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多時間微擾理論及其在非諧和振子之應用

Studies on the Multiple-Scale Perturbation Theory and Its Application to the Anharmonic Oscillator

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ABSTRACT — In this article of the works on the anharmonic oscillator, a multiple-time-scale perturbation method, which is originally introduced by Krylov and Bogoliubov to solve the problems in nonlinear mechanics, is applied. It is seen that the results obtained agree completely with those of Landau and Lifshitz.

The multiple-time-scale perturbation theory (MTSPT) was first introduced by Krylov and Bogoliubov [1] to solve problems in nonlinear mechanics. Frieman [2] and Sandri [3] extended this method and applied it to the kinetic theory of gases and plasmas. Recently, Y. C. Lee [4][5] et al. applied the MTSPT to the spontaneous radiation process. The results obtained by the MTS PT agree completely with the previous results. The main purpose of this paper is to study this powerful method and try to apply it to the familiar mechanic problems.

As we know, in the conventional time-dependent perturbation theory, an expansion is made in powers of the coupling constant ε. While ε itself may be small, it often happens that the nth order term diverges like $\varepsilon^n t^\alpha$ at large time t. When t is small, a few term in the expansion series is sufficient to describe the behavior of the systems, but at large t, an appropriate sum over an infinite number of terms must be carried out in order to obtain a finite result. Since it is usually difficult to sum an infinite number of terms, particularly when it is not even known which terms should be included or left out, it would be advantageous if some other perturbation series can be found, such that the first few orders will be correctly to describe the behavior of the system at large t. In the kinetic theory of gases and plasmas, this was achieved first by recognizing the existence of several distinct time scales in that theory, and then by constructing a MTSPT expansion based on those distinct time scales. In the radiative problems, there exists also two very distinct time scales, one corresponds to the inverse of the frequency of the radiative transition, ω_o^{-1}, the other is the inverse of the radiation line width, γ^{-1}. And $\frac{\gamma}{\omega_o} \sim \varepsilon^2 \left(\frac{e^2}{\hbar c}\right)\left(\frac{v^2}{c^2}\right)$ which is a very small quantity. In fact, it is the smallness of this ratio which renders the application of the MTSPT successful.

For the anharmonic oscillator, the dimensionless equation is given by Equation (1), we notice that if the dimensionless quantites $\frac{\alpha a}{\omega_o^2}$ and $\frac{\beta a^2}{\omega_o^2}$ are very small compared to 1, then the right hand side of the above equation can be neglected and the equation becomes that of a linear harmonic oscillator. Thus, in this case, we can use one expansion parameter essentially. But if the above assumption is not valid, we shall have two expansion parameters instead of one expansion parameter in this problem.

The general equation for the anharmonic oscillator is given by [6] $\frac{d^2x}{dt^2}+\omega_o^2 x=-\alpha x^2-\beta x^3$ where x denotes the displacement of the oscillator from its equilibrium position (x=0).

If we solve the above equation by a perturbation expansion, what is to be the dimensionless expansion parameter? Suppose at t=0, x=a. We can then reduce the above equation to a dimensionless form by setting $T=\omega_o t$ and $\underline{X}=x/a$ so that

$$\frac{d^2\underline{X}}{dT^2} + \underline{X} = -\frac{\alpha a}{\omega_o^2}\,\underline{X}^2 - \frac{\beta a^2}{\omega_o^2}\,\underline{X}^3 \tag{1}$$

