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線性時間變數系統的轉換

Transformations of Linear Time-Varying Systems

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Abstract —— This paper presents some results on the tran.formation of a certain class of time-varying systems. It is shown that by successive application of algebraic transformation and $t \leftrightarrow \tau$ transformation, the linear time varying system can be reduced to a linear time-invariant one.

I. Introduction

Recently, Wu has presented some new results on the solution, stability, and the transformation of a class of linear time-varying system [1-4]. It was first shown by Wu [1] that for a specific class of linear time-varying system, the state transition matrix $\Phi(t_1)$ can be computed by the product of $\exp(A_1 t)$ times $\exp(A_2 t)$, where A_1 and A_2 are 2 constant matrices that can be obtained from the system matrix A(t). It was also shown by Wu [3] that the stability of a linear time-varying system can be determined from the joint eigenvalues of A_1 and A_2. Furthermore, by successive application of algebraic transformation and $t \leftrightarrow \tau$ transformation, the linear time-varying system can be reduced to a linear time-invariant system. Wu's results is pleasent, however, he did not consider the case where a time-varying input is applied. This paper extends Wu's results to include input terms in the system equation. It is shown that a class of linear time-varying system with input can also be reduced to a linear time-invariant systems.

II. Main Results

Lemma 1 Let A_1 be an nxn constant matrix, let h(t) be a time function which has h(t), let g(t) and t_o be defined as

$$g(t)=\int_{t_o}^{t} h(e)de \qquad \text{with } g(t_o) = 0 \tag{1}$$

Then

$$A(t)=\exp(A_1 g(t))A_h(t_o)\exp(-A_1 g(t))h(t) \qquad \forall t \tag{2}$$

is the solution of the following differential equation

$$\dot{A}(t)=[h(t)A_1+\frac{\dot{h}(t)}{h(t)}]A(t)-h(t)A(t)A_1 \tag{3}$$

where $A_h(t_o)$ is defined by

$$A_h(t_o)=\lim_{t\to t_o}\frac{A(t)}{h(t)} \tag{4}$$

Lemma 1 can be proved by direct substitution.

Lemma 2 Let A_1 and B_1 be an nxn and nxr constant matrix, respectively. Let h(t) is a time function which has $\dot{h}(t)$. Let g(t) and t_o be defined as (l).

Then

$$B(t) = h(t)\exp[A_1 g(t)]B_1 \tag{5}$$

is the solution of the following matrix differential equation

$$\dot{B}(t) = [h(t)A_1 + \frac{\dot{h}(t)}{h(t)}]\, B(t) \tag{6}$$

Lemma 2 can also be proved by direct substitution. Theorem 1 below shows how to transform a class of linear time varying system into time-invariant one.

Theorem 1 Consider the linear time-varying system

$$\dot{X}(t) = A(t)X(t) + B(t)u(t) \tag{7}$$

If there exists a time function h(t) and 2 constant matrices A_1 and B_1 which satisfies the following equations

$$A_1A(t) - A(t)A_1 = \frac{\dot{A}(t)}{h(t)} - \frac{\dot{h}(t)}{h^2(t)} A(t) \tag{8}$$

and

$$A_1B(t) = \frac{\dot{B}(t)}{h(t)} - \frac{\dot{h}(t)}{h^2(t)} B(t) \qquad \forall t \tag{9}$$

By successive application of algebraic transformation

$$X(t) = T(t)\overline{X}(t) \tag{10}$$

and $t \leftrightarrow \tau$ transformation

$$\tau = g(t) = \int_{t_o}^{t} h(s)ds \quad \text{with } g(t_o) = 0 \tag{11}$$

The system (7) reduced to a linear time-invariant one

$$\dot{Z}(\tau) = A_2Z(\tau) + B_1U(\tau) \tag{12}$$

where $$A_2 + A_1 = A_h(t_o) \tag{13}$$

and $$T(t) = \exp(A_1 g(t)) \tag{14}$$

Proof With the algebraic transformation (10), one obtains

$$\overline{A}(t) = T^{-1}(t)[A(t)T(t) - \dot{T}(t)] \tag{15}$$

$$\bar{B}(t) = T^{-1}(t)B(t) \tag{16}$$

Substituting (14) and (2) into (15) yields

$$\bar{A}(t) = [A_h(t_o) - A_1]h(t)$$

$$= A_2h(t) \tag{17}$$

Substituting (14) and (5) into (16)

$$\bar{B}(t) = B_1h(t) \tag{18}$$

Hence, by $X(t) = T(t)\bar{X}(t)$, system (7) becomes

$$\dot{\bar{X}}(t) = A_2h(t)X(t) + B_1h(t)u(t) \tag{19}$$

by the following $t \leftrightarrow \tau$ transformation

$$Z(\tau) = \bar{X}(t) \tag{20}$$

system (19) becomes

$$\dot{Z}(\tau)\frac{d\tau}{dt} = A_2h(t)Z(\tau) + B_1h(t)u(\tau) \tag{21}$$

or

$$Z(\tau) = A_2Z(\tau) + B_1u(\tau) \tag{22}$$

Q. E. D.

In the following 2 theorems, existence of A_1 and $h(t)$ will be discussed.

