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Lorentz-Dirac方程式之限制條件

Constraints of the Lorentz-Dirac Equation

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Abstract — Baylis and Huschilt pointed out that the Lorentz-Dirac equation with the usual constraint ($\ddot{x}^{\mu}(\tau) \to 0$ as $\tau \to \infty$) for physical solution, may permit two or more solutions to some problems. We impose another constraint to make the physical solution unique.

I. Introduction

The Lorentz-Dirac equation [1],[2],

$$\dot{u}^{\mu} = \frac{e}{mc} F^{\mu\nu} u_{\nu} + \frac{1}{b}(\ddot{u}^{\mu} - \frac{1}{c^2} u^{\mu} \dot{u}^{\nu} \dot{u}_{\nu}) \tag{1}$$

is usually accepted to describe the classical motion of a charged particle in a force field including the radiative reaction. Here $1/b$ is equal to $2/3(e^2/mc^3)$, u^{μ} is the proper time derivative of the position x^{μ} : $\dot{x}^{\mu} = u^{\mu}$ and $\dot{u}^{\mu} = \frac{du^{\mu}}{d\tau}$. Eq. (1) is a third-order differential equation. For given initial values of $x^{\mu}(0)$ and $\dot{x}^{\mu}(0)$, Eq. (1) has infinitely many solutions. To ensure an acceptable physical solution, one usually regards the initial acceleration $\ddot{x}^{\mu}(0)$ as a parameter and imposes the additional constraint [1]

$$\lim_{\tau \to \infty} \ddot{x}^{\mu}(\tau) = 0 \tag{2}$$

to implicitly determine the initial acceleration parameter $\ddot{x}^{\mu}(0)$ and assume that one and only one such physical solution exists. In [3], Plass made rather extensive studies to determine the physical solution of Eqs. (1) and (2). He concluded that if the force field satisfies some general criteria, then there always exists a unique physical solution to Eqs. (1) and (2) for given initial values of $x^{\mu}(0)$ and $\dot{x}^{\mu}(0)$. He then argued that one should accept the equation of motion including the force of radiative reaction, Eqs. (1) and (2), as an exact equation for a charged point particle within the framework of classical theory. But recently, Baylis and Huschilt [4] pointed out that there are at least two solutions for a specific problem. The second solution of [4] exists only when the distance between the rest particle and the force field region is very small or the force field strength is very strong. Although we can argue that in the case of such a small distance or such a strong field strength, the classical description is no longer valid, we should also note that, as in the case of classical mechanics, when we consider the complete classical equation of motion, we always assume that the equation of motion is valid for all distances no matter how small and for all the force field strengths no matter how strong. We further assume that the solution if it exists is unique, that is, we assume that the mathematical description of classical theory is complete up to any small distance and any strong field strength in spite of the fact that in such a case the classical description is no longer adequate to describe the actual physical situation. Thus we should reexamine the constraint condition to determine whether it is complete or not.

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In section 2, we use a specific problem to demonstrate the following point, which is pointed out first by Baylis and Huschilt in [4]:

"The constraint condition is not complete, that is, for a given initial values of $\chi^\mu(0)$ and $\dot{\chi}^\mu(0)$, Eqs. (1) and (2) can have more than one solution."

After examining these solutions, we find out that the constraint condition can determine only the magnitude but not the direction of the initial acceleration $\ddot{\chi}^\mu(0)$. Thus we suggest in section 3 one more constraint condition. The consequences of these constraints are discussed in section 4.

