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# 規範場的量子化

# Quantization of Gauge Field

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**Abstract** –– Following Dirac's approach, we treat the interaction of the material field with the gauge field as a constrained systemand give the quantization rules for all field operators. We then show that these commutation rules lead to the Poincare algebra of four momenta and angular momenta and also give the geometric meaning of these commutation rules.

## Introduction

It is well known that in the canonical quantization procedure of electromagnetic field the difficulty we met is that the momentum cannonically conjugate to the $A^o$ vanishes identically, so that the Hamiltonian theory must be amended. It is originated from the use of more variables than there are independent degrees of freedom. The same situation occures in the gauge field theory as can easily be seen by the fact that electromagnetic field is a special case of it. There are so many efforts in trying to solve this quantization problem of "singular" Lagrangian system. One example is the direct path integral method (Fadeev and Popov, [8], [ 9 ]) for developing a Feynaman diagram expansion consisting with the constraints. Another popular procedure of finding a suitable canonical quantization scheme for Yang-Mills field is well known to be quite complex (Schwinger, [11], [12]; Mohapatra, [10]).

The problem of finding a consistent classical Hamiltonian dynamics corresponding to a singular Lagrangian system was apparently attacked first by Dirac ([3]). Subsequently Dirac ([4]), Anderson and Bergmann ([1]), Bergmann and Goldberg ([2]), and Dirac ([5]) refined Dirac's original methods. An expanded treatment of the general constrained Hamiltonian system appears in Dirac's lectures on quantum mechanics ([6]); see also Dirac ([7]). A more systematic treatment was given by Hanson, Regge and Teitelbolm, recently.

Before we try to quantize the field and gauge field operators by Dirac's methods, we should figure out some important physical quantities such as the four momenta and angular momenta. The zero-th component of the four momenta is then regared as the Hamiltonian of the system which is important in canonical quantization procedure. We use the concept of parallel translation to generalize the Nother's theory in connection space and then obtain the four momentums and angular momentums in an unambiguous way. After that, we transform the Lagrangian equations into Hamiltonian equations and constrained equations:

$$F^{oi}_{a|i} = \gamma^o_a$$

which contain no time derivatives. We then prove that these constrained equations are just the initial conditions of the dynamical equations and the varables $A^o_a$ are the Lagrange multipliers. From this we introduce the concept of "weak equality" of Dirac and the Possion brackets. These commutation rules are simpler than any got before and, furthermore, it is got before we take any gauge, so it is gauge independent. These Possion brackets will be checked to be consistent by the Lie algebra of the four momenta and angular momenta as Poincare algebra. All the commutation rules will also be given their geometric meaning.

## I. Lagrangian Formulism

Let us consider a field system q(x) with Lagrangian density

$$\mathcal{L}=\mathcal{L}\,(q^A, q^A_{\ \ |\mu}), \tag{1}$$

which is a scalar under local gauge transformation

$$q^A(x)\to q^A(x)+\epsilon^a(x)\, T^A_{(a)B}\, q^B(x)\,. \tag{2}$$

Since $\mathcal{L}$ is a gauge scalar, so we demand

$$\frac{\partial\mathcal{L}}{\partial q^A}\to\frac{\partial\mathcal{L}}{\partial q^A}-\epsilon^a(x)\, T^B_{(a)A}\,\frac{\partial\mathcal{L}}{\partial q^B}\,, \tag{3}$$

under the same local gauge transformation, i.e., $\frac{\partial\mathcal{L}}{\partial q^A}$ transforms like a contravariant gauge quantity. It is easy to see, if we define the covariant derivative of a contravariant quantity $\pi_A(x)$ by

$$\pi_{A|\mu}=\pi_{A;\mu}+A^a_\mu\, T^B_{(a)A}\,\pi_B\,, \tag{4}$$

$\pi_{A|\mu}$ will have the same transformation as (3) under local gauge transformation.

The Euler-Lagrange equations of (1) are

$$\frac{\partial\mathcal{L}}{\partial q^A}-\frac{\partial\mathcal{L}}{\partial q^B_{\ |\mu}}\,A^a_\mu\, T^B_{(a)A}-\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial q^A_{\ |\mu}}\Big)=0\,,$$

which can be written as

$$\frac{\partial\mathcal{L}}{\partial q^A}-\Big(\frac{\partial\mathcal{L}}{\partial q^A_{\ |\mu}}\Big)_{|\mu}=0\,. \tag{5}$$

(5) are the covariant forms of field equations.

