The Journal of National Chiao Tung University
Vol. 2, December 1976
pp. 157-170

廣義高氏測度之垂直與對等

A Dichotomy Theorem for Generalized Gaussian Measures

章金昌 Kim-Chheng Ouy

Department of Applied Mathematics, N.C.T.U.

(Received August 16, 1976)

ABSTRACT — Let $\{X_t|t\in T\}$ be a stochastic process defined on a measurable space (Ω,F) Let P and Q be two measures induced by $\{X_t|t\in T\}$. It is known [3] that if both P and Q are Gaussian, then P and Q are either perpendicular or equivalent. It will be shown that under certain conditions, such a dichotomy can be extended to the case where P and Q are generalized Gaussian.

I. Generalized Gaussian Stochastic Processes and Generalized Gaussian Measures

A random variable X is a generalized Gaussian random variable if and only if there exists a nonnegative real number α such that for each real number t,

$$E(e^{tX}) \le e^{\alpha^2 t^2/2} \tag{1}$$

The minimum of those α's satisfying (1) will be denoted by $\tau(X)$.

It follows from the definition that if X is a generalized Gaussian random variable, so is aX for all real number a.

If $X_1, X_2, \ldots, X_n$ are generalized Gaussian random variables, then by the Cauchy-Bunyakovsky-Schwartz (C.B.S.) inequality, $X=X_1+\ldots+X_n$ is a generalized Gaussian random variable with

$$E(e^{tX}) \le \exp[2^{n-1}(\alpha_1^2+\alpha_2^2+\ldots+\alpha_n^2)t^2/2] \tag{2}$$

where the α_i's satisfy (1).

If X is a generalized Gaussian random variable satisfying (1), then for each $\varepsilon>0$ [1]

$$P(|X|>\varepsilon) \le 2\exp(-\varepsilon^2/2\alpha^2) \tag{3}$$

Let (Ω,F) be a measurable space, T a closed interval of real numbers. Let $\{X(t),\ t\in T\}$ be a stochastic process defined on (Ω,F). For each finite subset $\{t_1, t_2, \ldots, t_n\}$ of T, an n-dimensional probability distribution

$$F_n[x_1,x_2,\ldots,x_n;\ t_1,t_2,\ldots,t_n]=P_n[X(t_1)\le x_1, X(t_2)\le x_2,\ldots,$$

$X(t_n) \le X_n]$

may be defined arbitrarily. If these finite dimensional distribution are subject to the following consistency conditions:

a. Symmetry: for every permutation $(j_1, j_2, \ldots, j_n)$ of $(1, 2, \ldots, n)$, we have

$$F_n[x_{j_1}, x_{j_2}, \ldots, x_{j_n};\ t_{j_1}, t_{j_2}, \ldots, t_{j_n}]$$

$$= F_n[x_1, x_2, \ldots, x_n;\ t_1, t_2, \ldots, t_n]$$

b. Compatibility: for $m<n$, we have

$$F_n[x_1, x_2, \ldots, x_m, \infty, \ldots, \infty;\ t_1, t_2, \ldots, t_n]$$

$$= F_m[x_1, x_2, \ldots, x_m;\ t_1, \ldots, t_m],$$

then we may extend the P_n's to a unique probability measure P defined on (Ω, F) such that

$$P[X(t_1) \le x_1, \ldots, X(t_n) \le x_n] = P_n[X(t_1) \le x_1, \ldots, X(t_n) \le x_n]$$

for any subset $\{t_1, t_2, \ldots, t_n\}$ of T.

Thus given any stochastic process, we may define this process in terms of its finite dimensional distributions, subject, of course, to the two consistency conditions mentioned above. For example, a Gaussian process is defined in this manner. Since there is a one-to-one correspondence between a subclass of probability distribution functions and moment generating functions (real Laplace Transforms), some stochastic processes can also be defined in terms of moment generating functions.

Definition 1. An n-dimensional random vector is an n-dimensional generalized Gaussian random vector if given any non-zero vector $(a_1, a_2, \ldots, a_n)$ of real numbers, $a_1X_1 + a_2X_2 + \ldots + a_nX_n$ is a generalized Gaussian random variable.

Definition 2. A stochastic process $\{X(t), t \in T\}$ is a generalized Gaussian process if and only if each finite subfamily

$[X(t_1), X(t_2), \ldots, X(t_n)]$ of $\{X(t), t \in T\}$ is generalized Gaussian random vector.

It is clear from (2) that $\{X(t), t \in T\}$ is a generalized Gaussian process if and only if each X(t) is a generalized Gaussian random variable. Hence

Definition 2'. A stochastic process is generalized Gaussian if and only if each X(t) is a generalized Gaussian random variable.

Examples:

a) Every Gaussian process with zero mean is generalized Gaussian. b) If

$\{X(t), t\varepsilon T\}$ is a stochastic process such that $E[X(t)]=0$ and $|X(t)| \leq M(t) < \infty$ for each $t\varepsilon T$, then $\{X(t), t\varepsilon T\}$ is generalized Gaussian.

Definition 3. The measure extending the finite dimensional distributions of a generalized Gaussian process is called a generalized Gaussian measure.

By what preceeds, any stochastic process defined on a measurable space (Ω,F) may induce a generalized Gaussian measure on (Ω,F), if each X_t is generalized Gaussian.

