The Journal of National Chiao Tung University
Vol. 2, December 1976
pp. 139-150

局部凸空間上的演化方程式

Nonlinear Evolution Equations in Locally Convex Spaces

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(Received August 7, 1976)

ABSTRACT — Let S be a locally convex topological vector space. The method of product integration is used to obtain solutions to the time dependent evolution equation $\mu'(t)=A(t)\mu(t)$, $t\geq 0$, where A is a function from $[0,\infty)$ to the set of nonlinear operators from S to itself, and μ is a function from $[0,\infty)$ to S.

I. Introduction

Let S be a sequentially complete Hausdorff locally convex topological vector space, we consider the existence problem for the time dependent nonlinear evolution equation:

$$\mu'(t)=A(t)\mu(t), \quad t\geq 0. \tag{1}$$

where A is a mapping from $[0,\infty)$ to the set of operators on S, μ is a continuous function from $[0,\infty)$ to S. When A(t) is linear, (1) has been developed by the following papers of T. Kato [2] , K. Yosida [6] and many others. More recently G. F. Webb [5] and Y. Komura [4] . M. G. Crandoll and A. Pazy [1] , and T. Kato [3] have considered the nonlinear form.

We consider (1) in the form $\mu(t)=F(t)A(t)\mu(t)$, where F is an absolutely continuous function from $[0,\infty)$ to the set of bounded linear operators on S, and the method of product integration is used to obtain the solutions.

II. Notations and Definitions

Throught this paper we let S be a sequentially complete Hausdorff locally convex topological vector space which possesses the property that there exists an open bounded neighborhood of O and let X be the family consisting of all convex, symmetric neighborhood containing 0. Furthermore, let V(p) be the intersection of all $V\varepsilon X$ such that $p\varepsilon V$. If F is a linear operator in S, and for any p, there exists a non-negative number k, such that $F(p)\varepsilon kV(p)$, then we say F is bounded. When F is a linear bounded operator in S, let $\|F\|=\inf\{k \mid F(p)\varepsilon kV(p),\ p\varepsilon S\}$. It is clear that for any $W\varepsilon X$, $p\varepsilon W$ implies $F(p)\varepsilon \|F\|W$.

Let B(S) be the set of all bounded linear operators on S, N(S) be the set of mappings (possibly nonlinear) from a subset of S to S, D(A) and R(A) be the domain and range of A, respectively.

Definition 1. Let F be a function from $[0,\infty)$ to B(S). F is said to be absolutely continuous provided that if $0\le\mu\le v$ and $c>0$, there exists $d>0$ such that if $\{(s_i,t_i)\}_{i=0}^{n}$ is a sequence of disjoint intervals in $[\mu,v]$ and $\Sigma_{i=0}^{n}(t_i-s_i)<d$, then $\Sigma_{i=0}^{n}\|F(t_i)-F(s_i)\|<c$. The greatest such number d is called the modulus of absolute continuity of F over $[\mu,v]$ with respect to c. Note that if F is an absolutely continuous function from $[0,\infty)$ to B(S), and $0\le\mu\le v$, then F is of bounded variation on $[\mu,v]$ i.e. there exists a number N such that if $\{s_i\}_{i=0}^{n}$ is a chain from μ to v, then $\Sigma_{i=1}^{n}\|F(s_i)-F(s_{i-1})\|\le N$. The least such number N is denoted by $\int_{\mu}^{v}\|dF\|$.

Definition 2. Let $A\varepsilon N(S)$ and let F be a function from $[0,\infty)$ to B(S). Define A to be m-dissipative with respect to F provided that for $0\le s<t$, $p,q\varepsilon D(A)$, the following hold:

for any $W\varepsilon X$ and $p-q\notin W$, $[I-(F(t)-F(s))A]p-[I-(F(t)-F(s))A]q\notin W$. (2)

$$R[I-(F(t)-F(s))A]=S. \tag{3}$$

Note that (2) and (3) imply that for $0\le s<t$, $[I-(F(t)-F(s))A]$ is one-to-one, so $[I-(F(t)-F(s))A]^{-1}$ is defined on S, and if p, $q\varepsilon S$ then for any $W\varepsilon X$, we have

$$\text{If } p-q\varepsilon W \text{ then } [I-(F(t)-F(s))A]^{-1}p-[I-(F(t)-F(s))A]^{-1}q\varepsilon W. \tag{4}$$

We call that $[I-(F(t)-F(s))A]^{-1}$ is non-expansive.

