= 0.023 \text{ mm}$$

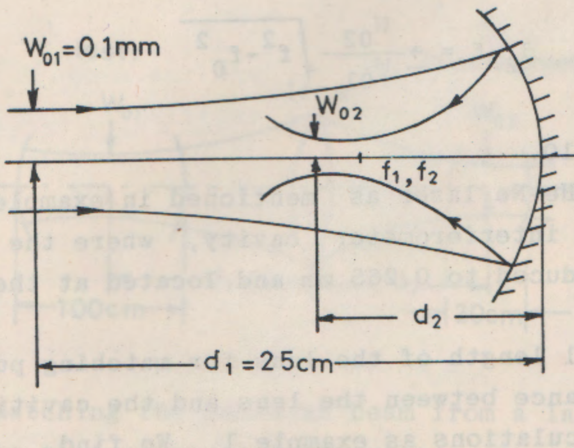

Fig. 9 Example 4. Calculation.

VI. Mode Matching [4,3]

In general, when a laser beam from a mode of one system is injected into another system the mode parameters of the two system should be matched. This can be done by selecting a suitable lens and by properly adjusting the waist spacing d_1 and d_2 to transform a beam with a given beam radius W_{01} into another beam of waist radius W_{02}. Same as shown in Fig.4.

We combine equations (4-7) and (4-8), obtain

(6-1) $$\frac{d_1-f}{d_2-f} = \frac{W_{01}^2}{W_{02}^2}$$

This is used to rewrite (4-8) in the form

(6-2) $$\frac{1}{W_{02}^2} = \frac{1}{W_{02}^2 f^2}(d_1-f)(d_2-f)+\frac{1}{f^2}\left(\frac{\pi W_{01}}{\lambda}\right)^2$$

Multiplying (6-2) by $W_{02}^2 f^2$ we arrive at.

(6-3) $$(d_1-f)(d_2-f) = f^2-f_0^2$$

Where we have defined

(6-4) $$f_o = \pi\left(\frac{W_{01}\ W_{02}}{\lambda}\right)$$

To arrive at the mode-matching formular we multiply or divide (6-3) by (6-1), extract the square root, and find

(6-5) $$d_1-f = \pm\frac{W_{01}}{W_{02}}\sqrt{f^2-f_0^2}$$

or

(6-6) $$d_2-f = \pm\frac{W_{02}}{W_{01}}\sqrt{f^2-f_0^2}$$

Example 5. Fig. 10.

A 1 m long He-Ne laser as mentioned in example, is to be matched to a 20 cm interferometer cavity, where the radius of beam waist will be reduced to 0.265 mm and located at the center of the cavity.

Find: A. The focal length of the lens for matching purpose.

B. The distance between the lens and the cavities.

With the same calculations as example 1. We find

W_{01}=0.43 mm (Wasit radius in laser cavity)

A. With equations (6-1) and (6-3), we can find the value of d_2, say for a lens of a given focal length, by eliminating d_1, to obtain the following quadratic equation

$$d_2^2-2fd_2+\frac{\pi^4 W_{01}^4}{\lambda^2}-f^2\left(\frac{W_{02}^2}{W_{01}^2}-1\right)=0$$

It is apparent that the above only has a real solution for Z_2, if

$$f > f_0$$

with expression equation (6-4) we get

The minimum length used for matching is f 56.5 cm

B. If a lens 92.5 cm is chosen, equations (6-5) and (6-6) enable the position of the centers of the laser interferometer cavities to be determined.

$$d_1=80 \pm \frac{0.43}{0.265}\sqrt{92.5-56.5}$$

$$=80\pm 9.74 \text{ cm}$$

$$d_2=80 \pm \frac{0.265}{0.43}\sqrt{92.5-56.5}$$

$$=80\pm 14 \text{ cm}$$

Laser = 89.74 cm Interferometer = 94 cm

The output of the laser must be positioned 52.74 cm to one side of the lens, the interferometer input mirror being 84 cm to the other side, for the two cavities to be matched to each other.

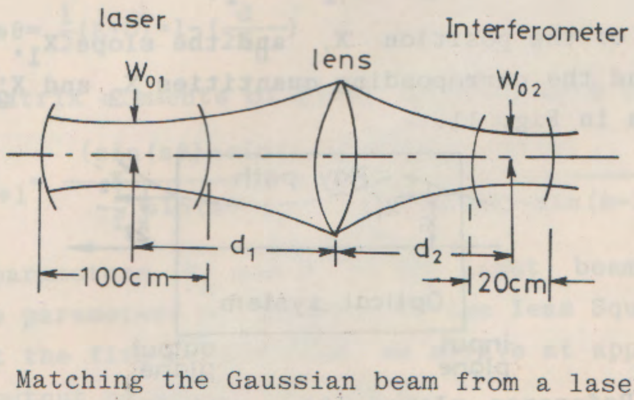

Fig. 10 Matching the Gaussian beam from a laser cavity to that of an interferometer.

