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多頻道 n－α 交互作用力的正確解法

Exact Method for Calculating the Multichannel Nucleon-Alpha Potential

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ABSTRACT — A two-channel ℓ-dependent n-α interaction potential is obtained by diagonalizing the corresponding Schrödinger equations by a similar transformation. The n-α elastic scattering cross section, expressed in terms of the potential parameters, is fitted by low-energy scattering data. The results are compared with the previous results.

I. Introduction

The interaction of α-particles with nuclear systems has been studied by a large number of workers, both experimentally and by phenomenological reasoning. The tight structure of the α-particle helps it to interact with many systems while retaining its identity, especially when the energies involved are not high. In particular, there is a limited range≈20 Mev for which an α-n interaction can be usefully regarded as a 2-body problem. This concept has received ample experimental confirmation.

Sack et al [1] fitted the s and p wave phase shifts using a phenomenological, two-body, local, energy-independent potential interaction, having a central term and a spin-orbit term. The spin-orbit term is required to be the derivative of the central term. Three different shapes are used in fitting the phase shifts but they concluded that the Gauss well gave the best over-all fit to the phase shifts. Although the accurate of this potential for use in the bound state calculation, the mathematical complexities involved in using the Gauss well, however, limited the use of it.

One of the earlist and most comprehensive phenomenological attempts to represent an α-n force by a two-body potential is that due to Gammel and Thaler [2] who were able to give almost precision fits to α-n scattering data up to about 40 Mev. However, in view

of the mathematical complexities involved in such a problem the exigencies of approximation may largely obscure any specific conclusions that could otherwise be made about such a "potential".

Mitra et al [3] took a view similar to the reference 2, but with a closer eye on the pratical possibility of subjecting the α-n "potential" to some simple 3-body systems. This necessarily implied a simple choice of the potential, so as to make the subsequent 3-body problem mathematically tractable. However, they considered an energy range of 40 Mev too large for neglecting the structure effects of the α-particle.

Representing the α-n interaction by an effective 2-body potential has some distinct advantages. In principle, it gives a concrete realization to the scheme of resonating group structures, with the alpha particle as the natural resonating group. In practice, an effective α-n potential is extremely useful for calculation of the energy levels in nuclear systems where alpha particles can be regarded as distinct entities, e.g. He^6, Li^6 and Be^9. In the most works as mentioned, an assumption is made that the alpha particle is a structureless elementary particle. While this assumption simplifies the calculation greatly, its justification is not really convincing.

To take into account the internal structure of the alpha particle, recently, the binding energies of Li^6 and Be^9 have been calculated on a multichannel three-body model in which the internal structure of the alpha particle is approximately represented by a 2-state system.[4,5] The n-α interaction is given by a 2-channel ℓ-dependent square-well potential which can be solved analytically in the resonance approximation. On fitting the low-energy scattering data, the potential is thus obtained. After employing such a potential in the calculations of the energy states of Li^6 and Be^9, the results are in good agreement with experiments.

The works mentioned in the last paragraph encourage us to develope a more realistic and simpler analytic method for finding the 2-channel α-n interaction without the limination of the resonance approximation.

In this paper we shall consider the α-n system as a multichannel problem by assuming that the internal structure of the alpha particle may be approximately represented by a 2-level system. This means that the alpha particle maintains its identity although it

may be either its ground state or in its excited state. This picture has effectively taken into account all the inelastic-channel contributions except for breakup of the alpha particle. In sec.II, we make a complete formulation of how the multichannel n-α potential is obtained. The results of the calculation are discussed in the sec. III.