Definition 3. Let F be a function from $[0,\infty)$ to B(S) and let A be a function from $[0,\infty)$ to N(S) such that A(t) is m-dissipative with respect to F for $t\ge 0$. If $0\le\mu\le v$ and $s=\{s_i\}_{i=0}^{n}$ is a chain from μ to v, denote $[I-(F(s_i)-F(s_{i-1}))A(s_{i-1})]^{-1}$ by $[A,F,s_i]$ and denote $\pi_{i=1}^{n}[A,F,s_i]$ by $\pi_s[A,F]$. Suppose that $p\varepsilon S$, $0\le\mu\le v$ and z is a point of S such that if t is a refinement of s, then $z-\pi_t$

$[A,F]p\varepsilon W$, for any $W\varepsilon X$. Define z to be the product integral of A with respect to F for p from μ to v and denote z by $\pi_{\mu}^{v}[A,F]p$

Definition 4. Let $0\leq\mu\leq v$, A sequence of chains $\{s(1)_i\}_{i=0}^{r_1}$, $\{s(2)_i\}_{i=0}^{r_2}, \dots$ from μ to v is said to be admissible provided that if n is a positive integer then the following hold:

(i) $s(n+1)$ is a refinement of $s(n)$,

(ii) either $\max\{s(n)_i - s(n)_{i-1} \mid 1\leq i\leq r_n\} < \frac{1}{n}$ or if $s(n)_{j-1} > \frac{1}{n}$ for some integer j in $[1,r_n]$, then there is no point $s(m)_i$ of $\{s(m)_i\}_{i=0}^{r_m}$ such that $s(n)_{j-1} < s(m)_i < s(n)_j$ for every integer $m>n$.

Definition 5. Let A be a function from $[0,\infty)$ to $N(S)$ let F be a function from $[0,\infty)$ to $B(S)$, and let $E\subset S$ be such that (i) $A(t)$ is m-dissipative with respect to F for $t\geq 0$, (ii) $E\subset D(A(t))$ for $t\geq 0$, and (iii) $[I-(F(t)-F(s)A(s)]^{-1}(E)\subset E$ for $0\leq s<t$. We define A to be product stable with respect to F on E, if the following are true.

If $p\varepsilon E$, and $0\leq v$, then there exists an open bounded set in X, say $W(F,p,v)$, such that if $0\leq\mu\leq x\leq y\leq v$ and s is a chain from μ to y, then $A(x)\pi_s[A,F]p\varepsilon W(F,p,v)$. (5)

If $p\varepsilon E$, $0\leq v$ and any $W\varepsilon X$, there exists $d>0$ such that if $0<y\leq d$, s is a chain from v to $v+y$ and $v\leq x\leq v+y$, then $A(x)\pi_s[A,F]p-A(v)p\varepsilon W$. (6)

If $p\varepsilon E$, $0\leq\mu\leq v$ and $s(1), s(2), \dots$ is an admissible sequence of chains from μ to v such that $\lim_{n\to\infty}\pi_{s(n)}[A,F]p$ exists, then $\lim_{n\to\infty}\pi_{s(n)}[A,F]p\varepsilon E$. (7)

III. Existence Theorem

THEOREM 1. Let A be a function from $[0,\infty)$ to $N(S)$. Let F be an absolutely continuous function from $[0,\infty)$ to $B(S)$, and let E be a subset of S such that

1. $A(t)$ is m-dissipative with respect to F for $t\geq 0$;
2. $E\subset D(A(t))$ for $t\geq 0$;
3. E is invariant under $[I-(F(t)-F(s))A(s)]^{-1}$ for $0\leq s<t$;
4. A is product stable with respect to F on E.

If $p\varepsilon E$ and $0\leq\mu\leq v$, then $\pi_{\mu}^{v}[A,F]p$ exists and $\pi^{v}_{\mu}[A,F]p\varepsilon E$.

Theorem 1 is proved by means of a series of lemmas under the hypotheses of Theorem 1.

LEMMA 1. Let $p\varepsilon E$, $0\leq\mu\leq v$, $\{s_i\}_{i=0}^{n}$ be a chain from μ to v and $W(F,p,v)$ be as in (5). If $1\leq j\leq n$ and $z=\pi_{i=1}^{j-1}[A,F,s_i]p$ then

$$\pi_{i=j}^{n}[A,F,s_i]z-z=\Sigma_{i=j}^{n}[F(s_i)-F(s_{i-1})]A(s_{i-1})\pi_{k=j}^{i}[A,F,s_k]z \quad (8)$$

and
$$\pi_{i=j}^{n}[A,F,s_i]z-z\varepsilon\Sigma_{i=j}^{n}\|F(s_i)-F(s_{i-1})\|W(F,p,v). \quad (9)$$

Proof: Since $\pi_{i=j}^{n}[A,F,s_i]z-z=\Sigma_{i=j}^{n}\{\pi_{k=j}^{i}[A,F,s_k]z-\pi_{k=j}^{i-1}[A,F,s_k]z\}$

$$=\Sigma_{i=j}^{n}[F(s_i)-F(s_{i-1})]A(s_{i-1})\pi_{k=j}^{i}[A,F,s_k]z$$

and (9) follows from (8) and (5).

