擬相似之弱收縮變換

# Quasi-Similarity of Weak Contractions

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ABSTRACT — Let T be a completely non-unitary (c.n.u.) weak contraction (in the sense of Sz.-Nagy and Foias). We show that T is quasi-similar to the direct sum of its $C_0$ part and $C_{11}$ part. As a corollary, two c.n.u. weak contractions are quasi-similar to each other if and only if their $C_0$ parts and $C_{11}$ parts are quasi-similar to each other, respectively. We also completely determine when c.n.u. weak contractions and $C_0$ contractions are quasi-similar to normal operators.

Recall that a contraction T on a Hilbert space H is called a weak contraction if its spectrum $\sigma(T)$ does not fill the open unit disc D and $1-T^*T$ is of finite trace. Contained in this class are all contractions T with finite defect index $d_T \equiv \dim \mathrm{Range}\,(1-T^*T)^{\frac{1}{2}}$ and with $\sigma(T) \neq \bar{D}$ (cf. [8], p. 323).

Assume that T is a weak contraction which is also completely non-unitary (c.n.u.), that is, T has no non-trivial reducing subspace on which T is a unitary operator. For such a contraction, Sz.-Nagy and Foias obtained a $C_0$-$C_{11}$ decomposition and then found a variety of invariant subspaces which furnish its spectral decomposition (cf. [8], chap. VIII). In this note we are going to supplement other interesting properties of such contractions. We show that a c.n.u. weak contraction is quasi-similar to the direct sum of its $C_0$ part and $C_{11}$ part. Although the proof is not difficult, some of its interesting applications justify the elaboration here. An immediate corollary is that two such contractions are quasi-similar to each other if and only if their $C_0$ parts are quasi-similar and their $C_{11}$ parts are quasi-similar to each other. This is in term used to show that two quasi-similar weak contractions have equal spectra. Another interesting consequence is that a c.n.u. weak contraction is

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quasi-similar to a normal operator if and only if its $C_o$ part is. The latter
can be shown to be equivalent to the condition that its minimal function is a
Blaschke product with simple zeros, thus completely settling the question of
when a c.n.u. weak contraction is quasi-similar to a normal operator.

Before we start to prove our main theorem, we provide some background work
for our notations and terminologies. The main reference is [8].

Let T be an arbitrary contraction on H. Let $H_o = \{h \epsilon H : T^n h \to 0\}$, $H_o' = \{h \epsilon H : T^{*n} h \to 0\}$, $H_1' = H \ominus H_o$ and $H_1 = H \ominus H_o'$. Note that $H_o$ and $H_o'$ are invariant for T and T*, res-
pectively. Consider the triangulations of T with respect to the orthogonal
decompositions $H = H_o \oplus H_1'$ and $H = H_1 \oplus H_o'$:

$$T = \begin{bmatrix} T_o & X \\ 0 & T_1' \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} T_1 & Y \\ 0 & T_o' \end{bmatrix}$$

The triangulations are of type $\begin{bmatrix} C_{o.} & * \\ 0 & C_{1.} \end{bmatrix}$ and $\begin{bmatrix} C_{.1} & * \\ 0 & C_{.o} \end{bmatrix}$, respectively (cf. [8],
p. 73). Recall that a contraction T is of class $C_{o.}$ (resp. $C_{.o}$) if $T^n h \to 0$ (resp.
$T^{*n} h \to 0$) as $n \to \infty$ for all h and T is of class $C_{1.}$ (resp. $C_{.1}$) if $T^n h \not\to 0$ (resp.
$T^{*n} h \not\to 0$) as $n \to \infty$ for all $h \neq 0$. T is of class $C_{oo}$ if $T \epsilon C_{o.} \cap C_{.o}$ and of class $C_{11}$ if
$T \epsilon C_{1.} \cap C_{.1}$. A c.n.u. contraction T is said to be of class $C_o$ if there exists
a non-zero function $u \epsilon H^\infty$ such that $u(T) = 0$. In this case we can choose u to be
a minimal inner function in the sense that u is an inner function such that
$u(T) = 0$ and u divides (in $H^\infty$) every other function $v \epsilon H^\infty$ for which $v(T) = 0$. Such
a function is called a minimal function for T and is denoted by $m_T$. If T is
a c.n.u. weak contraction, then in the previous triangulations $T_o$ is of class
$C_o$ and $T_1$ is of class $C_{11}$, called the $C_o$ part and the $C_{11}$ part of T (cf. [8],
p. 331). Note that in this case we have $H_o \vee H_1 = H$ and $H_o \cap H_1 = \{0\}$ (cf. [8], p.
332). For arbitrary operators T, T' on H, H', respectively, T<T' denotes that
T is a quasi-affine transform of T', that is, there exists a linear, one-to-
one and continuous transformation S from H onto a dense linear manifold in H'
(called quasi-affinity) such that ST=T'S. T and T' are quasi-similar if T<T'
and T'<T.