Define $\pi^\mu_A\equiv\frac{\partial\mathcal{L}}{\partial q^A_{\ |\mu}}$, and we assume that these $\pi$'s exist for all $q^A$, then $\pi^0_A$ are the conjugate variables of $q^A$. The currents which correspond to the local gauge transformation are

$$\mathcal{J}^\mu_a\equiv-\pi^\mu_A\, T^A_{(a)B}\, q^B\,, \tag{6}$$

and their covariant derivatives vanish in virtue of the equations of motion and gauge invariance:

$$\mathcal{J}^\mu_{a|\mu}\equiv\mathcal{J}^\mu_{a;\mu}+A^b_\mu\, f_b{}^c{}_a\,\mathcal{J}^\mu_c=0\,. \tag{7}$$

Two covariant differentiations do not in general commute. One finds

$$q^A_{\ |\mu|\nu}-q^A_{\ |\nu|\mu}=F^a_{\mu\nu}\, T^A_{(a)B}\, q^B\,. \tag{8}$$

It remains to find a free Lagrangian $\mathcal{L}_0$ for the gauge field. Clearly $\mathcal{L}_0$ must be separably invariant, and it is easy to see that this implies that it must contain $A^a_\mu$ only through the covariant combination $F^a_{\mu\nu}$. The simplest such Lagrangian is

$$\mathcal{L}_0 = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a , \tag{9}$$

where the tensor indices are raised with flat-space metric $\eta^{\mu\nu}$ with diagonal elements (1, -1, -1, -1) and the index a is lowered with the metric

$$g_{ab} = f_a{}^c{}_d \, f_c{}^d{}_b , \tag{10}$$

associated with the Lie group (except of course for a one-parameter group). With the choice (9) of $\mathcal{L}_0$, the equations of motion for the gauge fields are

$$F^{\mu\nu}_a{}_{|\nu} = \mathcal{J}^\mu_a . \tag{11}$$

Because of the antisymmetry of $F^{\mu\nu}_a$ one can define another current which is conserved in strict sense:

$$(\mathcal{J}^\mu_a + j^\mu_a)_{;\mu} = 0 , \tag{12}$$

where $$j^\mu_a \equiv -A^b{}_\nu \, f_b{}^c{}_a \, F^{\mu\nu}_c . \tag{13}$$

This extra current $j^\mu_a$ may be regarded as the current of the gauge fields itself. Note, however, that is not a covariant quantity. To obtain a strict conservation law one must sacrifice the covariance of the current.

$F^a_{\mu\nu}$ satisfy Bianchi identity

$$F^a_{\mu\nu|\lambda} + F^a_{\nu\lambda\,|\mu} + F^a_{\lambda\mu\,|\nu} = 0 . \tag{14}$$

We also can define the conjugate variables of $A^a_i$ to be $\dfrac{\partial \mathcal{L}_0}{\partial A^a_{i;0}}$ which are equal to $-F^{0i}_a$. There are no conjugate variable of $A^a_0$.

## II. Energy-momentum Tensor and Angular Momentum Tensor

As explained in Section 1, we are now dealing with a physical system $\mathcal{L}$ on a connection space M with connection $A^a_\mu$. The basic space-time M is no longer a simple Minkowski space and must be regarded as a generalized "curved" space with curvature $F^a_{\mu\nu}$. When we study the relationship between the values of a field at different points, the concept of parallel displacement must be extensively used. But the system still possesses many invariant properties which are treated on the flat space-time. We are going to investigate the invariant properties and their conserved quantities. In some sense, the following description can be regarded as a generalized Nother's theory in "curved" space.

The first invariance is the translation invariance. Under an infinitesimal translation

$$x'^\mu = x^\mu + \epsilon^\mu , \tag{15}$$

the Lagrangian changes by an amount

$$\delta \mathcal{L} = \epsilon^\mu \frac{\partial \mathcal{L}}{\partial x^\mu} \tag{16}$$

On the other hand, if $\mathcal{L}$ is translationally invariant, it has no explicit coordinate dependence and

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial q^A} \delta q^A + \frac{\partial \mathcal{L}}{\partial q^A{}_{|\mu}} \delta q^A{}_{|\mu} \tag{17}$$

