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多頻道 $\alpha-\alpha$ 交互作用力

The Alpha-Alpha Interaction for Coupled Channels

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ABSTRACT — We have obtained the alpha-alpha interaction for the coupled channels in which the internal structure of the alpha particle is approximately represented by a two-state system. In this model, the alpha-alpha interaction is represented by a 3X3 matrix. By using an unitary transformation, we obtain an exact expression for the phase shift in terms of the potential parameters which is determined from the two-body scattering data.

I. Introduction

The subject of the alpha-alpha interaction occupies so important a role in the nuclear structure problems that the entire understanding of this interaction is necessary. Many authors has attempted to construct the alpha-alpha potential in order to reproduces the experimental phase shifts. The earliest phenomenological alpha-alpha interaction was proposed by Haefner in 1951 [1] . He assumed an repulsive potential for small r (representing the effect of the Pauli Principle operating between nucleons of the alpha clusters), attractive for intermediate r and coulombic for larger r. Van der Spuy and Pienaar [2] made a alpha-alpha scattering analysis up to bombarding energy of about 6 Mev. They considered an square well potential to investigate the velocity-dependence of this interaction. In 1960, Igo [3] made an optical model analysis of the elastic alpha-alpha scattering from 23.1 to 47.1 Mev by using a complex potential. The real part of phase shift obtained for this potential were in good agreement with the preliminary values of Snyder [4] . Among all of the existing works, the most realistic alpha-alpha interaction is perhaps the one obtained by Darriulat et al [5] . This interaction is l-dependent where l denotes the order of the partial waves. It consists of a Saxon-Woods repulsive core and a complex Saxon-Woods attractive well of larger radius and can reproduce the

behavior of the phase shifts over the entire energy range from 0.15 to 120 Mev. However, all these approaches assume that the alpha particle is an elementary particle, and neglect its internal structure.

Recently, considerable interest has been shown in the cluster model of light nuclei [6-8] . Such a model often involves two or more alpha particles and the alpha particle is assumed to be a two-level system to take care of its internal structure approximately. In other words, the alpha particle can be either in its ground state or in its excited state. Therefore, one has to make a multichannel analysis for the alpha-alpha interaction that fits the scattering data. In a previous paper [6] we have obtained a alpha-alpha interaction by using the resonance approximation method. This potential is represented by a 3X3 matrix because of symmetry consideration, and is l-dependent with each matrix element represented by a square well. We obtained two sets of parameters which fit the two-body scattering data equally well but yield slightly different results in the three-body calculations.

In this paper, we present an exact calculation of the multichannel alpha-alpha interaction. In our model we still assume a square well potential form with different strength in each matrix element but with the same range for simplicity. By using an unitary transformation, the set of coupled equations can be transformed to a new system in which the Schrödinger equation becomes uncoupled, and thus can be solved exactly. Therefore, we can get an analytic expression for the phase shift in term of the potential parameters, which can be then obtained by fitting the experimental data.

In section II, we describe the general formalation. We briefly review the resonance approximation method in section III. In section IV, we solve the problem exactly. The results and discussion are given in section V.

II. The General Formulation

We assume that the alpha particle has only two nondegenerate internal states, the excited state $|\uparrow\rangle$ with energy $\varepsilon_{\uparrow}$ and the ground state $|\downarrow\rangle$ with energy $\varepsilon_{\downarrow}$. Furthermore we also assume that both these states have spin parity 0^+ for simplicity. We can treat a system of such two-state particles in analogy with a system of spin $\frac{1}{2}$ particles. Thus we introduce, for a system of two alpha parti-

cles, the operator R=R(1)+R(2) corresponding to the total spin operator. The z-component of R is called R_z such that

$$R_z(i)\,|\uparrow\rangle = \tfrac{1}{2}\,|\uparrow\rangle \qquad R_z(i)\,|\downarrow\rangle = -\tfrac{1}{2}\,|\downarrow\rangle \tag{1}$$

The internal states $|RM\rangle$ of the two-alpha system are given by

$$\begin{aligned} |11\rangle &= |\uparrow\uparrow\rangle \\ |10\rangle &= \frac{1}{\sqrt{2}}[\,|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\,] \\ |1{-1}\rangle &= |\downarrow\downarrow\rangle \\ |00\rangle &= \frac{1}{\sqrt{2}}[\,|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\,] \end{aligned} \tag{2}$$