Our main theorem is the following

THEOREM 1.   Let $T$ be a c.n.u. weak contraction on $H$.   Let $T_0$ and $T_1$ be the $C_0$ part and $C_{11}$ part of $T$.   Then $T$ is quasi-similar to $T_0 \oplus T_1$.

*Proof*: Let $S: H_0 \oplus H_1 \to H$ be defined by $S(h_0 \oplus h_1) = h_0 + h_1$.   Certainly $T$ is a continuous linear transformation.   Since $H_0 \vee H_1 = H$ and $H_0 \cap H_1 = \{0\}$, it is easily seen that $S$ is a quasi-affinity such that $S(T_0 \oplus T_1) = TS$.   Thus $T_0 \oplus T_1 \prec T$.   Note that $T^*$ is also a c.n.u.   weak contraction and $T_0'^*$ and $T_1'^*$ are the $C_0$ and $C_{11}$ parts of $T^*$ (cf. [8], p. 332).   As above, we have $T_0'^* \oplus T_1'^* \prec T^*$.   Hence $T \prec T_0' \oplus T_1'$, and $T_0 \oplus T_1 \prec T \prec T_0' \oplus T_1'$.   Let $V$ be the quasi-affinity from $H_0 \oplus H_1$ to $H_0' \oplus H_1'$ such that $V(T_0 \oplus T_1) = (T_0' \oplus T_1')V$.   Since $T_0$ and $T_1'$ are of class $C_{0.}$ and $C_{1.}$, respectively, it is easily seen that $VH_0 \subseteq H_0'$.   Say,

$$V = \begin{pmatrix} V_0 & Z \\ 0 & V_1 \end{pmatrix}$$

is the corresponding triangulation.   An easy calculation shows that $ZT_1 = T_0' Z$. Since $T_1$ is of class $C_{11}$ and $T_0'$ is of class $C_{00}$, we must have $Z = 0$.   Thus $V_0$ and $V_1$ are quasi-affinities satisfying $V_0 T_0 = T_0' V_0$ and $V_1 T_1 = T_1' V_1$.   Hence $T_0 \prec T_0'$ and $T_1 \prec T_1'$.   It follows from the uniqueness of the Jordan model for $C_0$ contractions that $T_0$ and $T_0'$ are quasi-similar to each other (cf. [2]).   To show that $T_1$ is quasi-similar to $T_1'$, note that $T_1$ and $T_1'$, being $C_{11}$ contractions, are quasi-similar to unitary operators, say $U_1$ and $U_1'$, respectively.   We have $U_1 \prec U_1'$.   By a theorem of Douglas [4], $U_1$ and $U_1'$ are unitarily equivalent.   Hence $T_1$ is quasi-similar to $T_1'$, and $T$ is quasi-similar to $T_0 \oplus T_1$.

An immediate corollary of Theorem 1 is

COROLLARY 1.   Let $T_1$ and $T_2$ be c.n.u.   weak contractions.   Then $T_1$ and $T_2$ are quasi-similar to each other if and only if their $C_0$ parts are quasi-similar and their $C_{11}$ parts are quasi-similar to each other.

*Proof*: The sufficiency follows immediately from Theorem 1.   The necessity can be proved by a similar argument as in Theorem 1.

In particular, for c.n.u. contractions with scalar-valued characteristic functions, we have

COROLLARY 2. For $j=1,2$, let $T_j$ be a c.n.u. contraction with the scalar-valued characteristic function $\psi_j \not\equiv 0$. Let $\psi_j = \psi_{ji} \psi_{je}$ be the canonical factorization into the product of its inner part $\psi_{ji}$ and outer part $\psi_{je}$, and let $E_j = \{e^{it} : |\psi_j(e^{it})| < 1\}$. Let $T_j = \begin{bmatrix} T_{j1} & X_j \\ 0 & T_{j2} \end{bmatrix}$ be the triangulation of type $\begin{bmatrix} C_{.1} & * \\ 0 & C_{.o} \end{bmatrix}$, $j=1,2$. Then the following are equivalent to each other:

(i)     $T_1$ is quasi-similar to $T_2$;

(ii)    $T_{11}$ is quasi-similar to $T_{21}$ and $T_{12}$ is unitarily equivalent to $T_{22}$;

(iii)   $\psi_{1i} = \psi_{2i}$ and $E_1$ and $E_2$ differ by a set of zero Lebesque measure.