As we mentioned previously as $\frac{\beta a^2}{\omega_o^2} : \frac{\alpha a}{\omega_c^2} = \frac{\alpha a}{\omega_o^2}$:1, we just need one expan- [illegible] r essnetially, namely, $\lambda=\frac{\alpha a}{\omega_o^2}$ If the above assumption is not [illegible] have two e[illegible]ns on paramet rs instead of one. Thus we write

$$\frac{d^2x}{dt^2}+\omega_o{}^2x=-\lambda\alpha x^2-\lambda^2\beta x^3;\; x(0)=a,\; \left(\frac{dx}{dt}\right)_{t=0}=0 \tag{2}$$

and shall let $\lambda=1$ at the end of calculation. Let us solve (2) by MTSP method, replace t by τ which is a collection of variables $\tau_o,\tau_1,\tau_2,\ldots$ where $\tau_n=\lambda^n t+C_n$ (with $c_o=0$). namely:

$$\frac{d}{dt}\rightarrow \sum_{n=0}\lambda^n\frac{\partial}{\partial\tau_n};\; \frac{d^2}{dt^2}\rightarrow \sum_{m=k=0}\lambda^{m+k}\frac{\partial^2}{\partial\tau_m\partial\tau_k} \text{ and } x(t)\rightarrow x(\tau)=\sum_{n=0}\lambda^n x_n(\tau)$$

Substituting these into Eq. (2), we thus obtain:

$$\left[\sum_{m=k=0}\lambda^{m+k}\frac{\partial^2}{\partial\tau_m\partial\tau_k}\sum_{n=0}\lambda^n x_n(\tau)\right]+\omega_o{}^2\sum_{n=0}\lambda^n x_n(\tau)=-\lambda\alpha\left[\sum_{m=k=0}\lambda^{m+k}x_m(\tau)\right.$$

$$\left.x_k(\tau)\right]-\lambda^2\beta\sum_{m=n=p=0}\lambda^{m+n+p}x_m(\tau)x_n(\tau)x_p(\tau) \tag{3}$$

When we introduce the variable c_1 into x, i.e. when $x(t)\rightarrow x(t,c_1)$. The differential equation $\partial^2x(t,c_1)/\partial t^2+\omega_o{}^2x(t,c_1)=-\lambda\alpha x^2(t,c_1)-\lambda^2\beta x^3(t,c_1)$ and the initial conditions $x(t=0,c_1=0)=a$; $\frac{\partial x(t,c_1=0)}{\partial t}\Big|_{t=0}=0$ can only determine $x(t,c_1=0)$. In order to specify $x(t,c_1)$ for any $c_1\neq 0$ we need to know, say what $\frac{\partial x(t,c_1)}{\partial c_1}$ is at any t & c_1 i.e. we impose $\partial x(t,c_1)/\partial c_1=f(t,c_1)$ since $x(t,c_1=0)$ is known, we can obtain $x(t,c_1)$ for any c_1 for a given t. Here, $\frac{\partial x(t,c_1)}{\partial c_1}$ is completely at our disposal. Similarly, when $x(t)\rightarrow x(t,c_1,c_2)$. However we may choose to determine $x(t,c_1)$ for $c_1\neq 0$ by imposing conditions on, say, $\frac{\partial^2x(t,c_1)}{\partial c_1{}^2}$ and on $\frac{\partial x(t,c_1)}{\partial c_1}\Big|_{c_1=0}$. Then $\frac{\partial^2x(t,c_1)}{\partial c_1{}^2}$ and $\frac{\partial x(t,c_1)}{\partial c_1}\Big|_{c_1=0}$ are completely at our disposal.

When, say, $\frac{\partial x(t,c_1,c_2)}{\partial c_1}$ is at our disposal, this means that $\frac{\partial x(t,\tau_1,\tau_2)}{\partial\tau_1}\left(\frac{\partial\tau_1}{\partial c_1}\right)_{t,c_2}+\frac{\partial x(t,\tau_1,\tau_2)}{\partial\tau_2}\left(\frac{\partial\tau_2}{\partial c_1}\right)_{t,c_2}$ is at our disposal. But $\left(\frac{\partial\tau_1}{\partial c_1}\right)_{t,c_2}=1$ and $\left(\frac{\partial\tau_2}{\partial c_1}\right)_{t,c_2}=0$, hence we also have $\frac{\partial x(\tau)}{\partial\tau_2}$ at our disposal. Here we have replaced $x(t)$ by $x(t,c_1,c_2,\ldots)$ or $x(\tau_o,\tau_1,\ldots)$ and thus the initial conditions are: $x_o(\tau_n=0)+\lambda x_1(\tau_n=0)+\lambda^2x_2(\tau_n=0)=a$ and $\left(\frac{\partial x(\tau)}{\partial\tau_o}+\frac{\partial x(\tau)}{\partial\tau_1}\lambda+\lambda^2\frac{\partial x(\tau)}{\partial\tau_2}+\ldots\right)_{\tau_n=0}=0$. Equating the coefficients of each power of λ in Equation (3), we obtain the zeroth order λ