We must be careful to find $\delta q^A$: $q^A(x+\epsilon)$ and $q^A(x)$ are vectors belonging to different vector spaces, one in $F_{x+\epsilon}$ and the other in $F_x$, so their direct difference has no meaning at all. We should parallelly displace the vector $q^A(x)$ to the point $x+\epsilon$ along the translation first and then to find the difference. The vector obtained by parallel displacement of $q^A(x)$ along translation $x \to x+\epsilon$ has components

$$q^A(x) + A^a_\mu T^A_{(a)B}\, q^B(x)\, \epsilon^\mu . \tag{18}$$

Hence

$$\begin{aligned} \delta q^A &= q^A(x+\epsilon) - (q^A(x) + A^a_\mu T^A_{(a)B}\, q^B\, \epsilon^\mu) \\ &= q^A{}_{|\mu}\, \epsilon^\mu , \end{aligned} \tag{19}$$

Similarly

$$\delta q^A{}_{|\nu} = q^A{}_{|\nu|\mu}\, \epsilon^\mu \tag{20}$$

Substituting (19) and (20) into (17), we have

$$\begin{aligned} \delta \mathcal{L} &= \epsilon^\nu \frac{\partial \mathcal{L}}{\partial x^\nu} \\ &= \epsilon^\mu \Big( \frac{\partial \mathcal{L}}{\partial q^A} q^A{}_{|\mu} + \frac{\partial \mathcal{L}}{\partial q^A{}_{|\nu}} q^A{}_{|\nu|\mu} \Big) \\ &= \epsilon^\mu \Big( \Big( \frac{\partial \mathcal{L}}{\partial q^A{}_{|\nu}} \Big)_{|\nu} q^A{}_{|\mu} + \frac{\partial \mathcal{L}}{\partial q^A{}_{|\nu}} q^A{}_{|\mu|\nu} \\ &\quad + \frac{\partial \mathcal{L}}{\partial q^A{}_{|\nu}} F^a_{\nu\mu}\, T^A_{(a)B}\, q^B \Big) \\ &= \epsilon^\mu \Big( \frac{\partial \mathcal{L}}{\partial q^A{}_{|\nu}} q^A{}_{|\mu} \Big)_{;\nu} - \epsilon^\mu F^a_{\nu\mu}\, j^\nu_a \\ &= \epsilon^\mu \Big( \frac{\partial \mathcal{L}}{\partial q^A{}_{|\nu}} q^A{}_{|\mu} \Big)_{;\nu} - \epsilon^\mu F^a_{\nu\mu}\, F^{\nu\lambda}_{a\,|\lambda} , \end{aligned} \tag{21}$$

where we have used the equations of motion (5), (11) and the identity (8).

*Lemma 1* $\quad F^a_{\nu\mu}\, F^{\nu\lambda}_{a\,|\lambda} = -(F^a_{\nu\mu}\, F^{\lambda\nu}_a - \delta^\lambda_\mu \mathcal{L}_0)_{;\lambda}\, .$ (22)

*Proof:*

$$F^a_{\nu\mu|\lambda}\, F^{\nu\lambda}_a = (F^a_{\nu\mu}\, F^{\nu\lambda}_a)_{|\lambda} - F^a_{\nu\mu|\lambda} F^{\nu\lambda}_a .$$

Using Bianchi identity (14), we have

$$F^a_{\nu\mu|\lambda}F^{\nu\lambda}_a = -F^a_{\mu\lambda|\nu}\ F^{\nu\lambda}_a + F^a_{\nu\lambda|\mu}\ F^{\nu\lambda}_a$$

$$= -F^a_{\nu\mu|\lambda}\ F^{\nu\lambda}_a + \frac{1}{2}\ (F^{\ a}_{\nu\lambda}\ F^{\nu\lambda}_a).$$

Hence

$$F^a_{\nu\mu|\lambda}\ F^{\nu\lambda}_a = \frac{1}{4}(F^{\ a}_{\nu\lambda}\ F^{\nu\lambda}_{\ a})_{|\mu}\ ,$$

and finally we have

$$F^a_{\nu\lambda}\ \ F^{\nu\lambda}_{\ a|\lambda} = (F^a_{\nu\mu}\ \ F^{\nu\lambda}_a)_{;\lambda}\ \ -\frac{1}{4}(F^{\ a}_{\nu\sigma}\ \ F^{\nu\sigma}_{\ a})$$

$$= -(F^a_{\nu\mu}\ F^{\lambda\nu}_a - \delta^{\lambda}_{\mu}\mathcal{L}_o)_{;\lambda}\ .$$

q.e.d.