II. Continuity in Probability and Separability of Stochastic Processes

Let T be a closed interval and $\{X(t), t\varepsilon T\}$ be a stochastic process defined on a measurable space (Ω,F). We may assume that F is the complete Borel field generated by $\{X(t), t\varepsilon T\}$. Let P and Q be two probability measures defined on (Ω,F) with respect to which the stochastic process $\{X(t), t\varepsilon T\}$ is a Brownian motion. It is known that P and Q are either equivalent: for $A\varepsilon F$, $P(A) = 0$ if and only if $Q(A)=0$; or mutually perpendicular: for some $A\varepsilon F$, $P(A)=0=Q(A')$ (' denotes complement).

If we look at Brownian motions with zero mean, we find that they possess the following features:

1. separability,
2. continuity in probability,
3. $E[\exp(tX(s))] \leq \exp(\sigma^2(s)t^2/2)$.

It will be seen that an extension to generalized Gaussian measure of the previous results concerning the equivalence of two Gaussian measures is possible if $\{X(t), t\varepsilon T\}$ is a stochastic process possessing properties 1, 2, and 3.

Definition 4 (Separability). Let A be the class of all closed intervals (finite or infinite). A stochastic process $\{X(t), t\varepsilon T\}$ will be called separable relative to A if there is a countable subset T_1 of T such that for each open interval I and each $A\varepsilon A$,

$$\{X(t)\varepsilon A,\ t\varepsilon I\cap T\}=\{X(t_i)\varepsilon A,\ t_i\varepsilon I\cap T_1\}\cup N \tag{4}$$

with $P(N)=0$.

Definition 5 (Continuity in Probability). A stochastic process is said to be continuous in probability if for every sequence

$\{s_n\}_{n=1}^{\infty} \subseteq T$, such that $\lim_{n\to\infty} s_n=t$, $\lim_{n\to\infty} X(s_n)=X(t)$ in probability.

Since Brownian motions have a continuous sample path a.s. $s_n \to t$ implies $X(s_n) \to X(t)$ a.s. . Thus Brownian motions are continuous in probability.

Proposition 1. If there is a countable subset $T_1 \subset T$ such that for all open intervals I,

$$\underset{t \in IT}{\text{g.l.b}}\ X(t) = \underset{t \in IT_1}{\text{g.l.b.}}\ X(t_i) \quad \text{a.s.}$$

$$\underset{t \in IT}{\text{l.u.b.}}\ X(t) = \underset{t \in IT_1}{\text{l.u.b.}}\ X(t_i) \quad \text{a.s.} \tag{5}$$

then X(t) is separable.

Proof: Let $A=[\alpha,\beta]$.

$$\{X(t_i) \in A,\ t_i \in IT_1\} = \{\alpha \leq X(t_i) \leq \beta,\ t_i \in IT_1\}$$

$$= \{\underset{t_i \in IT_1}{\text{g.l.b.}}\ X(t_i) \geq \alpha\} \cap \{\underset{t_i \in IT_1}{\text{l.u.b.}}\ X(t_i) \leq \beta\}$$

$$= [\{\underset{t \in IT}{\text{g.l.b.}}\ X(t) \geq \alpha\} \cup N_1] \cap [\{\underset{t \in IT}{\text{l.u.b.}}\ X(t) \leq \beta\} \cup N_2]$$

$$= [\{\underset{t \in IT}{\text{g.l.b.}}\ X(t) \geq \alpha\} \cap \{\underset{t \in IT}{\text{l.u.b.}}\ X(t) \leq \beta\}] \cup N$$

$$= \{X(t) \in A,\ t \in IT\} \cup N,$$

where $N_1 = \{\underset{t_i \in IT_1}{\text{g.l.b.}}\ X(t_i) \neq \underset{t \in IT}{\text{g.l.b.}}\ X(t)\} \cap \{\underset{t \in IT_1}{\text{g.l.b.}}\ X(t_i) \geq \alpha\}$

$$N_2 = \{\underset{t_i \in IT_1}{\text{l.u.b.}}\ X(t_i) \neq \underset{t \in IT}{\text{l.u.b.}}\ X(t)\} \cap \{\underset{t_i \in IT_1}{\text{l.u.b.}}\ X(t_i) \leq \beta\};$$

the changes regarding null sets N_1, N_2, N are obtained by using the fact that

$$(A \cup B) \cap (C \cup D) = (A \cap C) \cup [(B \cap C) \cup (A \cap D) \cup (B \cap D)].$$

By completeness of (Ω,F,P), $N \in F$ and $P(N)=0$. Thus $\{X(t), t \in t\}$ is separable.

COROLLARY. If $\{X(t), t \in T\}$ is a Brownian motion, then it is separable.

Proof: Let T_1 be a countable dense subset of T, then

$$\underset{t_i \in IT_1}{\text{g.l.b.}}\ X(t_i) = \underset{t \in IT}{\text{g.l.b.}}\ X(t) \quad \text{a.s.}$$

$$\underset{t_i \in IT_1}{\text{l.u.b.}}\ X(t_i) = \underset{t \in IT}{\text{l.u.b.}}\ X(t) \quad \text{a.s.}$$

since Brownian motions have continuous sample path a.s.

Proposition 2. Let $\{X(t), t \in T\}$ be a continuous in probability, separable stochastic process. If T_1 is any countable dense subset T, then T_1 satisfies the separability condition (4).