Theorem 2 [4] For the system (7), there exists a constant matrix A_1 that satisfied (8) if and only if the n^2 vector $\underline{a}$ formed in terms of elements of $\dot{A}(t)$, $\{\dot{a}_{ij}(t)\}$ as

$$\dot{\underline{a}}^T = [\dot{a}_{11},\dot{a}_{12},\ldots,\dot{a}_{1n},\ldots,\dot{a}_{n1},\ldots,\dot{a}_{nn}\} \tag{23}$$

is linearly dependent on the column vectors of the following Kronecker sum matrix

$$M(t) \triangleq -A(t) \times I + I \times A^T(t) \tag{24}$$

Theorem 3 [4] For the system (7), a necessary condition that there will exist a time function $h(t)$ which satisfies (8) is that the characteristic polynomial of $A(t)$ be of the form

$$\det[SI-A(t)] = S^n + \beta_1 S^{n-1}h(t) + \ldots + \beta_n h^n(t) \tag{25}$$

where β_i S are constants.

Remark:

For a given $A(t)$ and $h(t)$, if the constant matrix A_1 which satisfies (8) exists, then it is not unique [6]. Hence, a suitable A_1 must be choosen that will also satisfy (9).

III. Examples

Example 1. Consider the system

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{6}{t^2} & -\frac{6}{t} \end{bmatrix} X(t) + \begin{bmatrix} \frac{1}{t} \\ \frac{1}{t^2} \end{bmatrix} u(t) \qquad (26)$$

The characteristic polynomial of A(t) is

$$D(S) = \det(SI-A(t)) = S^2 + \frac{6}{t} S + \frac{6}{t^2} \qquad (27)$$

From theorem 3, we know that there exists an h(t).

$$\text{let} \quad h(t) = \frac{1}{t} \qquad (28)$$

$$\text{Hence} \quad g(t) = \int_1^t \frac{1}{S} dS = \ell n\, t \quad (t_o = 1) \qquad (29)$$

By solving (8) with $h(t) = \frac{1}{t}$, we obtain

$$A_1 = \begin{bmatrix} \alpha+1 & 0 \\ 0 & \alpha \end{bmatrix} \qquad (30)$$

where α is arbitrary constant

from (9) and (30), we obtain that

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad (31)$$

Hence

$$A_2 = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \qquad (32)$$

with the following transformation

$$X(t) = \exp[A_1 g(t)]\overline{X}(t)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{t} \end{bmatrix} \overline{X}(t) \qquad (33)$$

we obtain the following

$$\dot{\overline{X}}(t) = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \frac{1}{t} \overline{X}(t) + \frac{1}{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \qquad (34)$$

By applying $t \longleftrightarrow \tau$ transformation with the following form

$$\tau = g(t) = \ell n\, t \qquad (35)$$

The system (34) reduced to

$$\dot{Z}(\tau) = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} Z(\tau) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(\tau) \tag{36}$$

Q. E. D.

Example 2. Consider the following system

$$\dot{X}(t) = \begin{bmatrix} -1 + \frac{1}{2}\cos 2t & 1 - \frac{1}{2}\sin 2t \\ -1 - \frac{1}{2}\sin 2t & -1 - \frac{1}{2}\cos 2t \end{bmatrix} X(t) + \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} u(t) \tag{37}$$

the characteristic polynomial of A(t) is

$$D(S) = \det(SI-A(t)) = S^2 + 2S + \frac{7}{4} \tag{38}$$

from theorem 3, we know that there exists an h(t). For simplicity, we choose

$$h(t) = 1 \tag{39}$$

solving (8) with h(t) = 1 yields

$$A_1 = \begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix} \tag{40}$$

where α is arbitrary constant. In order A_1 must also satisfy (9), α must be zero. Therefor

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{41}$$

and

$$A_2 = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} \end{bmatrix} \tag{42}$$

with the algebraic transformation (10) where

$$T(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \tag{43}$$

the system (37) reduced to

$$\dot{\overline{X}}(t) = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} \end{bmatrix} \overline{X}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \tag{44}$$

IV. Conclusion

A class of linear time-varying system with input terms that can be reduced to a linear time-invariant system by suitablly applying transformation is given. Related problems as controller design and realization problem will be exploited in the future.

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