By using this lemma, (21) can be written as

$$\epsilon^{\nu}\ \frac{\partial\mathcal{L}}{\partial x^{\nu}} = \epsilon^{\mu}\ (\frac{\partial\mathcal{L}}{\partial q^{A}_{\ |\nu}}\ q^{A}_{\ |\mu}\ )_{;\nu} + \epsilon^{\mu}\ (F^a_{\lambda\mu}\ \ F^{\nu\lambda} - \delta^{\ \nu}_{\mu}\ \ \mathcal{L}_o)_{;\ \nu}\ .$$

So the quantity $\mathcal{J}^{\nu}_{\mu}\ + t^{\ \nu}_{\ \mu}$ , where

$$\mathcal{J}^{\nu}_{\mu} = \frac{\partial\mathcal{L}}{\partial q^{A}_{\ |\nu}}\ q^{A}_{\ |\mu}\ - \delta^{\nu}_{\mu}\ \mathcal{L}\ , \tag{23}$$

and $$t^{\nu}_{\ \ \mu} = F^a_{\lambda\mu}\ \ F^{\nu\lambda}_a - \delta^{\ \nu}_{\mu}\mathcal{L}_o\ , \tag{24}$$

satisfies $$(\mathcal{J}^{\nu}_{\mu}\ + t^{\nu}_{\ \mu}\ )_{;\ \nu} = 0\ . \tag{25}$$

We interpret $\mathcal{J}^{\nu}_{\ \mu}$ to be energy-momentum tensor of field and $t^{\nu}_{\mu}$ to be energy-momentum tensor of guage field. The conserved quantitues are

$$P_{\mu} + \rho_{\ \mu} = \int \mathcal{J}^{o}_{\mu}\ d^3x + \int\ t^{o}_{\mu}\ \ d^3x\ ,$$

which are total energy and momentum of the system.

The $\mathcal{J}^{o}_{\ o}$ and $t^{o}_{\ o}$

$$\mathcal{H} \equiv \mathcal{J}^{o}_{\ o} = \Gamma^{o}_{\ A}\ q^{A}_{\ \ |o} - \mathcal{L}\ , \tag{26}$$

$$\mathcal{K} \equiv t^{o}_{\ o} = F^{oi}_{a}\ F_{io} - \mathcal{L}_o\ , \tag{27}$$

called hamiltonian density of field and guage field, respectively, will play an important role in the following sections.

In a similar way we construct the analogous statement for invariance of this system under rotation

$$x'^{\nu} = x^{\nu} + \epsilon^{\nu\mu}\ x_{\mu}\ , \qquad \epsilon^{\nu\mu} = -\ \epsilon^{\mu\nu}\ . \tag{28}$$

As above, we have

$$\delta q^A = \frac{1}{2}\epsilon_{\sigma\lambda}(x^\lambda q^{A|\sigma} - x^\sigma q^{A|\lambda} + \Sigma^{\lambda\sigma}_{(A)} q^A), \tag{29}$$

$$\delta q^A_{\ |\mu} = \frac{1}{2}\epsilon_{\sigma\lambda}(x^\lambda q^{A\ |\sigma}_{\ |\mu} - x^\sigma q^{A|\lambda}_{|\mu} + \Sigma^{\lambda\sigma}_{(A)} q^A + \Omega^{\lambda\sigma\nu}_{\ \ \ \mu} q^A_{\ \nu}), \tag{30}$$

where $\Sigma^{\lambda\sigma}_{(A)}$ are the infinitismal rotation operators of $q^A$ (For example, if $q^A$ is a Dirac field, then $\Sigma^{\lambda\sigma}_{(A)} = \frac{1}{8}[\gamma^\lambda, \gamma^\sigma]$.) and

$$\Omega^{\lambda\sigma\nu}_{\ \ \ \mu} = \delta^\lambda_\mu \eta^{\sigma\nu} - \delta^\sigma_\mu \eta^{\lambda\nu}, \tag{31}$$

are infinitesmal operators of Lorentz group. Now from

$$\delta \mathcal{L} = \epsilon^{\mu\nu} x_\nu \frac{\partial \mathcal{L}}{\partial x^\mu}$$

$$= \frac{1}{2}\epsilon_{\sigma\lambda}\frac{\partial}{\partial x^\mu}((g^{\mu\nu} x^\lambda - g^{\mu\lambda} x^\sigma)\mathcal{L})$$

$$= \frac{\partial \mathcal{L}}{\partial q^A}\delta q^A + \frac{\partial \mathcal{L}}{\partial q^A_{|\mu}} q^A_{\ |\mu},$$