Proof: By Proposition 2, it suffices to show that T_1 satisfies (5).

For each $t \varepsilon T$, there is a sequence $\{t_i\}$ from T_1 such that $t_i \to t$. By continuity in probability, $\lim_{t_i \to t} X(t_i)=X(t)$ in probability. There is a subsequence $\{s_k\}$ such that $\lim_{s_k \to t} X(s_k)=X(t)$ a.s. .

For each open interval I and each $t \varepsilon IT$, there is a sequence $S=\{s_k\}$ from T_1 such that

$$\text{g.l.b.}_{t_i \varepsilon IT_1} X(t_i) \le \text{g.l.b.}_{s_k \varepsilon IS} X(s_k) \le \text{g.l.b.}_{|s_k - t|<1/n} X(s_k)$$

and

$$\text{l.u.b.}_{t_i \varepsilon IT_1} X(t_i) \ge \text{l.u.b.}_{s_k \varepsilon IS} X(s_k) \ge \text{l.u.b.}_{|s_k - t|<1/n} X(s_k)$$

for sufficiently large n. Hence

$$\text{g.l.b.}_{t_i \varepsilon IT_1} X(t_i) \le \lim_{n\to\infty} \text{g.l.b.}_{|s_k - t|<1/n} X(s_k)$$

$$\text{l.u.b.}_{t_i \varepsilon IT_1} X(t_i) \ge \lim_{n\to\infty} \text{l.u.b.}_{|s_k - t|<1/n} X(s_k) \ .$$

But $\lim_{s_k \to t} X(s_k)=X(t)$ a.s., so

$$\lim_{n\to\infty} \text{g.l.b.}_{|s_k - t|<1/n} X(s_k)=X(t) \quad \text{a.s.}$$

$$\lim_{n\to\infty} \text{l.u.b.}_{|s_k - t|<1/n} X(s_k)=X(t) \quad \text{a.s.} \ .$$

Thus $\text{g.l.b.}_{t_i \varepsilon IT_1} X(t_i) \le X(t)$ a.s. and $\text{l.u.b.}_{t_i \varepsilon IT_1} X(t_i) \ge X(t)$ a.s. .

This being true for each $t \varepsilon IT$, we have

$$\text{g.l.b.}_{t_i \varepsilon IT_1} X(t_i) \le \text{g.l.b.}_{t \varepsilon IT_1} X(t) \quad \text{a.s.}$$

$$\text{l.u.b.}_{t_i \varepsilon IT_1} X(t_i) \ge \text{l.u.b.}_{t \varepsilon IT_1} X(t) \quad \text{a.s.} \ .$$

Since the inequalities going in the opposite directions are obvious, the result follows.

Now, let T_1 be a countable dense subset of T. Consider the stochastic process $\{X(t_i), t_i \varepsilon T_1\}$. Let F_1 be the complete Borel field generated by $\{X(t_i), t_i \varepsilon T_1\}$. Define the finite dimensional distributions P_1^n of $\{X(t_i), t_i \varepsilon T_1\}$ by

$$P_1^n[X(t_1) \le x_1, \ldots, X(t_n) \le x_n]=P[X(t_1) \le x_1, \ldots, X(t_n) \le x_n] \ .$$

Let P_1 be the extension of the P_1^n's. We thus obtain a probability space (Ω, F_1, P_1).

THEOREM 1. If $\{X(t),\ t\varepsilon T\}$ is separable and continuous in probability, then $(\Omega,F_1,P_1)=(\Omega,F,P)$.

Proof:

(i) $F_1=F$: Since T_1 is dense in T, for each $t\varepsilon T$ there exists a sequence $\{t_k\}$ $\subseteq T_1$, such that $t_k\to t$. Since X(t) is continuous in probability, $X(t_k)\to X(t)$ in probability. Hence, there exists a subsequence t_{n_k} such that $X(t_{n_k})\to X(t)$ a.s. since $X(t_{n_k})$ is F_1-measurable, X(t) is also F_1-measurable. Hence $F\subseteq F_1$, but $F_1\subseteq F$, so $F_1=F$.

(ii) $P=P_1$: Since $\{X(t),t\varepsilon T\}$ is continuous in probability, for $\{t_1,t_2,\ldots,t_k\}\subseteq T$, and sequences $\{s_n^1\}, \{s_n^2\},\ldots, \{s_n^k\}$ from T_1, converging to $t_1,\ldots,t_k$, we have $[X(s_n^1),X(s_n^2),\ldots,X(s_n^k)]\to[X(t_1),X(t_2),\ldots,X(t_k)]$ a.s. P. (Take subsequences if necessary.)