II. Formal Formulation

Let us consider the case of any number of ℓ channels coupled by a square-well-potential matrix V of range r_o. Since the alpha particle is assumed to be a 2-state particle, the Schödinger equation for the n-α system, after partial wave decomposition, can be written as [6]

$$\Psi'' + A_\ell \Psi_\ell = 0 \quad , \tag{1}$$

where Ψ_ℓ is a colume matrix defined by

$$\Psi_\ell = M^{-\frac{1}{2}} u_\ell = M^{-\frac{1}{2}} \begin{pmatrix} u_{1\ell} \\ u_{2\ell} \end{pmatrix} ,$$

and u_ℓ is related to the wave function Ψ in the ordinary form of partial wave decomposition. And

$$A_\ell = \begin{cases} k^2 - 2\hbar^2 M^{\frac{1}{2}} V M^{-\frac{1}{2}} - \dfrac{\ell(\ell+1)}{r^2} \, 1 - \varepsilon & \text{for } r<r_o \\ k^2 - \varepsilon - \dfrac{\ell(\ell+1)}{r^2} \, 1 & r>r_o \end{cases}$$

$$k^2 \equiv \begin{pmatrix} k^2 & 0 \\ 0 & k^2 \end{pmatrix} , \qquad \varepsilon \equiv \begin{pmatrix} \varepsilon_\downarrow & 0 \\ 0 & \varepsilon_\uparrow \end{pmatrix} , \tag{2}$$

$$M \equiv \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} , \qquad V = \begin{pmatrix} V_1 & V_3 \\ V_3 & V_2 \end{pmatrix} ,$$

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

k^2 is related to the kinetic energy of the neucleon, μ the reduced mass of n-α system, and ε the excitation energy of alpha particle defined as in the references 4 and 5.

Let us define

$$K^2=k^2-\varepsilon=\begin{pmatrix} k^2-\varepsilon_{\downarrow} & 0 \\ 0 & k^2-\varepsilon_{\uparrow} \end{pmatrix} \equiv \begin{pmatrix} k_1^2 & 0 \\ 0 & k_2^2 \end{pmatrix} . \tag{3}$$

Thus A_ℓ can be written as

$$A_\ell = \begin{cases} k^2-2\hbar^{-2}M^{\frac{1}{2}}VM^{\frac{1}{2}}-\dfrac{\ell(\ell+1)}{r^2}\,1 & r<r_o \\ k^2-\dfrac{\ell(\ell+1)}{r^2}\,1 & r>r_o \end{cases} , \tag{4}$$

One can observe that Eq. (1) is still a coupled equation inside the width of the potential. In order to make the problem soluble, it is more convenient to make a transformation within the width r_o as follows,
Let

$$T=\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{5}$$

be an unitary matrix which diagonalizes A_ℓ by a similar transformation, i.e.

$$a^2 \equiv T^{-1}A_\ell\, T = \begin{pmatrix} a_{11}^2 & 0 \\ 0 & a_{22}^2 \end{pmatrix} . \tag{6}$$

If we define:

$$k_1^{12} = k_1^2 + \frac{2\mu}{\hbar^2}V_1 - \frac{\ell(\ell+1)}{r^2} ,$$
$$k_2^{12} = k_2^2 + \frac{2\mu}{\hbar^2}V_2 - \frac{\ell(\ell+1)}{r^2} , \tag{7}$$

then

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} k_1^{12} & \frac{2\mu}{\hbar^2}V_3 \\ \frac{2\mu}{\hbar^2} & k_2^{12} \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$
$$=\begin{pmatrix} a_{11}^2 & 0 \\ 0 & a_{22}^2 \end{pmatrix} \tag{8}$$