For $\mu\geq 0$, let K be a subset of $[0,\infty)$ such that $\mu\varepsilon K$, K is bounded, every nonempty subset of K has a smallest number in K, and if $\{x_i\}_{i=1}^{\infty}$ is an infinite increasing sequence in K, then $\lim_{i\to\infty}\{x_i\}\notin K$. Let Q_μ be the family of all such K's.

If $K\varepsilon Q_\mu$ and $x>\mu$, let $K_x=\{y\varepsilon K|\, y<x\}$ and let $K'=\{x|\, x$ is the limit of an increasing sequence in K$\}$.

LEMMA 2. Let $\mu\geq 0$, $K\varepsilon Q_\mu$ and let $v=\text{Sup } K$. There exists a sequence of chains $\{s(1)\}_{i=0}^{m_1}, \{s(2)\}_{i=0}^{m_2},\ldots,$ from μ to v such that for each positive integer n:

s(n+1) is a refinement of s(n). (10)

if $1\leq i\leq m_n$, then $s(n)_i\varepsilon K\cup K'$; if $s(n)_i\varepsilon K$, then $s(n)_i$ is the point of K which follows immediately $s(n)_{i-1}$; if $s(n)_i\varepsilon K'$, then $s(n)_{i-1}\varepsilon K$ and $\Sigma_{i,s(n)_i\varepsilon K'}|s(n)_i-s(n)_{i-1}|$ $<\min\{b_n,\frac{1}{n}\}$ where b_n is the modulus of absolute continuity of F on $[\mu,v]$ with respect to $\frac{1}{2^n}$. (11)

Moreover, if $p\varepsilon E$, there exists a unique point z of E such that if s(1), s(2),... is any sequence of chains from μ to v satisfying (10), (11) then $\pi_{s(1)}[A,F]p$, $\pi_{s(2)}[A,F]p,\ldots$ converges to z.

Proof: By the fact that K' is closed and the covering theorem, (10) and (11) can be proved. Now suppose $\{s(1)\}_{i=0}^{m_1}$, $\{s(2)\}_{i=0}^{m_2}$,... is any sequence of chains from μ to v satisfying (10), (11). Let $p\varepsilon E$, n be a positive integer and let $\{w_i\}_{i=0}^{m_n}$ be an increasing sequence such that $w_o=0$, $w_{m_n}=m_{n+1}$, and if i is an integer in $[1,m_n]$, then $s(n)_i=s(n+1)w_i$. If i is an integer in $[1,m_n]$, let $K_i=\pi_{j=w_{i-1}+1}^{w_i}[A,F,s(n+1)_j]$, $J_i=\pi_{j=1}^{i}[A,F,s(n)_j]$. Noting that if $s(n)_i\varepsilon K$, then $K_iJ_{i-1}=J_i$. We see that

$$\pi_{s(n+1)}[A,F]p-\pi_{s(n)}[A,F]p=\Sigma_{i=1}^{m_n}[\pi_{j=i}^{m_n}K_jJ_{i-1}p-\pi_{j=i+1}^{m_n}K_jJ_ip] \quad (12)$$

$$K_iJ_{i-1}p-J_ip\varepsilon W \text{ implies } \pi_{j=i}^{m_n}K_jJ_{i-1}p-\pi_{j=i+1}^{m_n}K_jJ_ip\varepsilon W. \text{ (By (4)) } (13)$$