This implies convergence in distribution; i.e., for

$$F_n(x_1,x_2,\ldots,x_k)=P_1[X(s_n^1)\le x_1,\ldots,X(s_n^k)\le x_k]$$

$$=P[X(s_n^1)\le x_1,\ldots,X(s_n^k)\le x_k]$$

$$F_n(x_1,x_2,\ldots,x_k)\to F(x_1,x_2,\ldots,x_k)=P[X(t_1)\le x_1,\ldots,X(t_k)\le x_k].$$

$$P_1[X(t_1)\le x_1,\ldots,X(t_k)\le x_k]=P[X(t_1)\le x_1,\ldots,X(t_k)\le x_k]$$

if $[X(s_n^1),\ldots,X(s_n^k)]\to[X(t_1),\ldots,X(t_k)]$ a.s. P_1. The set of convergence of $[X(s_n^1),X(s_n^2),\ldots,X(s_n^k)]$ is

$$C=\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{n'=n+1}^{\infty}\{\max_{n\le j\le n'}|X(s_n^i)-X(s_j^i)|<1/m,\ 1\le i\le k\};\ \text{say, } C=\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{n'=n+1}^{\infty}V(m,n,n').$$

$$P_1(C)=\lim_{m\to\infty}\lim_{n\to\infty}P_1[V(m,n,n')]$$

$$=\lim_{m\to\infty}\lim_{n\to\infty}P[V(m,n,n')]=1\ .$$

Hence $[X(s_n^1),X(s_n^2),\ldots,X(s_n^k)]\to[X(t_1),X(t_2),\ldots,X(t_k)]$ a.s. P_1. Since $P=P_1$ on the field generating F, $P=P_1$.

III. Stochastic Processes of Function Space Type

Let $\{X(t),t\varepsilon T\}$ be a real stochastic process defined on an arbitrary probability space (Ω,F,P). $\{X(t),\ t\varepsilon T\}$ may be considered as a subset of R^T, the set of all real-valued functions defined on T. Let F_o be the family of subsets of R^T of the form

$$\{f\varepsilon R^T: [f(t_1),f(t_2),\ldots,f(t_n)]\ \varepsilon\ A\}$$

where $t_1, t_2, \ldots, t_n \varepsilon T$, and A is an n-dimensional Borel set. It is easy to verify that F_o is a field. On F_o, define a probability measure P_o as follows:

$$P_o[f\varepsilon R^T: [f(t_1),f(t_2),\ldots,f(t_n)]\varepsilon\ A] = P[(X(t_1),X(t_2),\ldots,X(t_n))\ \varepsilon\ A\ .$$

Let $\tilde{F}$ be the smallest Borel field containing F_o. Then there exists a unique extension of P_o to a probability measure $\tilde{P}$ defined on $\tilde{F}$. On $R^T xT$, let there be defined a function $\tilde{X}$ by $\tilde{X}(f,t)=f(t)$. For each $t\varepsilon T$, $X(f,t)$ is a random variable defined on $(R^T,\tilde{F},\tilde{P})$, since if A is any Borel set, $\{f:\tilde{X}(f,t)\varepsilon A\}=\{f:f(t)\ \varepsilon A\}\varepsilon F_o$. Hence $\{\tilde{X}(t),t\varepsilon T\}$ is a stochastic process defined on $(R^T,\tilde{F},\tilde{P})$. The stochastic process $\{\tilde{X}(t),t\varepsilon T\}$ so defined is called a stochastic process of function space type.

By what precedes, a stochastic process $\{X(t),\ t\varepsilon T\}$ defined on an arbitrary probability space (Ω,F,P) induces a probability measure $\tilde{P}$ defined on $(R^T,\tilde{F})$ and a stochastic process $\{\tilde{X}(t),t\varepsilon T\}$. The induced probability measure P and stochastic process $\{\tilde{X}(t),t\ \varepsilon\ T\}$ inherit some characteristics of the measure P and the stochastic process $\{X(t),\ t\ \varepsilon\ T\}$:

1. If $\{X(t),\ t\ \varepsilon\ T\}$ is a generalized Gaussian process, so is $\{\tilde{X}(t),t\ \varepsilon\ T\}$.

Proof: It suffices to show that for each $t\varepsilon T$, $\tilde{X}(t)$ is a generalized Gaussian random variable.

Let $\quad A_{mn}=\{\omega:n/2^m\underline{<}X(\omega,t)\underline{<}(n+1)/2^m\}$

$\quad\quad \tilde{A}_{mn}=\{f:n/2^m\underline{<}X(f,t)<(n+1)/2^m\}$

$\quad\quad m=1,2,\ldots;\ n=0,\pm1,\pm2,\ldots\ .$

Let $X_m(\omega,t)=n/2^m$ if $\omega\varepsilon A_{mn}$; $\tilde{X}_m(f,t)=n/2^m$ if $f\varepsilon\tilde{A}_{mn}$. Then $X_m(t)\to X(t)$ a.s. P, and $\tilde{X}_m(t)\to\tilde{X}(t)$ a.s. $\tilde{P}$. Thus for each $s\varepsilon R$, $e^{X_m(t)s}\to e^{sX(t)}$ a.s. P and $e^{\tilde{X}_m(t)s}\to e^{s\tilde{X}(t)}$ a.s. $\tilde{P}$.

$$E[\exp[sX_m(t)]]=\Sigma e^{n/2^m}P[n/2^m\underline{<}X(t)<(n+1)/2^m]$$

$$E[\exp[s\tilde{X}_m(t)]]=\Sigma e^{n/2^m}\tilde{P}[n/2^m\underline{<}\tilde{X}(t)<(n+1)/2^m]\ .$$

Since $\quad P[n/2^m\underline{<}X(t)<(n+1)/2^m]=\tilde{P}[n/2^m\underline{<}\tilde{X}(t)<(n+1)/2^m]$

$$E[\exp[sX_m(t)]]=E[\exp[s\tilde{X}_m(t)]]\ .$$

For $s\underline{>}0$, $e^{sX_m(t)}\uparrow e^{sX(t)}$. So $\lim_{m\to\infty}E[\exp[sX_m(t)]]=E[\exp[sX(t)]]$. Similarly, $\lim_{m\to\infty}E[\exp[s\tilde{X}_m(t)]]=E[\exp[s\tilde{X}(t)]]$. So $E[\exp[s\tilde{X}(t)]]=E[\exp[sX(t)]]\underline{<}$ exp

$[s^2\alpha^2(t)/2]$.