*Proof*: Since $T_1$ and $T_2$ are c.n.u. weak contractions, the equivalence of (i) and (ii) follows from Corollary 1. Note that $T_{j1}$ is quasi-similar to the multiplication by $e^{it}$ on the space $L^2(E_j)$ and $T_{j2}$ is unitarily equivalent to the compression of the shift $S(\psi_{ji})$ on $H^2 \ominus \psi_{ji} H^2$, $j=1,2$. Thus the equivalence of (ii) and (iii) follows immediately.

The equivalence of (i) and (iii) in Corollary 2 is compatible with the result of Kriete [6] that $T_1$ is similar to $T_2$ if and only if $\frac{\psi_1}{\psi_2}, \frac{\psi_2}{\psi_1} \in H^\infty$ and $E_1$ and $E_2$ differ by a set of zero Lebesque measure.

COROLLARY 3. Let $T_1$ and $T_2$ be c.n.u. weak contractions. If $T_1$ and $T_2$ are quasi-similar to each other, then $\sigma(T_1) = \sigma(T_2)$.

*Proof*: For $j=1,2$, let $T_{jo}$ and $T_{j1}$ be the $C_o$ part and $C_{11}$ part of $T_j$. By Corollary 1, $T_{10}$ and $T_{11}$ are quasi-similar to $T_{20}$ and $T_{21}$, respectively. Since the spectrum of a $C_o$ contraction is completely determined by its minimal function (cf. [8], p. 126), and $T_{10}$ and $T_{20}$ have the same minimal function, we have $\sigma(T_{10}) = \sigma(T_{20})$.

To show that $\sigma(T_{11}) = \sigma(T_{21})$, let $U_j$ be the residual part of the minimal unitary dilation of $T_{j1}$, $j=1,2$ (cf. [8], p. 61). Note that $T_{j1}$ is quasi-similar to $U_j$ and $\sigma(T_{j1})$ lies entirely on the unit circle (cf. [8], p. 75 and p. 328). It follows that $\sigma(T_{j1}) = \sigma(U_j)$ (cf. [8], p. 311-312). By Douglas' theorem [4], $U_1$ and $U_2$ are quasi-similar implies that they are unitarily equivalent. Thus $\sigma(T_{11}) = \sigma(U_1) = \sigma(U_2) = \sigma(T_{21})$. Since $\sigma(T_j) = \sigma(T_{jo}) \cup \sigma(T_{j1})$ ([8], p. 332), we

have $\sigma(T_1) = \sigma(T_2)$, completing the proof.

We remark that the proof can be modified to show that quasi-similar work contractions (not necessarily c.n.u.) have equal spectra. This result is not new. It also follows from the facts that weak contractions are decomposable [5] and quasi-similar decomposable operators have equal spectra [3]. However our proof seems more direct.

In the remaining part of this note we are concerned with the question of when a c.n.u. weak contracrion is quasi-similar to a normal operator. The next theorem reduces the problem to the $C_o$ part of the c.n.u. weak contraction.

THEOREM 2. Let T be a c.n.u. weak contraction on H. Let $T_o$ be the $C_o$ part of T acting on the subspace $H_o \subseteq H$. Then T is quasi-similar to a normal operator if and only if $T_o$ is.

*Proof*: The sufficiency follows trivially from Theorem 1. To prove the necessity, we may assume that T is quasi-similar to a normal operator N on the space K with $\|N\| \leq \|T\| \leq 1$ (cf. [1], Proof of the sufficiency part of Theorem). Let $K = K_1 \oplus K_2$ be the direct sum of reducing subspaces for N such that $N_1 \equiv N|K_1$ is c.n.u. and $N_2 \equiv N|K_2$ is unitary. Let S be the quasi-affinity from H to K such that $ST = NS$. Since $T_o$ is of class $C_o$ and $N_2$ is of class $C_{11}$, it is easily seen that $SH_o \subseteq K_1$. Note that $\overline{SH}_o$ is an invariant subspace for $N_1$. Let $N_1' = N_1|\overline{SH}_o$. Then $S_1 \equiv S|H_o$ is a quasi-affinity from $H_o$ to $\overline{SH}_o$ satisfying $S_1 T_o = N_1' S_1$. Since $T_o$ is of class $C_o$, so is $N_1'$ (cf. [8], p. 125). By the uniqueness of the Jordan model for $C_o$ contractions, we have that $T_o$ is quasi-similar to $N_1'$ (cf. [2]). Since $N_1'$ is subnormal and $\sigma(N_1')$ has planar area zero (cf. [8], p. 126), it follows from Putnam's theorem [7] that $N_1'$ is normal. This completes the proof.