we find

$$(\mathcal{M}^{\mu\lambda\nu} + \eta^{\mu\lambda\nu})_{;\nu} = 0,$$

where

$$\mathcal{M}^{\mu\lambda\nu} \equiv x^\lambda \mathcal{J}^{\mu\nu} - x^\nu \mathcal{J}^{\mu\lambda} + \Pi^\mu_A \Sigma^{\lambda\nu}_{(A)} q^A, \tag{32}$$

$$\eta^{\mu\lambda\nu} \equiv x^\lambda t^{\mu\nu} - x^\nu t^{\mu\lambda}, \tag{33}$$

are the angular momentum tensors of field and gauge field, respectively. The conserved quantities are

$$M^{\lambda\nu} + \mathcal{M}^{\lambda\nu} \equiv \int \mathcal{M}^{o\lambda\nu} d^3x + \int \eta^{o\lambda\nu} d^3x, \tag{34}$$

which are the total angular momentum of the system.

From (33), it seems that the gauge field is a spin zero field. But it is not. We will show, after we define the commutation relations, that gauge field is still a spin one field.

The energy-momentum tensors $\mathcal{J}^\nu_\mu$ and $t^\nu_\mu$ are gauge scalars separately. It is different from the energy-momentum tensors we will get if we use the ordinary Nother theorem directly instead of the techniques of parallel displacement (see appendix A). In the latter case the sum of energy-momentum tensors of field and guage field is a gauge scalar but not separately. But from the physical point of view, we belive that the result we get here is a resonable one.

## III. Hamiltonian Formulism

We have got the hamiltonian density of field, that is

$$\mathcal{H} = \pi^o_A \, q^A{}_{|o} - \mathcal{L} \; . \tag{35}$$

The same as in the classical mechanics, $\mathcal{H}$ is a function of $q^A(x)$, $q^A{}_{|i}(x)$ and $\pi^o_A(x)$, so the hamiltonian

$$H = H\,[q^A(x), \pi^o_A(x)] = \int d^3x \, \mathcal{H}(q^A(x), q^A{}_{|i}(x), \pi^o_A(x)), \tag{36}$$

is a functional of $q^A(x)$ and $\pi^o_A(x)$. From the definition of $\pi^\mu_A(x)$ and the equations of motion (5), we have the following set of equations

$$\frac{\partial \mathcal{H}}{\partial q^A} = - \frac{\partial \mathcal{L}}{\partial q^A} = - \pi^\mu{}_{A|\mu} \; ,$$
$$\frac{\partial \mathcal{H}}{\partial \pi^o_A} = q^A{}_{|o} \; , \tag{37}$$
$$\frac{\partial \mathcal{H}}{\partial q^A{}_{|i}} = - \frac{\partial \mathcal{L}}{\partial q^A{}_{|i}} = - \pi^i{}_A \; .$$

Using the hamiltonian H, we can change (37) into equations more like hamiltonian equations

$$\frac{\delta H}{\delta q^A(\vec{x})} = \frac{\partial \mathcal{H}}{\partial q^A} - \left( \frac{\partial \mathcal{H}}{\partial q^A{}_{|i}} \right)_{|i} = - \pi^o{}_{A|o} \; ,$$
$$\frac{\delta H}{\delta \pi^o_A(\vec{x})} = \frac{\partial \mathcal{H}}{\partial \pi^o_A} = q^A{}_{|o} \; , \tag{38}$$

where $\frac{\delta}{\delta}$ means functional derivative. Apart from the covariant derivatives with respect to time on the right hand side, (38) are just the hamiltonian equations.

Now we turn to the hamiltonian density and hamiltonian of gauge field:

$$\mathcal{H} = F^{oi}_a \, F^a_{oi} + \frac{1}{4} \, F^{\mu\nu}_a F^a_{\mu\nu} \tag{39}$$

$$H = \int \mathcal{H} d^3x \, . \tag{40}$$

It is easy to show

$$\frac{\partial \mathcal{H}}{\partial A^b_i} = A^c_j f_c{}^a{}_b \, F^{ij}_b \; ,$$
$$\frac{\partial \mathcal{H}}{\partial A^a_{i;j}} = F^{ij}_a \; , \tag{41}$$
$$\frac{\partial \mathcal{H}}{\partial F^{oi}_a} = - F^a_{oi} \, .$$