Since if $s(n)_i\varepsilon K$ then $K_iJ_{i-1}p=J_ip$, we only consider that $s(n)_i\varepsilon K'$. By lemma 1, $K_iJ_{i-1}p-J_{i-1}p\varepsilon\Sigma_{j=w_{i-1}+1}^{w_i}\|F(s(n+1)_j)-F(s(n+1)_{j-1})\|W(F,p,v)$, and $[A,F,s(n)_i]J_{i-1}p-J_{i-1}p\varepsilon\|F(s(n)_i)-F(s(n)_{i-1})\|W(F,p,v)$, thus by (12) and (13), we get $\pi_{s(n+1)}[A,F]p-\pi_{s(n)}[A,F]p\varepsilon\frac{1}{2^{n-1}}W(F,p,v)$, i.e. $\pi_{s(1)}[A,F]p$, $\pi_{s(2)}[A,F]p$, ..., is a Cauchy sequence and so let z be its limit. We observe that s(1), s(2),...is an admissible sequence of chains from μ to v, so $z\varepsilon E$ by (7). Suppose that each of s(1), s(2),... and t(1), t(2),... is a sequence of chains from μ to v satisfying (10),(11), then there exists a sequence $s(n_1)$, $t(n_2)$, $s(n_3)$, $t(n_4)$, ... satisfying (10), (11), so that $\pi_{s(1)}[A,F]p$, $\pi_{s(2)}[A,F]p$,... and $\pi_{t(1)}[A,F]p$, $\pi_{t(2)}[A,F]p$... must converge to the unique z.

We need the following notations. Let the unique z of lemma 2 be denoted by z(K,p). Suppose that $p\varepsilon E$, $\mu\geq 0$, $K\varepsilon Q_\mu$, v=Sup K, and for any $W\varepsilon X$, by (6) there exists d>0 such that if $0<y\leq d$, s is a chain from v to v+y, and $y\leq x\leq v+y$, then $A(x)\pi_s[A,F]z(K,p)-A(v)z(K,p)\varepsilon W$. Let such a number d be denoted by d(K,p,W). If $p\varepsilon E$, $\mu\geq 0$, $W\varepsilon X$, define

$$A(p,\mu,W)=\{K\varepsilon Q_\mu|\text{if } q\neq\mu,\ q\varepsilon K \text{ then } q=\text{Sup } K_q+d(K_q,p,W)\} \quad (14)$$

We remark that if $K\varepsilon A(p,\mu,W)$ and $q\neq\mu$, $q\varepsilon K$, then $K_q\varepsilon A(p,\mu,W)$. Furthermore, if K and J are two members of $A(p,\mu,W)$, then there exists $q\varepsilon K$ such that $K_q=J$ or there exists $q\varepsilon J$ such that $J_q=K$.

LEMMA 3: If $p\in E$, $0\leq\mu\leq v$, $W\in X$, then there exists $K\in A(p,\mu,W)$ Such that $v\leq \text{Sup } K$.

Proof: Assume there exists no $K\in A(p,\mu,W)$ such that $v\leq\text{Sup } K$. Let $J=\cup_{K\in A(p,\mu,W)} K$. Then J is bounded and $J\in A(p,\mu,W)$. But $J\cup\{\text{Sup } J+d(J,p,W)\}$ is in $A(p,\mu,W)$, implies $[\text{Sup } J+d(J,p,W)]\in J$. It is a contradiction.

LEMMA 4. Suppose $p\in E$, $\mu\geq 0$ and $K\in Q_\mu$. We have the following:

(i) If $q\in K$, $q\neq\mu$, $v=\text{Sup } K_q$, then $[I-(F(q)-F(v))A(v)]^{-1}z(K_q,p)=z(K_q\cup\{q\},p)$.

(ii) If $x\in K$, $x\neq\mu$, $y\in K'$, $x<y$, $w=z(K_x\cup\{x\},p)$, and let $J=K_y-K_x$, then $z(K_y,p)=z(J,w)$.

Proof: (i) Suppose $s(1), s(2),\ldots$ is a sequence of chain from μ to v satisfying (10), (11) of Lemma 2 for K_q. Then $s(1)\cup\{q\}, s(2)\cup\{q\},\ldots$ is a sequence from μ to q satisfying (10), (11) for $K_q\cup\{q\}$. Thus $\pi_{s(1)\cup\{q\}}[A,F]p,\ldots$ converges to $z(K_q\cup\{q\}, p)$ and $\pi_{s(1)}[A,F]p,\ldots$ converges to $z(K_q,p)$. Since $[I-(F(q)-F(v))A(v)]^{-1}\pi_{s(n)}[A,F]p=\pi_{s(n)\cup\{q\}}[A,F]p$, by (4), we have (i).

(ii) Suppose $s(1), s(2),\ldots$ is a sequence of chains from μ to x satisfying (10), (11) for $K_x\cup\{x\}$. Let $t(1), t(2),\ldots$ be a sequence of chains from x to y satisfying (10), (11) for J. Define $r(n)=s(n)\cup t(n)$ for each positive integer n. Then a subsequence $r(n_1), r(n_2),\ldots$ satisfying (10), (11) for K_y. Thus $\pi_{r(n_1)}[A,F]p,\ldots,\pi_{s(n_1)}[A,F]p,\ldots$ and $\pi_{t(n_1)}[A,F]\ldots$ converge to $z(K_y,p)$, w and $z(J,w)$ respectively. Hence $z(K_y,p)=z(J,w)$.