Thus we must require that

$$a_{11}^2 = a_1 - \frac{\ell(\ell+1)}{r^2} \quad ,$$

$$a_{22}^2 = a_2^2 - \frac{\ell(\ell+1)}{r^2} \quad ,$$

$$\tan 2\alpha = \frac{V_3}{(k^{12}-k_1^{12})} \cdot \frac{4\mu}{\hbar^2} = \frac{4\mu}{\hbar^2} V_3 \frac{1}{[(k_2^2-k_1^2)+\frac{2\mu}{\hbar^2}(V_2-V_1)]} \quad , \tag{9}$$

where

$$\begin{aligned} a_1^2 &= (k_1^2+\tfrac{2\mu}{\hbar^2}V_1)\cos^2\alpha + (k_2^2+\tfrac{2\mu}{\hbar^2}V_2)\sin^2\alpha - \tfrac{4\mu}{\hbar^2}V_3\sin\alpha\cos\alpha \quad , \\ a_2^2 &= (k_1^2+\tfrac{2\mu}{\hbar^2}V_1)\sin^2\alpha + (k_2^2+\tfrac{2\mu}{\hbar^2}V_2)\cos^2\alpha + \tfrac{4\mu}{\hbar^2}V_3\sin\alpha\cos\alpha \quad . \end{aligned} \tag{10}$$

Eq. (1) thus becomes

$$T^{-1}\psi_\ell'' + T^{-1}A_\ell T T^{-1}\psi_\ell = 0 \quad . \tag{11}$$

Define $T^{-1}\psi_\ell = F_\ell$, and $\mathcal{A}^2 = \begin{pmatrix} a_1^2 & 0 \\ 0 & a_2^2 \end{pmatrix}$, then

$$F_\ell'' + (\mathcal{A}^2 - \frac{\ell(\ell+1)}{r^2}\mathbf{1})F_\ell = 0 \quad . \tag{12}$$

This equation can be solved exactly with the solution represented by

$$F_\ell = \begin{pmatrix} a_1 r j_\ell(a_1 r)B_{11} \\ a_2 r j_\ell(a_2 r)B_{22} \end{pmatrix}$$

Thus

$$\psi_\ell = TF_\ell = \begin{cases} \begin{pmatrix} \cos\alpha(a_1 r)j_\ell(a_1 r)B_{11} + \sin\alpha(a_2 r)j_\ell(a_2 r)B_{22} \\ -\sin\alpha(a_1 r)j_\ell(a_1 r)B_{11} + \cos\alpha(a_2 r)j_\ell(a_2 r)B_{22} \end{pmatrix}, & r<r_o \quad , \\ \begin{pmatrix} C_{11}[\cos\delta_\ell(k_1 r)j_\ell(k_1 r) - \sin\delta_\ell(k_1 r)n_\ell(k_1 r)] \\ C_{22}\, k_2 r\, h_\ell(k_2 r) \end{pmatrix}, & r>r_o \quad , \end{cases} \tag{13}$$

where n_ℓ and h_ℓ are the modified spherical Neumann and Hankel function respectly, and defined as

$$n_\ell(\rho) = (-)^{\ell+1} \left(\frac{\pi}{2\rho}\right)^{\frac{1}{2}} J_{-\ell-\frac{1}{2}}(\rho) \quad ,$$
$$h_\ell(\rho) = j_\ell(\rho) - i\eta_\ell(\rho) \qquad (14)$$

The equation (13) comes from the fact that the only incoming waves are in the first channel, i.e. we assume that the alpha particle originally stays in its ground state instead of excited state. Therefore the excited state channel exists only in the outgoing waves. The asymptotic behavior will be in the following form:

$$\Psi_\ell^{(1)} \to C_{11} \sin\left(kr - \tfrac{1}{2}\ell\pi + \delta_\ell\right) \quad ,$$
$$\Psi_\ell^{(2)} \to C_{22}\, e^{-i\left[kr - \frac{1}{2}(\ell+1)\pi\right]} \quad . \qquad (15)$$