Hence, we have

$$\frac{\delta H}{\delta A^b_i(\vec{x})} = F^{ik}_b{}_{|k} \; , \tag{42}$$

$$\frac{\delta H}{\delta F^{oi}_a(\vec{x})} = - F^a_{oi} \, . \tag{43}$$

So the equations

$$F^a_{oj} = \frac{\partial A^a_j}{\partial x^o} - \frac{\partial A^a_o}{\partial x^j} - f_c{}^a{}_b A^c_o A^b_j \ , \tag{44}$$

can be rewritten as

$$\frac{\partial A^a_j}{\partial x^o} = -\frac{\delta \mathscr{H}}{\delta F^{oj}_a(\vec{x})} + \frac{\partial A^a_o}{\partial x^j} + A^c_o f_c{}^a{}_b A^b_j \ . \tag{45}$$

On the other hand, the space components of equations of motion of gauge field (11)

$$F^{i\nu}_{a\ |\nu} = \mathcal{J}^i_a \ ,$$

can be written to be

$$\frac{\partial F^{oj}_a}{\partial x^o} = \frac{\delta \mathscr{H}}{\delta A^a_j(\vec{x})} - A^b_o f_b{}^c{}_a F^{oj}_c - \mathcal{J}^j_a \ . \tag{46}$$

But, from (36), we have

$$\mathcal{J}^j_a = -\frac{\delta H}{\delta A^a_j(\vec{x})} \ . \tag{47}$$

So (46) is changed to the form

$$\frac{\partial F^{oj}_a}{\partial x^o} = \frac{\delta(H+\mathscr{H})}{\delta A^a_j(\vec{x})} - A^b_o f_b{}^c{}_a F^{oj}_c \ . \tag{48}$$

Summarily, we have the following set of dynamic equations for field and guage field:

$$\frac{\partial q^A(\vec{x})}{\partial x^o} = \frac{\delta(H+\mathscr{H})}{\delta \pi^o_A(\vec{x})} + A^b_o T_{(b)}{}^A{}_B q^B \ ,$$

$$\frac{\partial \pi^o_A(x)}{\partial x^o} = -\frac{\delta(H+\mathscr{H})}{\delta q^A(x)} - A^b_o T_{(b)}{}^B{}_A \pi^o_B \ ,$$

$$\frac{\partial A^a_j(\vec{x})}{\partial x^o} = -\frac{\delta(H+\mathscr{H})}{\delta F^{oj}_a(\vec{x})} + \frac{\partial A^a_o}{\partial x^j} + A^b_o f_b{}^a{}_c A^c_o \ ,$$

$$\frac{\partial F^{oj}_a(\vec{x})}{\partial x^o} = \frac{\delta(H+\mathscr{H})}{\delta A^a_j(\vec{x})} - A^b_o f_b{}^c{}_a F^{oj}_c \ . \tag{49}$$

Besides these, we have constraint equations, which are the time components of equations (11):

$$\mathcal{F}_c(\vec{x}) \equiv F^{oi}_{c;i}(\vec{x}) + A^f_i(\vec{x}) f_f{}^d{}_c F^{oi}_d(\vec{x}) + \pi^o_A(\vec{x}) T_{(a)}{}^A{}_B q^B(\vec{x})$$

$$= 0 \ . \tag{50}$$

*Theorem 1* Constraint equations (50) are just the initial conditions of equations of motion (49). That is, the solutions of (49) $\{q^A(\vec{x},t),\ \pi^o_A(\vec{x},t),\ A^a_i(\vec{x},t),\ F^{oi}_a(\vec{x},t)\}$ satisfy (50) automatically if initially $\{q^A(\vec{x},t_o),\ \pi^o_A(\vec{x},t_o),\ A^a_i(\vec{x},t_o),\ F^{oi}_a(\vec{x},t_o)\}$ satisfy (50) for arbitrary $A^b_o$.