LEMMA 5. Suppose $p\in E$, $0\leq\mu\leq v$, and $W\in X$. There exists a chain s from μ to v such that if t is a refinement of s, then $\pi_s[A,F]p-\pi_t[A,F]p\in W$.

Proof: By Lemma 3, one of the following two cases must hold:

Case 1: There exists $K\in A(p,\mu,W)$ such that $v=\text{Sup } K$

Case 2: There exists $K\in A(p,\mu,W)$ such that $v<\text{Sup } K$ and there doesn't exist $q\in K$ such that $v<q<\text{Sup } K$.

First, we consider case 1. Let $s=\{s_i\}_{i=0}^m$ be a chain from μ to v such that if i is an integer in [1,m], then s has the properties: (i') if $s_i\in K$ then s_i is the first point of K which follows s_{i-1}, (ii') if $s_i\notin K$, then $s_i\in K'$ and $s_{i-1}\in K$ and (iii') $\Sigma_{i,s_i\in K'}(s_i-s_{i-1})<b$, where b is the modulus of absolute con-

tinuity of F on $[\mu, v]$ with respect to c.

Define $\{q_i\}_{i=0}^{m}$ as following: $q_o=p$; if $s_i \varepsilon K$, $q_i=z(K_{s_i} U\{s_i\},p)$; if $s_i \varepsilon K'$, $q_i=z(K_{s_i},p)$. We observe that if $s_i \varepsilon K$, $i \neq 0$, by Lemma 4 (i) and (14), we have:

$$q_i=z(K_{s_i} U\{s_i\},p)$$

$$=\{I-[F(\mathrm{Sup}K_{s_i}+d(K_{s_i},p,W)-F(\mathrm{Sup}K_{s_i})]A(\mathrm{Sup}K_{s_i})\}^{-1}z(K_{s_i},p)$$

$$=[I-(F(s_i)-F(s_{i-1}))A(s_{i-1})]^{-1}q_{i-1}. \tag{15}$$

Furthermore, if $s_i \varepsilon K$, $i \neq 0$, r is a chain from s_{i-1} to y where $s_{i-1} \leq y \leq s_i$ and $x \varepsilon [s_{i-1}, y]$, then by (14) and (6) we have:

$$A(x)\pi_r[A,F]q_{i-1}-A(s_{i-1})q_{i-1} \varepsilon W. \tag{16}$$

If $s_i \varepsilon K'$, we have, by Lemma 4 (ii)

$$q_i=z(K_{s_i},p)=z(K_{s_i}-K_{s_{i-1}},q_{i-1}). \tag{17}$$

And bv Lemma 1 together with (iii') above, we get

$$\Sigma_{i,s_i \varepsilon K'}\{\pi_{r(i)}[A,F]q_{i-1}-q_{i-1}\} \varepsilon cW(F,p,v), \text{ where } r(i)$$

$$\text{is a chain from } s_{i-1} \text{ to } s_i, \text{ and} \tag{18}$$

$$\Sigma_{i,s_i \varepsilon K'}(q_i-q_{i-1})=\Sigma_{i,s_i \varepsilon K'}\{z(K_{s_i}-K_{s_{i-1}},q_{i-1})-q_{i-1}\} \varepsilon cW(F,p,v). \tag{19}$$

Suppose now that $t=\{t_j\}_{j=0}^{n}$ is a refinement of $s=\{s_i\}_{i=0}^{n}$. Then there exists an increasing sequence $\{w_i\}_{i=0}^{m}$ such that $w_o=0$, $w_m=n$ and if $1 \leq i \leq m$, $s_i=t_{w_i}$. Let $K_i=\pi_{j=w_{i-1}+1}^{w_i}[A,F,t_j]$ for $1 \leq i \leq m$. Then by (15), (16) and (8), we have

$$\Sigma_{i,s_i \varepsilon K}(K_i q_{i-1}-q_i)=\Sigma_{i,s_i \varepsilon K}(K_i q_{i-1}-[A,F,s_i]q_{i-1}) \tag{20}$$

By (4) we have that

$$[I-(F(s_i)-F(s_{i-1}))A(s_{i-1})]K_i q_{i-1}-q_{i-1} \varepsilon W \text{ implies } K_i q_{i-1}$$
$$-[A,F,s_i]q_{i-1} \varepsilon W$$