of as: $\frac{\partial^2 x_o(\tau)}{\partial \tau_o^2}+\omega_o^2 x_o(\tau)=0$ and $x_o(\tau_n=0)=a$, therefore

$$x_o(\tau)=a(\tau_1,\tau_2,\ldots)\cos[\omega_o\tau_o+\psi(\tau_1,\tau_2,\ldots)] \quad (4)$$

the 1st order as: $\frac{\partial^2 x_1}{\partial \tau_o^2}+2\frac{\partial^2 x_o}{\partial \tau_o \partial \tau_1}+\omega_o^2 x_1=-\alpha x_o^2$; with (4), therefore $\frac{\partial^2 x_1}{\partial \tau_o^2}+\omega_o^2 x_1 = -2[-\omega_o\frac{\partial a}{\partial \tau_1}\sin(\omega_o\tau_o+\psi)-\omega_o a\frac{\partial\psi}{\partial\tau_1}\cos(\omega_o\tau_o+\psi)]-\alpha a^2\cos^2(\omega_o\tau_o+\psi)$. Thus the term $\frac{\partial}{\partial\tau_o}(\frac{\partial x_o}{\partial\tau_1})$ gives rise to terms proportional to $\cos(\omega_o\tau_o+\psi)$ or to $\sin(\omega_o\tau_o+\psi)$, which yields secular behavior. We thus impose $\frac{\partial x_o(\tau)}{\partial\tau_1}=0$ to prevent from this secular behavior by making use of the τ_1-freedom. Therefor

$$x_o(\tau)=a(\tau_2,\ldots)\cos[\omega_o\tau_o+\psi(\tau_2,\ldots)] \quad (5)$$

and $\frac{\partial^2 x_1}{\partial \tau_o^2}+\omega_o^2 x_1=-\alpha\frac{a^2(\tau_2,\ldots)}{2}[\cos 2(\omega_o\tau_o+\psi(\tau_2,\ldots))+1]$ the particular solution can be obtained

$$x_1(\tau)=-\frac{\alpha a^2(\tau_2,\ldots)}{2\omega_o^2}+\frac{\alpha a^2(\tau_2,\ldots)}{6\omega_o^2}\cos 2[\omega_o\tau_o+\psi(\tau_2,\ldots)] \quad (6)$$