II. Specific Problem

We consider a particle of mass m and charge e moving in the electric field $\vec{E}(\vec{r})$,

$$\vec{E}(\vec{r}) = \begin{cases} 0 & , 0 \leq r < r_1 \\ E_0 \dfrac{\vec{r}}{r} & , r_1 \leq r \leq r_2 \\ 0 & , r_2 < r \end{cases} \tag{3}$$

which may be regarded as the field created by an ideal spherical capacitor. In this case, Eq. (1) becomes

$$\dot{\vec{u}} = \frac{e}{m}\left(1+\frac{u^2}{c^2}\right)^{1/2} \vec{E}(\vec{r}) + \frac{1}{b}\left\{\ddot{\vec{u}} - \frac{\vec{u}}{c^2}\left[\dot{\vec{u}}\cdot\dot{\vec{u}} - \frac{\frac{1}{c^2}(\vec{u}\cdot\dot{\vec{u}})^2}{1+\frac{u^2}{c^2}}\right]\right\} \tag{4}$$

where $u^2 = \vec{u}\cdot\vec{u}$ and $\vec{u} = \dfrac{d\vec{r}(\tau)}{d\tau}$.

Consider the following initial conditions

$$\vec{r}(0)=0 \ , \quad \dot{\vec{r}}(0)=\vec{u}(0)=0 \ . \tag{5}$$

The obvious physical solution of Eqs. (2) and (4) is $\vec{r}(\tau)=0$. But if we assume the initial acceleration $\dot{\vec{u}}(0)$ to be

$$\dot{\vec{u}}(0) = a\vec{b} \tag{6}$$

where $a > 0$, and $\vec{b}\cdot\vec{b} = 1$, then we can integrate Eq. (4) step by step and easily see that

$$\vec{r}(\tau) = r(\tau)\,\vec{b}$$

Eq. (4) becomes

$$\dot{u} = \frac{e}{m}\left(1+\frac{u^2}{c^2}\right) E_0 + \frac{1}{b}\left[\ddot{u} - \frac{u}{c^2}\,\frac{\dot{u}^2}{1+\frac{u^2}{c^2}}\right] , \quad (r_1 \leq r \leq r_2) \tag{7}$$

This nonlinear differential equation can be solved by introducing the new dependent variable $\omega(\tau)$ define by the equation

$$u(\tau) = c \sinh\left(\frac{\omega(\tau)}{c}\right)$$

It can be proved [4] that if the distance r_1 is sufficiently small or the field strength E_0 is sufficiently large, Eqs. (7) and (2) give us a physical solution with the initial acceleration $a > 0$. The unit vector $\vec{b}$ can be any unit vector, this means that for this specific problem, we have infinitely many physical solutions so long as r_1 is sufficiently small or E_0 is sufficiently large.

Now let us consider the following initial conditions

$$\vec{r}(0)=0 \quad , \quad \vec{u}(0)=u_0\,\vec{e}_1 \tag{8}$$

where $u_0>0$, and $\vec{e}_1$ is a unit vector. If we assume the initial acceleration $\dot{\vec{u}}(0)$ to be

$$\dot{\vec{u}}(0)=a_1\vec{e}_1+a_2\vec{e}_2$$

where $\vec{e}_1\cdot\vec{e}_2=0$ and $\vec{e}_2\cdot\vec{e}_2=1$. Then the motion of the charged particle will be restricted in the $(\vec{e}_1,\vec{e}_2)$ plane. if u_0 is sufficiently small, we have good reason to believe that Eqs. (7) and (2) have solution for a wide range of $\dot{\vec{u}}(0)$. For the special case $a_2=0$, it can be proved that for sufficiently small u_0, Eqs. (7) and (2) have two solutions, one with $a_1>0$ and the other with $a_1<0$.

III. Additional Constraint

From the analysis of section 2, we see the following facts:

(a) In general, the LD equation, Eq. (1), restricted by the condition, Eq. (2), has more than one physical solution for given initial position and velocity.

(b) Among the physical solutions for given initial position and velocity, every solution corresponds to a direction of the initial acceleration $\dot{\vec{u}}(0)$.

(c) The constraint condition, Eq. (2), seems to determine only the magnitude but not the direction of the initial acceleration.

In order to make the LD equation a complete mathematical description, we can assume one more constraint to the LD equation. We suggest that the following additional constraint should be added to the LD equation.