Since $[I-(F(s_i)-F(s_{i-1}))A(s_{i-1})]K_i q_{i-1}-q_{i-1}$

$$=K_i q_{i-1}-q_{i-1}-\{(F(s_i)-F(s_{i-1}))A(s_{i-1})\}K_i q_{i-1} \quad (21)$$

$$=\Sigma_{j=w_{i-1}+1}^{w_i}(F(t_j)-F(t_{j-1}))A(t_{j-1})\pi_{k=w_{i-1}+1}^{j}[A,F,t_k]q_{i-1}$$

$$-\{(F(s_i)-F(s_{i-1}))A(s_{i-1})\}K_i q_{i-1}$$

$$=\Sigma_{j=w_{i-1}+1}^{w_i}(F(t_j)-F(t_{j-1}))\{A(t_{j-1})\pi_{k=w_{i-1}+1}^{j}[A,F,t_k]q_{i-1}$$

$$-A(s_{i-1})q_{i-1}+A(s_{i-1})q_{i-1}-A(s_{i-1})\pi_{k=w_{i-1}+1}^{w_i}[A,F,t_k]q_{i-1}\}$$

$$\varepsilon 2\Sigma_{j=w_{i-1}+1}^{w_i}\|F(t_j)-F(t_{j-1})\|W=2(\int_{s_{i-1}}^{s_i}\|dF\|)W. \quad \text{(By (16))} \quad (22)$$

Hence, by (20), (21), (22), we have:

$$\Sigma_{i,s_i\varepsilon K}(K_i q_{i-1}-q_i)\varepsilon 2(\int_\mu^v\|dF\|)W. \quad (23)$$

Moreover, using (17), (18), (19), we have

$$\Sigma_{i,s_i\varepsilon K'}(K_i q_{i-1}-q_i)=\Sigma_{i,s_i\varepsilon K'}\{(K_i q_{i-1}-q_{i-1})$$

$$+(q_{i-1}-z(K_{s_i}-K_{s_{i-1}},q_{i-1})\}\varepsilon 2cW(F,p,v) \quad (24)$$

By (23), (24), (4) and the identity: $\Sigma_{i=1}^m(K_i q_{i-1}-q_i)=\Sigma_{i,s_i\varepsilon K}(K_i q_{i-1}-q_i)$ $+\Sigma_{i,s_i\varepsilon K'}(k_i q_{i-1}-q_i)$ we have:

$$\pi_t[A,F]p-q_m=\Sigma_{i=1}^m[\pi_{j=i}^m K_j q_{i-1}-\pi_{j=i+1}^m K_j q_i]\varepsilon 2(\int_\mu^v\|dF\|)W$$

$$+2cW(F,p,v). \quad (25)$$

Since W and c are arbitrary, we have the lemma for case 1.

Suppose case 2 holds and case 1 doesn't hold. Then Sup $K\varepsilon K$, $K_{\text{Sup }K}\varepsilon A(p$ $\mu,W)$ and $\text{Sup}(K_{\text{Sup }K})<v$. Let $q=z(K_{\text{Sup }K},p)$ and $w=\text{Sup}(K_{\text{Sup }K})$, then $w<v$. By (14), we have Sup $K=w+d(K_{\text{Sup }K},p,W)$. Furthermore, let $r=[I-(F(v)-F(w))A(w)]^{-1}q$. By (25) there exists a chain $\{s_i\}_{i=0}^m$ from μ to v such that if t is a refinement of s, then

$$\pi_t[A,F]p-q\varepsilon 2\{\int_\mu^w\|dF\|W+cW(F,p,v)\} \quad (26)$$

Define $s'=\{s'_i\}_{i=0}^{m+1}$ by $s'_i=s_i$ if $0\le i\le m$, $s'_{m+1}=v$. Let $\{t_i\}_{i=0}^{n}$ be a refinement of s', and let $t_k=w$. We have

$$\pi_t[A,F]p-r=\pi_{i=k+1}^{n}[A,F,t_i]\pi_{j=1}^{k}[A,F,t_j]p-\pi_{i=k+1}^{n}[A,F,t_i]q$$