To write down the Hamiltonian for the alpha-alpha system, we first note that the potential energy operator in the $|RM\rangle$ representation takes the form

$$v(r,RM) = \begin{pmatrix} v_1 & v_5 & v_6 & 0 \\ v_5 & v_2 & v_7 & 0 \\ v_6 & v_7 & v_3 & 0 \\ 0 & 0 & 0 & v_4 \end{pmatrix} \tag{3}$$

where r is the interparticle distance. The Schrodinger equation in the center of mass frame is

$$\left[-\frac{\hbar^2}{2\mu}\,\nabla_r^2 + \mathcal{H} + v\right]\underline{\psi} = E\,\underline{\psi} \tag{4}$$

where μ is the reduced mass of the two alpha particles, E is their relative energy, and $\mathcal{H}$ represents the internal structure energy operator, in $|RM\rangle$ representation, takes the form

$$\mathcal{H} = \begin{pmatrix} 2\varepsilon_\uparrow & 0 & 0 & 0 \\ 0 & \varepsilon_\uparrow+\varepsilon_\uparrow & 0 & 0 \\ 0 & 0 & 2\varepsilon_\downarrow & 0 \\ 0 & 0 & 0 & \varepsilon_\uparrow+\varepsilon_\downarrow \end{pmatrix} \tag{5}$$

Since the alpha particles are assumed to have spin zero in any state, matrix element of the spin-orbit force and the tensor force vanish identically. Therefore, we only have to consider the central force. Eq.(5) implies that if the two alpha particles are both in their ground states initially, the two-alpha system can never be found in the antisymmetric state. Consequently the Schrodinger equation (4) reduces to a set of three coupled equations, and we can write the total wave function as

$$\underline{\Psi}(r,RM) = \sum_{\ell} \frac{2\ell+1}{k_o r} i^{\ell} P_{\ell}(\cos\theta) \begin{pmatrix} u_{1\ell} \\ u_{2\ell} \\ u_{3\ell} \end{pmatrix} \quad (6)$$

The set of three coupled equations are

$$\frac{d^2}{dr^2} \begin{pmatrix} u_{1\ell} \\ u_{2\ell} \\ u_{3\ell} \end{pmatrix} + [k_o^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2} U(r)] \begin{pmatrix} u_{1\ell} \\ u_{2\ell} \\ u_{3\ell} \end{pmatrix} = 0 \quad (7)$$

where $U(r)=\frac{2\mu}{\hbar^2} V-\varepsilon$. ε is an diagonal matrix coming from the internal structure Hamiltonian $\mathcal{H}$ and has the form

$$\varepsilon = \frac{2\mu}{\hbar^2} \begin{pmatrix} 2\varepsilon_{\downarrow} & 0 & 0 \\ 0 & \varepsilon_{\uparrow}+\varepsilon_{\downarrow} & 0 \\ 0 & 0 & 2\varepsilon_{\uparrow} \end{pmatrix} \quad (8)$$

Beyond some enough distance r_o the nuclear part of the alpha-alpha interaction V becomes negligible and then the Coulomb part dominates. The required solution of eq.(7) are those which vanish at the origin and behaves asympototically as an incoming Coulomb distorted plane wave plus outgoing Coulomb, plus nuclear distorted spherical wave.

III. The Resonance Approximation Method

We briefly outline the method here. The details can be found in reference 6. The coupled equations in eq.(7) can be written as

$$\frac{d^2u_{1\ell}}{dr^2}+[k_1^2-\frac{\ell(\ell+1)}{r^2} - \frac{2\mu}{\hbar^2} v_1]u_{1\ell}-\frac{2\mu}{\hbar^2}(v_5u_{2\ell}+v_6u_{3\ell})=0$$

$$\frac{d^2u_{2\ell}}{dr^2}+[k_2^2-\frac{\ell(\ell+1)}{r^2} - \frac{2\mu}{\hbar^2} v_2]u_{2\ell}-\frac{2\mu}{\hbar^2}(v_5u_{1\ell}+v_7u_{3\ell})=0 \quad (9)$$