Notice that Theorem 2 is compatible with the result that T is similar to a normal operator if and only if $T_o$ is similar to a normal operator and $T_1'$ is similar to a unitary operator. This is true even for an arbitrary c.n.u. contraction (cf. [9], Theorem 3).

Since the $C_o$ part of a c.n.u. weak contraction is a $C_o$ contraction, the next theorem furnishes the complete solution to the previously posed question.

THEOREM 3.   Let T be a $C_o$ contraction on the space H with the minimal function $m_T$.   Then T is quasi-similar to a normal operator if and only if $m_T$ is a Blaschke product with simple zeros.

*Proof: Necessity.*   Let T be quasi-similar to the normal operator N on the space K and let S be the quasi-affinity from H to K such that ST=NS.   As before we may assume that $||N|| \leq ||T|| \leq 1$ (cf. [1]).   Now we show that N must be c.n.u. Indeed, for any $k \in K$ and $\varepsilon > 0$, let $h \in H$ be such that $||k-Sh|| < \varepsilon$.   Since $ST^n h = N^n Sh \to 0$ as $n \to \infty$, we have $||N^n Sh|| < \varepsilon$ for all $n \geq N_o$.   Hence $||N^n k|| \leq ||N^n k - N^n Sh|| + ||N^n Sh|| \leq ||N||^n$ $||k-Sh|| + ||N^n Sh|| < \varepsilon + \varepsilon = 2\varepsilon$ for all $n \geq N_o$.   This shows that $N^n k \to 0$ for all $k \in K$   and hence N is c.n.u.   Since N is quasi-similar to a $C_o$ contraction, N is also a $C_o$ contraction with the same minimal function $m_N = m_T$ (cf. [8], p. 125).  Let $m_T = Bs$, where $B(\lambda) = \pi_i \frac{\bar{\lambda}_i}{|\lambda_i|} (\frac{\lambda_i - \lambda}{1 - \bar{\lambda}_i \lambda})^{n_i}$ is a Blaschke product and s is a singular  function.

Note that $\lambda_i$ is a characteristic value of N with index $n_i$ (cf. [8], p. 135). Since N is a normal operator, $n_i = 1$ for all i, that is, each $\lambda_i$ is an eigenvalue of N.   Let $K_i$ be the corresponding eigenspace.   Then $\bigvee_i K_i$ reduces N and the normal operator $N_1 \equiv N | (\bigvee_i K_i)^\perp$ has no eigenvalue.   Hence the minimal function of the $C_o$ contraction $N_1$ must be s (cf. [8], p. 129).   It follows that $\sigma(N_1)$ is contained in the unit circle, and thus $N_1$ is a  unitary  operator.   Since  N  is c.n.u., we must have $(\bigvee_i K_i)^\perp = \{0\}$ and $K = \bigvee_i K_i$.   Hence $m_T = B$  is a Blaschke product with simple zeros (cf.  [8], p. 135).

*Sufficiency.*   Assume that $m_T$ is a Blaschke product with  simple zeros, say, $m_T(\lambda) = \pi_i \frac{\bar{\lambda}_i}{|\lambda_i|} \frac{\lambda_i - \lambda}{1 - \bar{\lambda}_i \lambda}$, where the distinct $\lambda_i$'s satisfy $|\lambda_i| < 1$ and $\sum_i (1 - |\lambda_i|) < \infty$. For each i let $H_i = \{h \in H : (T - \lambda_i)h = 0\}$.   Then $T|H_i$ is a normal operator and the system $\{H_i\}_{i=1}^\infty$ of invariant subspaces satisfies

$$H = H_i \dotplus \bigvee_{j \neq i} H_j \quad \text{for each } i,$$

and

$$\bigcap_i (\bigvee_{j \geq i} H_j) = \{0\}$$

(cf. [8], p. 135 and p. 131).   That is, $\{H_i\}_{i=1}^\infty$ is a basic system of invariant

subspaces for T.  By a result of Apostol [1], T is quasi-similar to a normal operator, completing the proof.

## References

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