By matching the wavefunctions at $r=r_o$, the phase shift δ_ℓ can thus be expressed in terms of the square well potential parameters as:

$$\begin{aligned}\cot\delta_\ell = \Big\{ & \left[\cos\alpha a_1 r_o j_\ell(a_1 r_o) B_{11}/B_{22} + \sin\alpha a_2 r_o j_\ell(a_2 r_o)\right] \\ & \left[\eta_\ell(k_1 r_o) + r_o \eta'_\ell(k_1 r_o)\right] - r_o \eta_\ell(k_1 r_o)\{\left[\cos\alpha a_1\right. \\ & \left. j_\ell(a_1 r_o) + \cos\alpha a_1 r_o j'_\ell(a_1 r_o)\right] B_{11}/B_{22} + \left[\sin\alpha\right. \\ & \left. a_2 j_\ell(a_2 r_o) + \sin\alpha a_2 r_o j'_\ell\ (a_2 r_o)\right]\}\Big\} / \Big\{\left[j_\ell(k_1 r_o)\right. \\ & \left. + r_o j'_\ell(k_1 r_o)\right]\left[\cos\alpha a_1 r_o j_\ell(a_1 r_o) B_{11}/B_{22} + \sin\alpha a_2 r_o\right. \\ & \left. j_\ell(a_2 r_2)\right] - r_o j_\ell(k_1 r_o)\cdot\{\left[\cos\alpha a_1 j_\ell(a_1 r_o) + \right. \\ & \left. \cos\alpha a_1 r_o j'_\ell(a_1 r_o)\right] B_{11}/B_{22} + \left[\sin\alpha a_2 j_\ell(a_2 r_2) + \right. \\ & \left. \sin\alpha\ a_2 r_o j'_\ell(a_2 r_o)\right]\}\Big\} \quad , \end{aligned} \qquad (16)$$

where

$$\begin{aligned} B_{11}/B_{22} = & \{r_o h_\ell(k_2 r_o)\left[\cos\alpha a_2 j_\ell(a_2 r_o) + \cos\alpha a_2 r_o j'_\ell(a_2 r_o)\right] - \\ & -\left[h_\ell(k_2 r_o) + r_o h'_\ell(k_2 r_o)\right]\cos\alpha a_2 r_o j_\ell(a_2 r_o)\}/ \\ & \{(r_o h_\ell(k_2 r_o)\left[\sin\alpha a_1 j_\ell(a_1 r_o) + \sin\alpha a_1 r_o j'_\ell(a_1 r_o)\right] \\ & -\sin\alpha a_1 r_o j_\ell(a_1 r_o)\left[h_\ell(k_2 r_o) + r_o h'_\ell(k_2 r_o)\right]\} \end{aligned} \qquad (17)$$

The differential cross section for elastic scattering is given by

$$\frac{d\sigma}{d\Omega} = \left|\sum_{\ell} \frac{(2\ell+1)}{k} e^{i\delta_\ell} \sin\delta_\ell P_\ell(\cos\theta)\right|^2 \qquad (18)$$

The n-α scattering differential cross sections for 10 different energies are fitted by an ℓ-dependent potential. With a set of initially chosen potential parameters, the numerical program calculates the square deviations of the cross section at 8 different angles. One set of parameters obtained in this way is listed in Table I. The α excitation energy $\varepsilon_\uparrow$ is taken to be 20 Mev.

Table I. Parameters of nα potential.

ℓ	V_1 (Mev)	V_2 (Mev)	V_3 (Mev)	r_o (fm)
0	47.207	30.843	0.276	3.660
1	22.085	45.941	0.369	3.587

III. Results and Discussion

While the potential parameters obtained in this exact soluble method as shown in the table I is compared with the first set solution of Reference 4 which is obtained by a resonance approximation method, we find that they are nearly the same. In fact it is expected that this small deviation can not to effect too much on the binding energies of the three-body system. Therefore our present work shows two important facts that (1) the resonance approximation is a very good approximation in the α-N system (2) with a weak coupling V_3, the data can be fitted without spin-orbit potential. This is in contrast to the single-channel potential in which a term of LS coupling is required [7] . It is also noted that the smallness of the coupling between the two channels does not mean that a single channel square-well potential for n-α interaction will produce the same result since V_1 alone does not fit the scattering data anymore.

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