$$\frac{d^2u_{3\ell}}{dr^2}+[k_3^2-\frac{\ell(\ell+1)}{r^2} - \frac{2\mu}{\hbar^2} v_3]u_{3\ell}-\frac{2\mu}{\hbar^2}(v_6u_{1\ell}+v_7u_{2\ell})=0$$

where the wave numbers are defined by

$$k_1^2=k_o^2-\frac{2\mu}{\hbar^2} 2\varepsilon_{\downarrow}$$

$$k_2^2 = k_o^2 - \frac{2\mu}{\hbar^2}(\varepsilon_\downarrow + \varepsilon_\uparrow) \qquad (10)$$

$$k_3^2 = k_o^2 - \frac{2\mu}{\hbar^2} 2\varepsilon_\uparrow$$

To solve the problem, we first expand the wave function in term of the complete set of functions satisfying the homogeneous equation corresponding to eq.(9). By looking at the elastic scattering around a particular resonance energy, we obtain the elastic scattering amplitude before symmetrization is

$$f(\theta) = f_{nuc}(\theta) + f_c(\theta)$$

with the nuclear part

$$f_{nuc}(\theta) = \sum_\ell \frac{2\ell+1}{2ik} e^{2i(\eta_\ell+\phi_\ell)} P_\ell(\cos\theta)\left[(1-e^{-2i\eta_\ell}) - \frac{i\Gamma_\ell}{(E-E_c)+\frac{1}{2}i\Gamma_\ell}\right] \qquad (11)$$

and the Coulomb part

$$f_c(\theta) = -\frac{\xi}{2k} \frac{\exp[-i\xi\ell n(\sin^2\frac{1}{2}\theta)]}{\sin^2\frac{1}{2}\theta}, \qquad \xi = \frac{4e^2}{h\nu} \qquad (12)$$

The differential cross section for the two alpha particles is, after proper symmetrization, given by

$$\frac{d\sigma}{d\Omega} = |f(\theta) + f(\pi-\theta)|^2 \qquad (13)$$

The first term in eq.(11) represents the direct channel contribution and the second term contains the contribution from all the coupled channels which is responsible for the resonance at the energy E_c with width Γ_ℓ.

IV. The Exact Solution

Eq.(7) may be written as

$$\frac{d^2u}{dr^2} + \left[k_o^2 - \frac{\ell(\ell+1)}{r^2} - U(r)\right] u = 0 \qquad (14)$$

where

$$u = \begin{pmatrix} u_{1\ell} \\ u_{2\ell} \\ u_{3\ell} \end{pmatrix} \qquad (15)$$

If we transform u to a new system u' by an unitary matrix A as

$$u' = Au = \begin{pmatrix} u'_{1\ell} \\ u'_{2\ell} \\ u'_{3\ell} \end{pmatrix} \tag{16}$$

Equation (14) becomes

$$[(\frac{d^2}{dr^2} + k_o^2 - \frac{\ell(\ell+1)}{r^2}]u' - U'u' = 0 \tag{17}$$

where

$$U' = AUA^+ \tag{18}$$

In order to decoupled the equations, we have to find the unitary transformation to diagonalize U. This transformation matrix $A=(a_{ij})$ and the eigenvalues $\lambda_i (i=1,2,3)$ of U can be obtained by the usual diagonalization method. Thus, in the new system, we have

$$\frac{d^2u'_{1\ell}}{dr^2} + [\, k_o^2 - \frac{\ell(\ell+1)}{r^2} + \lambda_1 \,]\, u'_{1\ell} = 0$$

$$\frac{d^2u'_{2\ell}}{dr^2} + [\, k_o^2 - \frac{\ell(\ell+1)}{r^2} + \lambda_2 \,]\, u'_{2\ell} = 0 \tag{19}$$

$$\frac{d^2u'_{3\ell}}{dr^2} + [\, k_o^2 - \frac{\ell(\ell+1)}{r^2} + \lambda_3 \,]\, u'_{3\ell} = 0$$