VII. Transformation of the Gaussian Laser Beam - The AB CD Law [4,9-11]

In section II the propagation of a Gaussian beam through a homegeneous medium is described by equations (2-8) and (2-9), and these two relations can be combined into a single equation (2-2). Once the value of $\tilde{q}(Z)$ at some plane say (Z_1) is known, we can find its value at any other plane Z_2 by

(7-1) $$\tilde{q}(Z_2) = \tilde{q}(Z_1) + (Z_1 - Z_2)$$

We find then in passing through a thin lens, $\tilde{q}(Z)$ transforms according to

(7-2) $$\frac{1}{\tilde{q}_2} = \frac{1}{\tilde{q}_1} - \frac{1}{f}$$

where $\tilde{q}_1$ is the complex beam parameter of the incoming Gaussian beam and $\tilde{q}_2$ that of the outgoing wave. Equations (7-1) and (7-2) shows that q transforms in the same way as the radius of curvature R of a spherical wave. It follows that the values of q at any two planes are related each other, by

(7-3) $$\tilde{q}_2 = \frac{A\tilde{q}_1 + B}{C\tilde{q}_1 + D}$$

where A, B, C, D are the elements of the matrix relating a paraxial ray at input plane to output plane as follows:

(7-4) $$r_2 = Ar_1 + Br_1'$$

$$r_2' = Cr_1 + Dr_1'$$

the relations of the position X, and the slope X_1, of ray in the input plane, and the corroponding quantities X_2 and X_2' in the output plane are shown in Fig. 11.

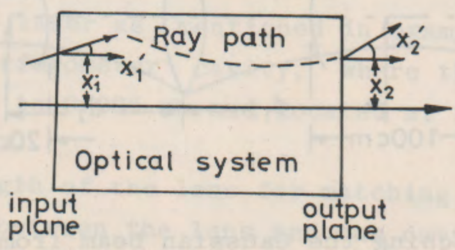

Fig. 11 Reference planes of an optical system. A typical ray path is indicated.

Example: To illustrate power of the ABCD law, we consider the propagation of a Gaussian laser beam through a sequence of thin lenses of equal focal length to and lens spacing d. as shown in Fig. 12. The matrix relating a ray in plane s+1 to the plane s=1 is

$$(7\text{-}5)\qquad \begin{vmatrix} X_{s+1} \\ X'_{s+1} \end{vmatrix} = \begin{vmatrix} A_T & B_T \\ C_T & D_T \end{vmatrix} \begin{vmatrix} X_1 \\ X'_1 \end{vmatrix}$$

and

$$\begin{vmatrix} A_T & B_T \\ C_T & D_T \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}^s$$

with A=1, B=d, $C=-\frac{1}{f_0}$ and $D=1-\frac{D}{f_0}$ in this thin lens system.

The matrix elements A_T, B_T, C_T and D_T can be computed with the help of Sylvester's theorem[9] and are well known [10,11]. One has

$$(7\text{-}6)\qquad \begin{vmatrix} A_T & B_T \\ C_T & D_T \end{vmatrix} = \frac{1}{\sin\theta} \begin{vmatrix} \sin(s\theta)-\sin(s-1)\theta & d\sin(s\theta) \\ -\frac{1}{f_0}\sin(s\theta) & (1-\frac{d}{f_0})\sin(s\theta)-\sin(s-1)\theta \end{vmatrix}$$

where

$$(7\text{-}7)\qquad \mathrm{Cos}\theta = \frac{1}{2}(A+D) = 1-(\frac{d}{2f_0})$$

and then use matrix elements of (7-6) in (7-3) with the result.

$$(7\text{-}8)\qquad \tilde{q}_{s+1} = \frac{\{\sin(s\theta)-\sin(s-1)\theta\}\tilde{q}_1 + d\sin(s\theta)}{-\frac{1}{f_o}\sin(s\theta)\tilde{q}_1 + (1-\frac{d}{f_o})\sin(s\theta)-\sin(s-1)\theta}$$

If the parameters W_1 and R_1 of the input beam do not differ much from the parameters of the mode of the lens Squence. And we neglect all but the first order term, we arrive at approximated formula for the output parameters W_2 and R_2.

(7-9) $$\frac{1}{R_2}+\frac{1}{2f_o}=\left(\frac{1}{R_1}+\frac{1}{2f_o}\right)\cos 2s\theta+\frac{\lambda}{\pi}\left(\frac{1}{W_1^2}-\frac{1}{W_m^2}\right)\sin 2s\theta$$

(7-10) $$\frac{1}{W_2^2}-\frac{1}{W_m^2}=-\frac{\pi}{\lambda}\left(\frac{1}{R_1}+\frac{1}{2f}\right)\sin 2s\theta+\left(\frac{1}{W_1^2}-\frac{1}{W_m^2}\right)\cos 2s\theta$$

where

$$W_m^2=\frac{d\lambda}{\pi\sin\theta}$$

VIII. Conclusion

In this paper we have shown that the analytical formulations of Gaussian laser beam propagation corresponds to simple geometrical properties, and all the discussions are based on the nontruncating diffraction limited optical systems. To avoid such a truncation problem, a circular aperture centered on the Gaussian beam should have a physical radius (r_a) at least $1.7\times r_e$, where r_e is the $1/e^2$ radius of the Gaussian beam at the aperture. A safer limit is $r_a \geq r_e$. For the latter case, truncation effect are negligible.

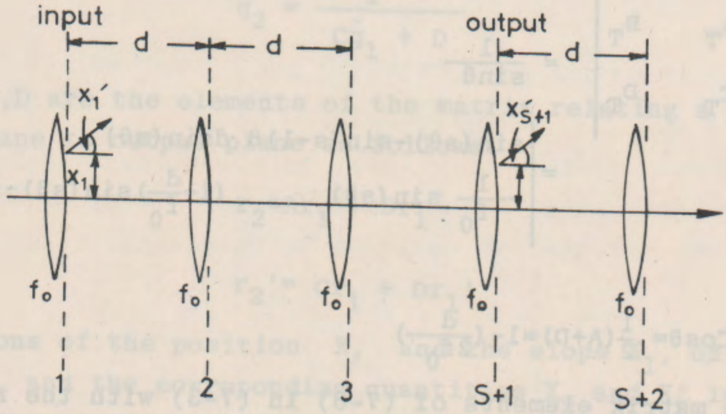

Fig. 12 Sequence of lenses of equal focal length and spacing.

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