For $s<0$. Let $X'_m(t)=(n+1)/2^m$ if $\omega\varepsilon A_{mn}$; $\tilde{X}'(f,t)=(n+1)/2^m$ if $f\varepsilon\tilde{A}_{mn}$. Then $\exp[sX'_m(t)] \uparrow \exp[sX(t)]$. By an argument similar to that above

$$E[\exp[sX(t)]]=E[\exp[s\tilde{X}(t)]] \leq \exp[s^2\alpha^2(t)/2]$$

2. For each $t\varepsilon T$, $E[X(t)]=E[\tilde{X}(t)]$.

The proof of this is similar to that in 1.

3. $\{X(t),\ t\varepsilon T\}$ and $\{\tilde{X}(t),\ t\varepsilon T\}$ have the same convariance function.

Proof: First, consider the stochastic process $\{X(t),\ t\varepsilon T\}$ such that $X(t)=\chi_{A_t}$, $A_t\varepsilon F$ for each $t\varepsilon T$. In this case, we have

$$\begin{aligned} E[X(s)X(t)] &= P[X(s)=1,\ X(t)=1] \\ &= \tilde{P}[f:\ f(s)=1,\ f(t)=1] \\ &= \tilde{P}[\tilde{X}(s)=1,\ \tilde{X}(t)=1] \end{aligned}$$

Since for each $t\varepsilon T$,

$$1=P[X(t)=1]+P[X(t)=0]=\tilde{P}[\tilde{X}(t)=1]+\tilde{P}[\tilde{X}(t)=0],$$

$$\tilde{P}[\tilde{X}(t)=1,\ \tilde{X}(s)=1]=E[\tilde{X}(s)\tilde{X}(t)]$$

Thus $$E[X(s)X(t)]=E[\tilde{X}(s)\tilde{X}(t)].$$

If each $X(t)$ is a simpl function, i.e. $X(t)=\Sigma_{i=1}^{n_t} a_i(t)\chi_{A_{t_i}}$. Then

$$\begin{aligned} E[X(s)X(t) &\ \Sigma_{i=1}^{n_s}\Sigma_{j=1}^{n_t} a_i(s)a_j(t)P[X(s)=a_j(s),X(t)=a_j(t)] \\ &= \Sigma_{i=1}^{n_s}\Sigma_{j=1}^{n_t} a_i(s)a_j(t)\tilde{P}[\tilde{X}(s)=a_i(s),\tilde{X}(t)=a_j(t)] \\ &= E[\tilde{X}(s)\tilde{X}(t)] \end{aligned}$$

For the general case, let $\{X_n(t)\}_{n=1}^{\infty}$ be a sequence of simple functions such that $X_n(t) \uparrow X(t)$ for each $t\varepsilon T$. To each sequence $\{X_n(t)\}_{n=1}^{\infty}$ corresponds a sequence of simple functions $\{\tilde{X}_n(t)\}_{n=1}^{\infty}$ such that

$$E[X_n(s)X_n(t)]=E[\tilde{X}_n(s)\tilde{X}_n(t)]$$

To prove the assertion, it remains to show that $\tilde{X}_n(t) \uparrow \tilde{X}(t)$ in probability $\tilde{P}$ for each $t\varepsilon T$.

$$\begin{aligned} P[|\tilde{X}_n(t)-\tilde{X}(t)|<\varepsilon] &= \Sigma_{i=1}^{n_t} P(a_i-\varepsilon<\tilde{X}(t)<a_i+\varepsilon)P(\tilde{X}_n(t)=a_i) \\ &= \Sigma_{i=1}^{n_t} P(a_i-\varepsilon<X(t)<a_i+\varepsilon)P(X_n(t)=a_i) \\ &= P[|X_n(t)-X(t)|<\varepsilon] \to 1. \end{aligned}$$

Hence $E[X(s)X(t)]=E[\tilde{X}(s)\tilde{X}(t)]$.

4. If $\{X(t), t\varepsilon T\}$ is continuous in probability, so is $\{\tilde{X}(t), t\varepsilon T\}$.

Proof: Assume $t_n \to t$.

Since $\tilde{P}[f: |X(f,t)-X(f,t_n)|>\varepsilon]=\tilde{P}[f:|f(t)-f(t_n)|>\varepsilon]$

$$=P[|X(t)-X(t_n)|>\varepsilon].$$

$X(t_n) \to X(t)$ in probability implies $\tilde{X}(t_n) \to \tilde{X}(t)$ in probability.

5. If P and Q are two probability measures induced by the stochastic process $\{X(t), t\varepsilon T\}$, and $\tilde{P}$ and $\tilde{Q}$ are two probability measures induced on the path space $(R^T, \tilde{F})$ by P and Q, respectively; then the equivalence of P and Q implies the equivalence of $\tilde{P}$ and $\tilde{Q}$ and vice versa.

Proof: Let $A=\{A\varepsilon\tilde{F}, \tilde{P}(A)=0\}$,

$B=\{A\varepsilon\tilde{F}, \tilde{P}(A)\neq 0\}$.