Therefore, the set of equations become uncoupled. The solutions of eq.(19) satisfying the boundary conditions at the origin are

$$\begin{aligned} u'_{1\ell} &= c_1 r j_\ell(k_1 r) \\ u'_{2\ell} &= c_2 r j_\ell(k_2 r) \\ u'_{3\ell} &= c_3 r j_\ell(k_3 r) \end{aligned} \tag{20}$$

where

$$\begin{aligned} k_1^2 &= k_o^2 + \lambda_1 \\ k_2^2 &= k_o^2 + \lambda_2 \\ k_3^2 &= k_o^2 + \lambda_3 \end{aligned} \tag{21}$$

From eq.(16), we get, for $r<r_o$, the wave function

$$u_{i\ell} = \sum_{j=1}^{3} C_j a_{ij} r j_\ell (k_j r), \quad i=1,2,3 \tag{22}$$

For $r>r_o$, the potential is Coulombic, we have

$$\begin{aligned} u_{1\ell} &= D_1[\cos\delta_\ell F_\ell(k_1 r)+\sin\delta_\ell G_\ell(k_1 r)] \\ u_{2\ell} &= D_2 H_\ell (k_2 r) \\ u_{3\ell} &= D_3 H_\ell (k_3 r) \end{aligned} \tag{23}$$

where F_ℓ and G_ℓ are the regular and irregular Comlomb wavefunctions, $H_\ell = G_\ell + iF_\ell$ representing the outgoing wave in the channel and δ_ℓ is the nuclear phase shift. Matching the wavefunctions and their first derivatives at $r=r_o$, we get

$$\begin{aligned} &\sum_{i=1}^{3} c_i a_{1i} j_\ell(k_i r_o) = D_1[\cos\delta_\ell F_\ell(k_1 r_o)+\sin\delta_\ell G_\ell(k_1 r_o)] \\ &\sum_{i=1}^{3} c_i a_{ji} j_\ell(k_i r_o) = D_j H_\ell(k_j r_o), \quad j=2,3 \\ &\sum_{i=1}^{3} c_i a_{1i} j'_\ell(k_i r_o) = D_1[\cos\delta_\ell F'(k_1 r_o)+\sin\delta_\ell G'_\ell(k_1 r_o)] \\ &\sum_{i=1}^{3} c_i a_{ji} j'_\ell(k_i r_o) = D_j H (k_j r_o), \quad j=2,3 \end{aligned} \tag{24}$$

After some straight forward algebra, we obtain

$$\tan\delta_\ell = \frac{F'_\ell(k_1 r_o) - \underline{\phi} F_\ell(k_1 r_o)}{G'_\ell(k_1 r_o) - \underline{\phi} G_\ell(k_1 r_o)} \tag{25}$$

where

$$\underline{\phi} = \frac{d}{dr} \log \psi$$

$$\psi = a_{11} j_\ell(k_1 r_o) + \beta\, a_{12} j_\ell(k_2 r_o) + \gamma\, a_{13} j_\ell(k_3 r_o)$$

$$\beta = \frac{a_{21}a_{33}d_1 d_{33} - a_{23}a_{31}d_{23}d_2}{a_{22}a_{33}d_{22}d_{33} - a_{23}a_{32}d_{32}d_{23}}$$

$$\gamma = \frac{a_{22}a_{31}d_{22}d_2 - a_{21}a_{32}d_1 d_{32}}{a_{22}a_{32}d_{22}d_{33} - a_{23}a_{32}d_{23}d_{32}}$$

$$d_{ij}=j_\ell(k_j r_o)H'_\ell(k_i r_o)-j'_\ell(k_j r_o)H_\ell(k_i r_o)$$

$$d_1=H_\ell(k_2 r_o)j'_\ell(k_1 r_o)-j_\ell(k_1 r_o)H'_\ell(k_2 r_o)$$

$$d_2=H_\ell(k_3 r_o)j'_\ell(k_1 r_o)-H'_\ell(k_3 r_o)j_\ell(k_1 r_o)$$

Eq.(25) is an exact expression for the nuclear phase shift as a function of the energy for the coupled channels. From this equation we can obtain the alpha-alpha interaction parameters by fitting the experimental scattering data.