"The direction of the initial acceleration $\dot{\vec{u}}(0)$ should be in the direction of $\dot{\vec{u}}_\infty(\tau_0)$, that is, $\dot{\vec{u}}(0)=\alpha\,\dot{\vec{U}}_\infty(\tau_0)$ with $\alpha>0$, where $\vec{u}_\infty(\tau)$ is the classical solution without radiative reaction, i.e., solution corresponding to the case $b\rightarrow\infty$, and $\tau_0=\min\{\tau \mid \dot{\vec{u}}_\infty(\tau)\neq 0\}\geqslant 0$. If $\tau_0=\infty$, then $\dot{\vec{u}}(0)$ should be zero." (9)

This additional constraint is very reasonable, for (1) it is a covariant constraint, (2) it is sufficient to isolate one solution for the specific problem discussed in section 2 and (3) the particle only begins accelerating noticeably over a time interval of the order b^{-1} <u>before</u> the force is applied and b^{-1} is very small.

Now let us consider the specific problem discussed in section 2. After introducing the additional constraint, Eq. (9), the problem becomes to solve the differential equation, Eq. (4), with constraint conditions, Eqs. (2) and (9).

Consider the case of the initial conditions, Eq. (5), then the exact solution for the classical motion without radiative reaction is

$$\vec{\gamma}_\infty(\tau)=0 \quad , \quad \vec{u}_\infty(\tau)=0 \quad , \quad \dot{\vec{u}}_\infty(\tau)=0 \; .$$

Thus the initial acceleration $\dot{\vec{u}}(0)$ should be zero according to the constraint condition, Eq. (9), and the unique physical solution is

$$\vec{\gamma}(\tau)=0 \quad , \quad \vec{u}(\tau)=0 \quad , \quad \dot{\vec{u}}(\tau)=0 \; .$$

The other solution discussed in section 2 become nonphysical solutions. We will come back to this point later.

Consider the case of initial conditions, Eq. (8), the classical solution without radiative reaction can be easily obtained and

$$\vec{u}_\infty(\tau_o) = \frac{c}{m}\left(1+\frac{u_o^2}{c^2}\right)^{1/2} E_o \vec{e}_1$$

Thus the initial acceleration $\dot{\vec{u}}(0)$ should be

$$\dot{\vec{u}}(0) = a_1 \vec{e}_1 \quad , \quad a_1 > 0$$

The problem becomes an one-dimensional problem. It can be proved that only one solution exists.

IV. Discussion

(a) For a given initial velocity and inital position, the LD equation, Eq. (1), with the constraint condition, Eq. (2), does not given us a unique physical solution in general. The constraint condition, Eq. (2), seems to determine only the magnitude but not the direction of the initial accerleration.

(b) If we impose another constraint condition, Eq. (9), which can be used to determine the direction of the initial acceleration, we find out that it is sufficient to isolate one solution among the many physical solutions in the problem discussed in section 2.

(c) The additional constraint, Eq. (9), is a covariant constraint.

(d) After imposing the constraint, Eq. (9), the unique solution under the initial conditions, Eq. (5), is $\vec{r}(\tau)=0$. The other nonphysical solutions correspond to the limit solutions when $\dot{\vec{u}}(0) \to 0$. From this discussion, we know that the limit solutions may exist, but they are now not physical solutions. This means that if we write the unique solution as

$$\vec{r}(\tau\,;\vec{r}(0);\dot{\vec{u}}(0))$$

then the limit solution

$$\lim_{a\to 0+} \vec{r}(\tau\,;\vec{r}(0);\, a\vec{b})$$

may exist, but is not equal to $\vec{r}(\tau\,;\vec{r}(0);0)$. For the specific problem discussed in section 2, we have,

$$\lim_{a\to 0+} \vec{r}(\tau\,;0;a\vec{b})$$

exists but is not equal to $\vec{r}(\tau\,;0\,;0)$, this is, $\vec{r}(\tau\,;0;\dot{\vec{u}}(0))$ is not continuous at $\dot{\vec{u}}(0) = 0$.

(e) It is still not an easy problem to prove that the LD equation, Eq. (1), with constraint conditions Eqs. (2) and (9), does have unique solution for general force field.

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