*Proof:*

We have to prove

$$\frac{d}{dx^o}\mathcal{F}_c(\vec{x})=0\,,$$

i.e.

$$\left(\frac{\partial F^{oi}_c}{\partial x^o}\right)_{;i}+A^b_i f_f{}^d{}_c\left(\frac{\partial F^{oi}_d}{\partial x^o}\right)+\left(\frac{\partial A^f_i}{\partial x^o}\right)f_f{}^d{}_c F^{oi}_d+\frac{\partial \mathcal{P}^o_A}{\partial x^o}T^A_{(c)B}\,q^B+\mathcal{P}^o_A T^A_{(c)B}\frac{\partial q^B}{\partial x^o}=0\,. \qquad (51)$$

The proof is divided into the following three parts:

1. $$\left(\frac{\delta H}{\delta A^c_i(\vec{x})}\right)_{;i}+A^f_i f_f{}^d{}_c\left(\frac{\delta H}{\delta A^c_i(\vec{x})}\right)$$

$$-\frac{\delta H}{\delta q^A(\vec{x})}T^A_{(c)B}\,q^B+\mathcal{P}^o_A T^A_{(c)B}\frac{\delta H}{\delta \mathcal{P}^o_A(\vec{x})}$$

$$=-\mathcal{J}^i_{c;i}-A^f_i f_f{}^d{}_c\,\mathcal{J}^i_c+\mathcal{P}^o_{A|o}\,T^A_{(c)B}\,q^B+\mathcal{P}^o_A T^A_{(c)B}\,q^B{}_{|o}$$

$$=-\mathcal{J}^i_{c|i}-\mathcal{J}^o_{c|o}=0.$$

2. $$\left(\frac{\delta \mathcal{H}}{\delta A^c_i}\right)_{;i}+A^f_i f_f{}^d{}_c\left(\frac{\delta \mathcal{H}}{\delta A^d_i}\right)-\frac{\delta \mathcal{H}}{\delta F^{oi}_f}\,f_f{}^d{}_c F^{oi}_d$$

$$=F^{ik}_{c|k;i}+A^f_i f_f{}^d{}_c F^{ik}_{d|k}+F^f_{oi}f_f{}^d{}_c F^{oi}_d$$

$$=F^{ik}_{c|k|i}+F^f_{oi}f_f{}^d{}_c F^{oi}_d$$

$$=0\,.$$

Since we can prove $F^{ik}_{c|k|i}=0$ by direct computation and $F^f_{oi}f_f{}^d{}_c F^{oi}_c=0$ by the antisymmetric property of $f_f{}^d{}_c$.

3. $$(-A^b_o f_b{}^d{}_c F^{oi}_d)_{;i}+A^f_i f_f{}^d{}_c(-A^b_o f_b{}^e{}_d F^{oi}_e)$$

$$+\left(\frac{\partial A^b_o}{\partial x^i}+A^b_o f_b{}^f{}_e A^e_i\right)f_f{}^d{}_c F^{oi}_d$$

$$+(-A^b_o T^B_{(b)A}\,\mathcal{P}^o_B)\,T^A_{(c)D}\,q^D\,.$$

$$+\mathcal{P}^o_A T^A_{(c)B}\,(A^b_o T^B_{(b)D}\,q^D)$$

$$=-\frac{\partial A^b_o}{\partial x^i}f_b{}^d{}_c\,F^{oi}_d-A^b_o F_b{}^d{}_c F^{oi}_{d\;;i}$$

$$-A^b_o(f_b{}^e{}_d f_f{}^d{}_c-f_b{}^d{}_f f_d{}^e{}_c)A^f_i\,F^{oi}_e$$

$$-A^b_o f_b{}^d{}_c\,\mathcal{P}^o_B T^B_{(d)D}\,q^D+\frac{\partial A^b_o}{\partial x^i}f_b{}^e{}_c F^{oi}_e$$

$$=-A^b_o f_b{}^d{}_c(F^{oi}_{d|i}+\mathcal{P}^o_B\,T^B_{(d)D}\,q^D)$$

$$=0\,.$$

Suming up 1, 2 and 3, we prove (51). q.e.d.

From this theorem we know that if we wish to solve the field equations, we can just forget the constraint equations (50) temporarily until we get a solution of dynamical equations (49), then we put the initial values of q, $\mathcal{P}$, A, F to satisfy (50). This means that it is legal to treat q, $\mathcal{P}$, A, F as independent varibales and the define the Possion bracket by the usual method.