V. Result and Discussion

From the resonance approximation method, we have obtained two sets of potential parameters [6] that fits the low energy differential cross sections up to 6.5 Mev equally well. They are listed in Table I.

Table I. Parameters for the alpha-alpha interaction

			V_1 (Mev)	V_2 (Mev)	V_3 (Mev)	V_5 (Mev)	V_6 (Mev)	V_7 (Mev)	r_o (fm)
Resonance Method	Sol.I	0	23.3	23.32	24.50	8.19	0.5	2.52	4806
		2	24.9	29.50	29.40	4.98	10.2	11.56	4.64
	Sol.II	0	23.3	23.32	20.80	6.80	3.86	2.69	4.806
		2	24.7	29.50	25.70	4.50	8.30	14.20	4.64
Exact Solution		0	22.4	23.32	24.70	6.90	4.50	2.50	4.81
		2	24.6	29.50	26.80	4.81	8.53	13.52	4.64

For the exact solution, we first try to fit the phase shifts for both s and d waves using the same potential but failed. Then we take recourse to an l-dependent potential and use a different set of parameters for each partial wave. With a set of initially chosen potential parameters, the numerical program calculates the square deviation of the phase shift at sixteen different energies from 0.5 Mev to 20 Mev. Iteration procedure then starts to minimize the square sum deviation by optimizing the parameters. The results are shown in Fig. 1. The parameters obtained in this way are listed in Table I. Here, the alpha excitation energy is taken to be 20 Mev.

Compare the results of exact calculation and the resonance approximation method, we see that there are only a little changes

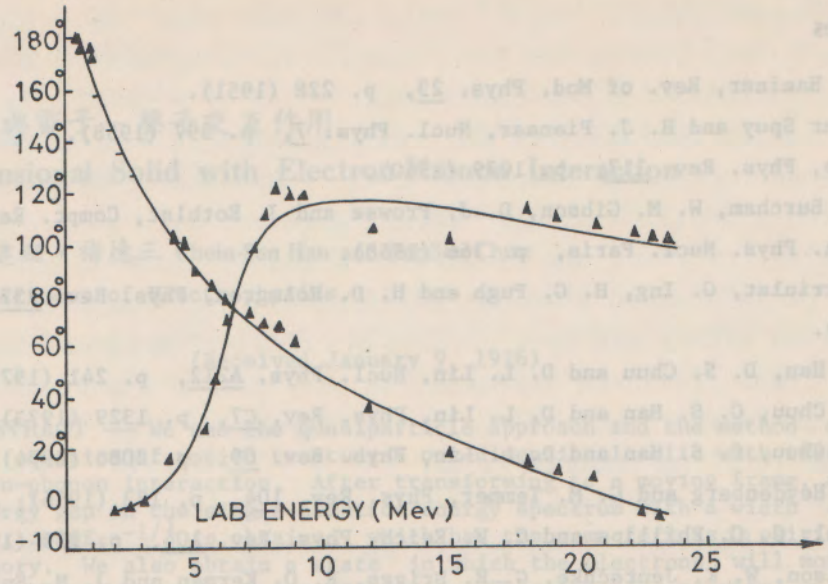

Fig. 1. Alpha-Alpha scattering phase shifts as a function of laboratory energy. The smooth curves are calculated from eq.(25). The solid triangles represent the experimental data [9].

in the diagonal matrix elements, but about 10% in V_5 and 20% in V_6 and V_7. This shows that the resonance method is indeed a good approximation. Furthermore, from the calculations of the three-body alpha model of C^{12}, we know that the contribution to the binding energy of C^{12} is mainly due to the diagonal matrix elements and the off diagonal elements give only about 15%. Therefore, the exact solution will change the three-body calculation less than 10%.

The alpha-alpha interaction that we have constructed from the experimental scattering data shows the same feature obtained by some other authors, namely, the interaction is l-dependent but is independent of the incident energy. It is observed that with the consideration of the internal structure of alpha particles, the data can be fitted without the repulsive core. This is in contrast to the single-channel interaction in which a repulsive term is required. Therefore, although the internal structure of alpha particle gives only a weak coupling, yet it is significant in the alpha-alpha interaction used in the nuclear structure problems such as the alpha particle model of the light nuclei.

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