Define a set function Q' on $\tilde{F}$, by $Q'(A)=0$ if $A\varepsilon A$, $Q'(A)=Q(A)$ if $A\varepsilon B$, $Q'(A \cup B)=Q(B)$ if $A\varepsilon A$, $B\varepsilon B$. It is easy to verify that Q' is a probability measure.

Since P and Q are equivalent, $\tilde{P}$ and $\tilde{Q}$ have the same null sets in F_o. Hence $\tilde{Q}$ and Q' agree on F_o which generates $\tilde{F}$. So, $\tilde{Q}=Q'$.

The reverse implication is proved by interchanging the role of P, Q and $\tilde{P}$, $\tilde{Q}$.

Remark: Since separability is characterized by the Borel fields, and the Borel field $\tilde{F}$ is constructed independently of the Borel field F, separability of (Ω, F, P) does not carry over to $(R^T, \tilde{F}, \tilde{P})$. However $(R^T, F, \tilde{P})$ can be replaced by $(R^T, \tilde{F}_1, \tilde{P}_1)$ by enlarging the Borel field to make $\{X(t), t\varepsilon T\}$ separable with respect to the new probability space $(R^T, \tilde{F}_1, \tilde{P}_1)$.

To obtain $(R^T, \tilde{F}_1, \tilde{P}_1)$, let C be a subset of R^T with outer measure 1 relative to $\tilde{F}$, i.e.

$$\tilde{P}^*(C)=\text{g.l.b.}\ \{\tilde{P}(A);\ C \subseteq A,\ A\varepsilon\tilde{F}\}=1$$

Let $\tilde{F}_1$ be the family of subsets of R^T of the form:

$$(A_1 \cap C) \cup (A_2 \cap C'),\ A_1, A_2 \varepsilon \tilde{F}.$$

On $\tilde{F}_1$, define a set function $\tilde{P}_1$ by

$$\tilde{P}_1(A)=\tilde{P}[(A_1 \cap C) \cup (A_2 \cap C')]=\tilde{P}(A_1)$$

It is easy to verify that $\tilde{P}_1$ is a probability measure on $\tilde{F}_1$, and agrees with $\tilde{P}$ on $\tilde{F}$.

Proposition 3. With $(R^T, \tilde{F}, \tilde{P})$ replaced by $(R^T, \tilde{F}_1, \tilde{P}_1)$, $\{\tilde{X}(t), t\varepsilon T\}$ is separable (with respect to closed sets) depending on a proper choice of C.

Proof: Let I be an open interval, and S the family of all possible sequences from $I \cap T$. Let $\{t_i(I)\}$ be a countable subset of $I \cap T$, such that

$$\underset{S}{\text{l.u.b.}} \int_{R^T} \tan^{-1} \underset{s_i \in IT}{\text{l.u.b.}} X(s_i)\, dP = \int_{R^T} \tan^{-1} \underset{t_i \in IT}{\text{l.u.b.}} X(t_i)\, dP$$

$$\underset{S}{\text{g.l.b.}} \int_{R^T} \tan^{-1} \underset{s_i \in IT}{\text{g.l.b.}} X(s_i)\, dP = \int_{R^T} \tan^{-1} \underset{t_i \in IT}{\text{g.l.b.}} X(t_i)\, dP$$

Then for all $\{s_i\} \varepsilon S$,

$$\underset{i}{\text{l.u.b.}}\ X(s_i) \le \underset{i}{\text{l.u.b.}}\ X(t_i(I))$$

$$\underset{i}{\text{g.l.b.}}\ X(s_i) \le \underset{i}{\text{g.l.b.}}\ X(t_i(I))$$

Now let I be the family of all open intervals with rational end-points. Let $\{t_i\} = \bigcup_{I \varepsilon I} t_i(I)$. Then for each $t \varepsilon T$,

$$\lim_{\varepsilon \downarrow 0} \underset{|t_i - t| < \varepsilon}{\text{g.l.b.}} X(t_i) \le X(t) \le \lim_{\varepsilon \downarrow 0} \underset{|t_i - t| < \varepsilon}{\text{l.u.b.}} X(t_i) \qquad (6)$$

a.s. $\tilde{P}$.

Let C be the set of all functions in R^T such that (6) is satisfied simultaneously for all $t \varepsilon T$. Then if $X(t) = f(t) \varepsilon C$,

$$\underset{t \varepsilon IT}{\text{l.u.b.}} X(t) = \underset{t_i \varepsilon IT}{\text{l.u.b.}} X(t_i)$$

To complete the proof, it suffices to show that $\tilde{P}^*(C) = 1$. Let B be any arbitrary set in $\tilde{F}$, which is a finite or countable union of sets from $\tilde{F}_o$, and contains C. If $\tilde{P}(B) = 1$, then $\tilde{P}^*(C) = 1$. Let

$$B = \bigcup_{i=1}^{\infty} \{f \varepsilon R^T : a_i < f(t_i) < b_i\}$$

and assume that $C \supset B$. Let B_o be a subset of B such that (6) is satisfied. Since the set of functions such that

$$\lim_{\varepsilon \downarrow 0} \underset{|t_i - \tau_i| < \varepsilon}{\text{g.l.b.}} f(t_i) \le f(\tau_i) \le \lim_{\varepsilon \downarrow 0} \underset{|t_i - \tau_i| < \varepsilon}{\text{l.u.b.}} f(t_i)$$

has probability one, $\tilde{P}(B_o) = 1$. Hence $\tilde{P}(B) = 1$.