$$+\pi_{i=k+1}^{n}[A,F,t_i]q-[I-(F(v)-F(w))A(w)]^{-1}q, \qquad (27)$$

$$[I-(F(v)-F(w))A(w)]\pi_{i=k+1}^{n}[A,F,t_i]q-q$$

$$=\pi_{i=k+1}^{n}[A,F,t_i]q-q-(F(v)-F(w))A(w)\pi_{i=k+1}^{n}[A,F,t_i]q$$

$$=\Sigma_{i=k+1}^{n}(F(t_i)-F(t_{i-1}))A(t_{i-1})\pi_{j=k+1}^{i}[A,F,t_j]q-A(w)q+A(w)q$$

$$-(F(v)-F(w))A(w)\pi_{j=k+1}^{n}[A,F,t_j]q$$

$$=\Sigma_{i=k+1}^{n}(F(t_i)-F(t_{i-1}))\{[A(t_{i-1})\pi_{j=k+1}^{i}[A,F,t_j]q-A(w)q]$$

$$+[A(w)q-A(w)\pi_{j=k+1}^{n}[A,F,t_j]q\}\varepsilon 2(\textstyle\int_w^v\|dF\|)W. \qquad (28)$$

By (4), (26), (27) and (28), we get $\pi_t[A,F]p-r\varepsilon 2[\int_\mu^w\|dF\|W+cW(F,p,w)+\int_w^v\|dF\|W]$. Thus, we have the lemma established for case 2.

By Lemma 5 and the fact that S is sequentially complete, we see that if $p\varepsilon E$, $0\le\mu\le v$, the product integral $\pi_\mu^v[A,F]p$ of A with respect to F for p from μ to v, exists. It follows from (7) that $\pi_\mu^v[A,F]p\varepsilon E$. Hence theorem 1 is established.

By virtue of theorem 1, we get the following:

Definition 6. Let A be a function from $[0,\infty)$ to N(S), let F be an absolutely continuous function from $[0,\infty)$ to B(S), and E be a subset of S such that 1, 2, 3 and 4 of theorem hold. If $0\le\mu\le v$, define the mapping $U(v,\mu)$ from E to E by $U(v,\mu)p=\pi_\mu^v[A,F]p$. For each $p\varepsilon E$, we say U is the evolution operator of A with respect to F on E.

It is easy to show that U has the following properties:

(i) $U(\mu,\mu)p=p$, for $p\varepsilon E$, $\mu\ge 0$.

(ii) if $p,q\varepsilon E$, $0\le\mu\le v$, and for any $W\varepsilon X$, $p-q\varepsilon W$ imply $U(v,\mu)p-U(v,\mu)q\varepsilon W$.

(iii) $U(v,w)U(w,\mu)p=U(v,\mu)p$ for $p\varepsilon E$, $0\le\mu\le w\le v$.

(iv) $U(v,\mu)p-p\varepsilon\int_\mu^v\|dF\|W(F,p,v)$ for $p\varepsilon E$, $0\le\mu\le v$.

If we have $E=S$ and $\mu=0$, let $T(x)p=U(x,0)p$. Then T is a mapping from $[0,\infty) \times S$ into S, and $\{T(x)\,|\,x\geq 0\}$ is a semi-group generated by A. A is called the infinitesimal generator of $\{T(x)\,|\,x\geq 0\}$ with respect to F.

We will use the facts, as in E. Hille and R. S. Phillips [7], that F has a strong derivative F' almost everywhere and $\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\|F'\|=\|F'(t)\|$ for almost all $t\geq 0$.

THEOREM 2. Let A be a function from $[0,\infty)$ to $N(S)$, let F be an absolutely continuous function from $[0,\infty)$ to $B(S)$, and let $E\subset S$ be such that 1, 2, 3 and 4 of theorem 1 are satisfied, let U be the evolution operator of A with respect to F on E, and let F have strong derivative almost everywhere on $[0,\infty)$. If $p\in E$ and $0\leq\mu$, then for almost all $t\geq\mu$, $\frac{d}{dt}U(t,\mu)p=F'(t)A(t)U(t,\mu)p$.

Proof: Let $p\in E$, $0\leq\mu$, $W\in X$, $t\geq\mu$ be such that $F'(t)$ exists and $\lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\|F'\|=\|F'(t)\|$. There exists $d>0$ such that if $0<h<d$, then