For the 2nd order, with $\frac{\partial x_1}{\partial\tau_1}=0$ & $\frac{\partial x_o}{\partial\tau_1}=0$, we obtain $\frac{\partial^2 x_2}{\partial\tau_o^2}+\omega_o^2 x_2=-2\frac{\partial^2 x_o}{\partial\tau_o\partial\tau_2}-2\alpha x_o x_1-\beta x_o^3$. Substitute (5) (6) into above equation we have: $\frac{\partial^2 x_2}{\partial\tau_o^2}+\omega_o^2 x_2=2\omega_o\frac{\partial a}{\partial\tau_2}\sin(\omega_o\tau_o+\psi)+(2\omega_o a\frac{\partial\psi}{\partial\tau_2}+\frac{5\alpha^2 a^3}{6\omega_o^2}-\frac{3\beta a^3}{4})\cos(\omega_o\tau_o+\psi)-(\frac{\alpha^2 a^3}{6\omega_o^2}+\frac{\beta a^3}{4})\cos 3(\omega_o\tau_o+\psi)$. To avoid secular behavior, we again impose conditions such that $\frac{\partial a}{\partial\tau}=0$ and $2\omega_o a\frac{\partial\psi}{\partial\tau_2}+\frac{5\alpha^2 a^3}{6\omega_o^2}-\frac{3}{4}\beta a^3=0$. Thus $\frac{\partial\psi(\tau_2,\ldots)}{\partial\tau_2}=\frac{3\beta a^2}{8\omega_o}-\frac{5\alpha^2 a^2}{12\omega_o^3}$ or $\psi(\tau_2,\ldots)=\frac{a^2}{24\omega_o}(9\beta-10\frac{\alpha^2}{\omega_o^2})^2\tau_2+\omega_o(\tau_3,\ldots)$. We can set $\psi_o(\tau_3,\ldots)=0$ in view of our initial condition, therefore:

$$\psi(\tau_2,\ldots)=\frac{a^2}{24\omega_o}(9\beta-10\frac{\alpha^2}{\omega_o^2})\tau_2\equiv\Delta\omega\tau_2 \text{ where } \Delta\omega=\frac{a^2}{24_o}(9\beta-\frac{10\alpha^2}{\omega_o^2}), \text{ i.e.}$$

$$x_o(\tau)=a\cos[(\omega_o\tau_o)+\Delta\omega\tau_2] \quad (7)$$

$$x_1(\tau)=-\frac{\alpha a^2}{2\omega_o^2}+\frac{\alpha a^2}{6\omega_o^2}\cos(\omega_o\tau_o+\Delta\omega\tau_2) \quad (8)$$

Then $\frac{\partial^2 x_2}{\partial \tau_o^2} + \omega_o^2 x_2 = -(\frac{\alpha^2 a^3}{6\omega_o^2} + \frac{\beta a^3}{4})\cos 3(\omega_o \tau_o + \psi)$ or

$$x_2(\tau) = \frac{a^3}{16\omega_o^2}(\frac{\alpha^2}{3\omega_o^2} + \frac{\beta}{2})\cos 3(\omega_o \tau_o + \Delta\omega\tau_2) \quad (9)$$

If we compare Equation (7), (8), (9) with the Equation (28-10), (28-12), (28-14) of reference 6, it is readily to see that the results obtained by the MTSPT agree completely with those of Landau & Lifshitz [7].

The main concept of the MTSPT based based on the many distinct perturbation expansion parameters and the freedom to impose some conditions to avoid the secular behaviors. The freedom of imposing additional conditions can be seen as follows. As in our case the displacement x originally is a single variable function of time t, but as we have seen this function can be replaced by a many-variable function in the language of the MTSPT. As we do so, the equations satisfied by the $x(t,c_1,\ldots)$ is just the original differential equation. It is obviously we don't have sufficient conditions to solve the arbitrary assumed many-variable function $x(t,c_1,\ldots)$. Now since the variables $c_1, c_2,\ldots$ are freely imposed, it is freely to choose the additional conditions which will give the values of the variable $c_1, c_2,\ldots$ and the related function to determine $x(t,c_1,c_2,\ldots)$. This just means that if the original single variable function is a figure in the space x-t. Then it is freely to impose some additional variables c_1, $c_2,\ldots$ to extend $x(t)$ from the one-dimensional space to a many-degree of freedom space under the condition that we must require that $x(t,c_1,c_2,\ldots)$ reduces to the original function $x(t)$ as $c_1, c_2,\ldots$ are all set equal to zero.

The MTSPT has proved by [1]~[5] to be a very useful method in treating the problems in kinetic theory, plasma and radiative process. Our application to the familiar mechanic problems and the agreement we achieved, once more to manifest its importance and powerfulness in the physical system even though the simple and easy problems as ours.

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