By what preceeds, if a stochastic process $\{X(t), t \varepsilon T\}$ defined on an arbitrary measurable space (Ω, F) is generalized Gaussian, separable, and continuous in probability with respect to both measure space (Ω, F, P) and (Ω, F, Q) induced by $\{X(t),\ t \varepsilon T\}$, the equivalence of P and Q is equivalent to the equivalence of $\tilde{P}$ and $\tilde{Q}$, probability measures induced by P and Q, respectively, on the path space R^T. So from now on we may assume that $(\Omega, F, P) = (R^T, \tilde{F}, \tilde{P})$.

IV. A Dichotomy Theorem for Generalized Gaussian Measures

THEOREM 2. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of independent generalized Gaussian random variables such that $\sup_k \tau(X_k)=\alpha<\infty$. Let $\{\phi_k\}_{k=1}^{\infty}$ be a sequence of uniformly bounded real valued functions defined on a closed interval T, and $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that $\Sigma_{k=1}^{n} a_k^2/n\to 0$, as $n\to\infty$. Then $Y_n(t)=\Sigma_{k=1}^{n} a_k\phi_k(t)X_k/\sqrt{n}$ converges a.s. to a stochastic process $\{Y(t),\ t\varepsilon T\}$ as $n\to\infty$ and $\{Y(t),\ t\varepsilon T\}$ is a Gaussian process.

Proof: First, we show that $Y_n(t)\to Y(t)$ a.s. for each $t\varepsilon T$. For each k, each n and each t, $a_k\phi_k(t)X_k/\sqrt{n}$ is generalized Gaussian with

$$E[\exp(sa_k\phi_k(t)X_k/\sqrt{n})]\le \exp(s^2a_k^2\alpha^2M^2/2n)$$

where M is the bound for ϕ_k. Since the convergence set of the sequence $Y_n(t)$ is $C=\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{j=n+1}^{\infty}\{|Y_n(t)-Y_j(t)|\le 1/m\}$, it suffices to show that $\lim_{m\to\infty}\lim_{n\to\infty}P\{|Y_n(t)-Y_j(t)|\le 1/m\}=1$. Now for $j>n$,

$$Y_j(t)-Y_n(t)=\Sigma_{k=1}^{n}(a_k/\sqrt{j}-a_k/\sqrt{n})\phi_k(t)X_k+\Sigma_{k=n+1}^{j}(a_k/\sqrt{j})X_k\phi_k(t)$$

and
$$E[\exp[s(Y_j(t)-Y_n(t))]]\le\exp(s^2\alpha^2M^2A_{nj}/2)\text{ where}$$

$$A_{nj}=\Sigma_{k=1}^{n}(a_k/\sqrt{j}-a_k/\sqrt{n})^2+\Sigma_{k=n+1}^{j}a_k^2/j=\Sigma_{k=1}^{j}a_k^2/j+\Sigma_{k=1}^{n}a_k^2/n - 2\Sigma_{k=1}^{n}a_k^2/\sqrt{nj}\ .$$

Hence $P(|Y_j(t)-Y_n(t)|\ge 1/m)\le\exp(-1/2m^2\alpha^2M^2A_{nj})$ for all j, n. Since $\lim_{n\to\infty}A_{nj}=0$, the assertion follows.

To show that $\{Y(t),\ t\varepsilon T\}$ is a Gaussian process, we must show that if $c_1, c_2, \ldots, c_n$ are real numbers and $t_1, t_2, \ldots, t_n\varepsilon T$, $c_1Y(t_1)+c_2Y(t_2)+\ldots+c_nY(t_n)$ is a Gaussian random variable.

Let
$$Z_m=c_1\Sigma_{k=1}^{m}a_k\phi_k(t_1)X_k+c_2\Sigma_{k=1}^{m}a_k\phi_k(t_2)X_k+\ldots+c_n\Sigma_{k=1}^{m}a_k\phi_k(t_n)X_k$$
$$=\Sigma_{k=1}^{m}a_k[c_1\phi_k(t_1)+c_2\phi_k(t_2)+\ldots+c_n\phi_k(t_n)]X_k.$$

Then $\lim_{m\to\infty}Z_m/\sqrt{m}=c_1Y(t_1)+c_2Y(t_2)+\ldots+c_nY(t_n)$. So, it suffices to show that the limit distribution of $Z_m/\sqrt{m}$ is normal. But this is the case if $\max_{1\le k\le m}|a_k[c_1\phi_k(t_1)+c_2\phi_k(t_2)+\ldots+c_n\phi_k(t_n)]X_k/\sqrt{m}|\to 0$ in probability ([2] p. 316).