$$\frac{1}{h}\int_t^{t+h}\|F'\|<\|F'(t)\|+1 \tag{29}$$

and if $0<h<d$, s is a chain from t to $t+h$, $t\leq x\leq t+h$, by (6), then

$$A(x)\pi_s[A,F]U(t,\mu)p-A(t)U(t,\mu)p\in W. \tag{30}$$

Let $0<h<d$. There is a chain $r=\{r_i\}_{i=0}^n$ from t to $t+h$ such that

$$\pi_r[A,F]U(t,\mu)p-U(t+h,h)U(t,\mu)p\in hW. \tag{31}$$

Then
$$\frac{1}{h}\{U(t+h,\mu)p-U(t,\mu)p\}-F'(t)A(t)U(t,\mu)p$$
$$=\frac{1}{h}\{U(t+h,t)U(t,\mu)p-\pi_r[A,F]U(t,\mu)p\}+\frac{1}{h}\{\pi_r[A,F]U(t,\mu)p-U(t,\mu)p$$
$$-[F(t+h)-F(t)]A(t)U(t,\mu)p\}+\frac{1}{h}[F(t+h)-F(t)]A(t)U(t,\mu)p$$
$$-F'(t)A(t)U(t,\mu)p$$
$$=\frac{1}{h}\{U(t+h,t)U(t,\mu)p-\pi_r[A,F]U(t,\mu)p\}$$
$$+\frac{1}{h}\{\Sigma_{i=1}^n[F(r_i)-F(r_{i-1})][A(r_{i-1})\pi_{j=1}^i[A,F,r_j]U(t,\mu)p-A(t)U(t,\mu)p]\}$$
$$+\{\frac{1}{h}[F(t+h)-F(t)]-F'(t)\}A(t)U(t,\mu)p.$$

Let $A(t)U(t,\mu)p\in W'$. Since there exists $h>0$, such that $\|\frac{1}{h}(F(t+h)-F(t)-$

$F'(t)\|W'\subset W$, by (29) and $\frac{1}{h}\Sigma_{i=1}^{n}\int_{r_{i-1}}^{r_i}\|F'\|=\frac{1}{h}\int_t^{t+h}\|F'\|$ and (30), (31), we have

$$\frac{1}{h}[U(t+h,\mu)p-U(t,\mu)p]-F'(t)A(t)U(t,\mu)p\varepsilon W+(\|F'(t)\|+1)W+W$$

Thus $\frac{d}{dt}U(t,\mu)p=F'(t)A(t)U(t,\mu)p.$

Now we consider an example. Let A be a function from $[0,\infty)$ to N(S) such that

$$D(A(t))=S \qquad \text{for } t\geq 0 \qquad (32)$$

A is continuous as a function from $[0,\infty)xS$ to S, and A is bounded on bounded subset of $[0,\infty)xS$. (33)

A is m-dissipative with respect to $F(t)=tI\varepsilon B(S)$ for $t\geq 0$. (34)

Let E=S and we see that the conditions 1, 2 and 3 are satisfied. Now to show 4, we let $p\varepsilon S$, $0\leq\mu\leq x\leq y\leq v$, $\{s_i\}_{i=0}^{m}$ be a chain from μ to y. For each i-1, let $A(s_{i-1})p\varepsilon W_{i-1}\varepsilon X$. Then by (4) we have:

$$\pi_{i=1}^{m}[A,F,s_i]p-p=\Sigma_{i=1}^{m}\{\pi_{j=i}^{m}[A,F,s_j]p-\pi_{j=i+1}^{m}[A,F,s_j]p\}$$

$$\varepsilon\Sigma_{i=1}^{m}(s_i-s_{i-1})W_{i-1}$$

Thus $\pi_s[A,F]p-p\varepsilon\Sigma_{i=1}^{m}(s_i-s_{i-1})W_{i-1}$. Hence $\pi_s[A,F]p$ is bounded. By (33) $A(x)\pi_s[A,F]p$ is bounded, i.e. (5) holds.

Let $p\varepsilon S$, $0\leq\mu\leq v$, $\{t_i\}_{i=0}^{n}$ be a chain from v to v+y' for some y'>0. If $A(t_{i-1})p\varepsilon W'_{i-1}$, let $W'=U_{i=1}^{n}\{W'_{i-1}\}$. Then $\pi_t[A,F]p-p\varepsilon y'W'$. Thus given any $W\varepsilon X$, by (33), there exists $V\varepsilon X$ such that if $\pi_t[A,F]p-p\varepsilon V$ and $v\leq x\leq v+y'$, then $A(x)\pi_t[A,F]p-A(v)p\varepsilon W$, i.e. for sufficient small y' such that $y'W'\subset V$, we get (6). (7) is clear for S is sequentially complete. Therefore 4 is satisfied. Hence A is product stable with respect to F=tI. If we apply theorems 1, 2 then the evolution operator U of A with respect to F exists on S for $t\geq\mu$, and $\frac{d}{dt}U(t,\mu)p=A(t)U(t,\mu)p$ for $p\varepsilon S$.

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