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Extensions of Operators and Vector Measures

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ABSTRACT — Let X be an $\mathcal{L}^\infty$-space. Then every weakly compact operator on X has a weakly compact extension to any Banach space containing X. As an application, every strongly additive map on a σ-algebra Σ has a strongly additive extension to any σ-algebra containing Σ.

Let X be an $\mathcal{L}^\infty$-space in the sense of Lindenstrauss and Pełczynski [3]. An interesting linear extension property of operators on X is given in Theorem 1. In the light of the relationship between vector-measures and operators on specific $\mathcal{L}^\infty$-spaces, which was exploited by Diestel in [1], we derive some applications of Theorem 1 to extensions of vector measures. To be precise, Theorem 2 asserts that every strongly additive map on a σ-algebra has a strongly additive extension to any σ-algebra containing Σ. An analogus extension theorem for bounded additive map is stated in Theorem 3.

THEOREM 1. Let X be an $\mathcal{L}^\infty$-space. Then for every Banach space Z containing X, every Banach space Y, and every weakly compact operator $T:X\to Y$, there exists a weakly compact extension $\tilde{T}: Z\to Y$.

Proof. Let Y be a Banach space and $T:X\to Y$ a weakly compact operator. Observe that the second adjoint operator $T^{**}:X^{**}\to Y^{**}$ is a linear extension of T. Now since T is weakly compact, T^{**} is weakly compact and $T^{**}(X^{**})\subset Y$.

Let Z be a Banach space containing X, then X^{**} can be regarded as a subspace of Z^{**}. Moreover, X is an $\mathcal{L}^\infty$-space, hence X^{**} is an injective space [3, p.291] and therefore, X^{**} is complemented in Z^{**}. Let P be a projection of Z^{**} onto X^{**}. It follows that T^{**} can be extended to a bounded linear operator $\hat{T}$ on Z^{**} with range in Y. Explicitly, $\hat{T}=T^{**}P$. The restriction $\tilde{T}$ of $\hat{T}$ to Z is then the desired bounded linear extension of T.

Remark. A slight modification of the proof shows that if X, Y are Banach spaces such that X^{**} is injective and Y is complemented in Y^{**} (e.g., any L-space or any dual space), then a bounded linear operator from X to Y can always be extended to each space containing X.

Let Ω be a set, Σ a σ-algebra of subsets of Ω, and $B(\Omega,\Sigma)$ the Banach space of bounded, scalar-valued, Σ-measurable functions defined on Ω; it is shown in [2] that $B(\Omega,\Sigma)$ is isometrically isomorphic to a $C(S)$ space with S a σ-Stonian space, i.e., the closure of open F_σ-subsets of S are open. Let $\mu:\Sigma\to Y$ be a bounded additive set function; then μ defineds via integration, a bounded linear operator $\int d\mu: B(\Omega,\Sigma)\to Y$. Conversely, every bounded linear operator on $B(\Omega,\Sigma)$ to Y is uniquely represented as an integral with respect to some bounded additive map μ.

We say that μ is strongly additive if, whenever given a sequence of pairwise disjoint sets $A_n\in\Sigma$, we have $\mu(A_n)\to 0$ as $n\to\infty$. We mention a result of Diestel [1]: If $\int d\mu : B(\Omega,\Sigma)\to Y$ is weakly compact then μ is strongly additive. Conversely, if μ is strongly additive then $\int d\mu$ is weakly compact. This together with a result of Rosenthal [4, p.32] implies that if Y contains no subspace isomorphic to ℓ^∞ then μ is strongly additive.

THEOREM 2. Let Σ_o, Σ be σ-algebras of subsets of a set Ω with $\Sigma_o\subseteq\Sigma$. Let $\mu_o:\Sigma_o\to Y$ be strongly additive. Then μ_o can be extended to a strongly additive map $\mu:\Sigma\to Y$.

Proof. $\mu_o:\Sigma_o\to Y$ is assumed to be strongly additive, hence $\int d\mu_o: B(\Omega,\Sigma_o)\to Y$ is weakly compact. Note that $B(\Omega,\Sigma_o)$ is an $\mathcal{L}^\infty$-space since it is isometrically isomorphic to a $C(S)$ space. Also $\Sigma_o\subseteq\Sigma$, so that $B(\Omega,\Sigma_o)\subset B(\Omega,\Sigma)$. By Theorem 1, $\int d\mu_o$ can be extended to a weakly compact operator on $B(\Omega,\Sigma)$, which is described as an integration with respect to some bounded additive set function μ. μ is then strongly additive and clearly is an extension of μ_o.

On account of remark of Theorem 1, we have

THEOREM 3. Let Σ_o, Σ be σ-algebras of subsets of a set Ω with $\Sigma_o\subseteq\Sigma$. Assume that Y is complemented in Y^{**}. Then every bounded additive map $\mu_o: \Sigma_o\to Y$ can be extended to a bounded additive $\mu: \Sigma\to Y$.

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