Let $A_k=c_1\phi_k(t_1)+c_2\phi_k(t_2)+\ldots+c_n\phi_k(t_n)$, $C=|c_1|+|c_2|+\ldots+|c_n|$. Then $|A_k|\le CM$ for all k. Hence, with $D=C^2M^2\alpha^2$,

$$E[\exp(ta_kA_kX_k/\sqrt{m})]\leq\exp(t^2a_k^2A_k^2\alpha^2/2m)\leq\exp(t^2a_k^2D^2/2m)$$

So, $$P(|a_kA_kX_k/\sqrt{m}|>\varepsilon)\leq\exp(-\varepsilon^2m/2D^2a_k^2)\leq\exp(-\varepsilon^2m/2D^2\Sigma_{k=1}^m a_k^2)$$

Now $$P(\max_{1\leq k\leq m}|a_kA_kX_k/\sqrt{m}|\leq\varepsilon)=P(|a_1A_1X_1/\sqrt{m}|\leq\varepsilon,\ |a_2A_2X_2/\sqrt{m}|\leq\varepsilon,\ldots,$$

$$|a_mA_mX_m/\sqrt{m}|\leq\varepsilon)=\prod_{k=1}^m P(|a_kA_kX_k/\sqrt{m}|\leq\varepsilon)$$

$$=\prod_{k=1}^m(1-P(|a_kA_kX_k/\sqrt{m}|>\varepsilon))$$

$$\geq(1-\exp(-\varepsilon^2m/2D^2\Sigma_{k=1}^m a_k^2))^m\to 1.$$

Let $\{X(t),t\in T\}$ be a stochastic process defined on a measurable space (Ω,F). We assume that F is generated by $\{X(t),\ t\in T\}$. Let P and Q be two probability measures induced by $\{X(t),\ t\in T\}$. We make the following assumptions:

(1) P and Q are generalized Gaussian;

(2) $\sup_t \tau_P(t)<\infty$ and $\sup_t \tau_Q(t)<\infty$, where $\tau_P(t)$ and $\tau_Q(t)$ are the minimums of those $\alpha_P(t)$ and $\alpha_Q(t)$ such that $E_P[\exp(sX(t))]\leq\exp(\alpha_P^2(t)s^2/2)$ and $E_Q[\exp(sX(t))]\leq\exp(\alpha_Q^2(t)s^2/2)$.

(3) There exists a countable dense subset $S=\{t_i\}_{i=1}^\infty$ of T such that $\{X(t_i)\}_{i=1}^\infty$ is a sequence of independent random variables with respect to both P and Q.

(4) $\{X(t),\ t\in T\}$ is separable and continuous in probability with respect to both P and Q.

Then by assumption 4 and theorem 16, if S is a countable dense subset of T, the probability spaces (Ω,F_1,P_1) and (Ω,F_1,Q_1) generated by $\{X(s),s\in S\}$ are the same as (Ω,F,P) and (Ω,F,Q) respectively. Let $S=\{t_k\}_{k=1}^\infty$ and $Y_k=X(t_k)$. Then by assumption 3, S may be so chosen that $\{Y_k\}_{k=1}^\infty$ is a sequence of independent random variables. From the previous theorem we obtain:

THEOREM 3. Let $\{a_k\}_{k=1}^\infty$ be a sequence of real numbers such that $\Sigma_{k=1}^n a_k^2/n\to 0$. Let $\{\phi_k\}_{k=1}^\infty$ be a sequence of uniformly bounded functions defined on T. Then under the assumptions 1-4, $Z(t)=\lim_{n\to\infty}\Sigma_{k=1}^n a_k\phi_k(t)Y_k/\sqrt{n}$ is a Gaussian process defined on both probability spaces (Ω,F,P) and (Ω,F,Q).

Now let B be the set of all sequences of real numbers such $\Sigma_{k=1}^\infty a_k^2/n<\infty$. Let C be the set of all sequences of uniformly bounded real valued functions defined on T. Let A=BxC. Then by Theorem 3 $\{X(t),t\in T\}$ defines a map from A into the set of all Gaussian process defined on both probability spaces (Ω,F,P) and

(Ω,F,Q). For each $\alpha\varepsilon A$, let F_α be the Borel field generated by the Gaussian process $Z_\alpha(t)$. Let $F' = \bigcup_{\alpha\varepsilon A} F_\alpha$, and

$$N_P=\{A\varepsilon F: P(A)=0\}$$
$$N_Q=\{A\varepsilon F: Q(A)=0\}$$
$$M_P=\{A\varepsilon F': P(A)=0\}$$
$$M_Q=\{A\varepsilon F': Q(A)=0\}\ .$$

In addition to assumptions 1-4, we assume

(5) $N_P=M_P$ and $N_Q=M_Q$.

THEOREM 4 (Dichotomy theorem). Under the assumptions 1-5, P and Q are either equivalent of perpendicular.

Proof: Let $P_\alpha=P|F_\alpha$, and $Q_\alpha=Q|F_\alpha$. Then by the dichotomy theorem for Gaussian measures P_α and Q_α are either equivalent or perpendicular. If for some $\alpha\varepsilon A$, P_α and Q_α are perpendicular, then P and Q are obviously perpendicular. Suppose P_α and Q_α are equivalent for all $\alpha\varepsilon A$, then if $P(A)=0$, there is an $\alpha\varepsilon A$ such that $P_\alpha(A)=0$ and so $Q_\alpha(A)=0$. Assumption 5 implies that $Q(A)=0$. Hence $P(A)=0 \Rightarrow Q(A)=0$. Similarly $Q(A)=0 \Rightarrow P(A)=0$.

References

1. Chow, Y. S.,"Some convergence theorems for independent random variables", Ann. Math. Stat., 1482-1493 (1966).
2. Loeve, M. Probability Theory Van Nostrand (1963).
3. Feldman, J. "Equivalence and perpendicularity of Gaussian processes", Pacific J. Math. 699-708 (1958).