*Definition:* For any two dynamic varibales

$$\Phi[\, q^A(\vec{x}),\ \mathcal{P}^o_A(\vec{x}),\ A^a_i(\vec{x}),\ F^{oi}_a(\vec{x})\,],$$

$$\Psi[\, q^A(\vec{x}),\ \mathcal{P}^o_A(\vec{x}),\ A^a_i(\vec{x}),\ F^{oi}_a(\vec{x})\,],$$

the Possion bracket is defined to be

$$\{\Phi, \Psi\} = \int d^3x \Big( \frac{\delta\Phi}{\delta q^A(\vec{x})} \frac{\delta\Psi}{\delta \mathcal{P}^o_A(\vec{x})} - \frac{\delta\Phi}{\delta \mathcal{P}^o_A(\vec{x})} \frac{\delta\Psi}{\delta q^A(\vec{x})} - \frac{\delta\Phi}{\delta A^a_i(\vec{x})} \frac{\delta\Psi}{\delta F^{oi}_a(\vec{x})} + \frac{\delta\Phi}{\delta F^{oi}_a(\vec{x})} \frac{\delta\Psi}{\delta A^a_i(\vec{x})} \Big). \tag{52}$$

From this definition, we then have

$$\{q^A(\vec{x}),\ \mathcal{P}^o_B(\vec{x}')\} = \delta^A_B(\vec{x}-\vec{x}'), \tag{53}$$

$$\{A^a_i(\vec{x}),\ F^{oj}_b(\vec{x}')\} = -\delta^{aj}_{bi}(\vec{x}-\vec{x}'), \tag{54}$$

and

$$\{\mathcal{F}_a(\vec{x}), \mathcal{F}_b(\vec{x}')\} = f_a{}^c{}_b \mathcal{F}_c(\vec{x})\, \delta(\vec{x}-\vec{x}'). \tag{55}$$

*Note:*

(i). The "commutation rules" are independent of the choice of gauge.

(ii). There is something that we have to be careful about in working with the Poisson bracket formalism: we have the constraints (50), but must not use these constraints before working out a Poisson bracket. If we did, we would get a wrong result. So we take it as a rule that Poisson brackets must all work out before we make use of constraint equations. To emphasize this in the formalism, we write the constraints (50) as equations with an equality sign " $\Subset$ " departing from the usual.

Thus they are written

$$\mathcal{F}_a(\vec{x}) \Subset 0. \tag{56}$$

Dirac ([ 6 ]) called such equations weak equations, distinguishing them from the usual or strong equations.

It is further observed that (49) can be written as

$$\frac{\partial q^A(\vec{x})}{\partial x^o} = \frac{\delta}{\delta \mathcal{P}^o_A(\vec{x})} (H + \mathcal{P} + \int A^b_o(x') \mathcal{F}_b(\vec{x}') \, d^3x'),$$

$$\frac{\partial \wp^{o}_{A}(\vec{x})}{\partial x^{o}} = -\frac{\delta}{\delta q^{A}(\vec{x})}\left(H+\wp + \int A^{b}_{o}(x')\mathcal{F}_{b}(\vec{x}')\, d^{3}x'\right),$$

$$\frac{\partial A^{a}_{j}(\vec{x})}{\partial x^{o}} = -\frac{\delta}{\delta F^{oj}_{a}(\vec{x})}\left(H+\wp + \int A^{b}_{o}(x')\mathcal{F}_{b}(\vec{x}')\, d^{3}x'\right), \tag{57}$$

$$\frac{\partial F^{oj}_{a}(\vec{x})}{\partial x^{o}} = \frac{\delta}{\delta A^{a}_{j}(\vec{x})}\left(H+\wp + \int A^{b}_{o}(x')\mathcal{F}_{b}(\vec{x}')\, d^{3}x'\right).$$

Subject to the rule we describe above, the Poisson bracket is quite definite, and we have the possibility of writing our equations of motion (57) in a very concise form.

$$\frac{\partial q^{A}(\vec{x})}{\partial x^{o}} = \{q^{A}(\vec{x}),\ H+\wp + \int A^{b}_{o}(x')\mathcal{F}_{b}(\vec{x}')\, d^{3}x'\},$$

$$\frac{\partial \wp^{o}_{A}(\vec{x})}{\partial x^{o}} = \{\wp^{o}_{A}(\vec{x}),\ H+\wp + \int A^{b}_{o}(x')\mathcal{F}_{b}(\vec{x}')\, d^{3}x'\}, \tag{58}$$

$$\frac{\partial A^{a}_{j}(\vec{x})}{\partial x^{o}} = \{A^{a}_{j}(\vec{x}),\ H+\wp + \int A^{b}_{o}(x')\mathcal{F}_{b}(\vec{x}')\, d^{3}x'\},$$

$$\frac{\partial F^{oj}_{a}(\vec{x})}{\partial x^{o}} = \{F^{oj}_{a}(\vec{x}),\ H+\wp + \int A^{b}_{o}(x')\mathcal{F}_{b}(\vec{x}')\, d^{3}x'\}.$$

So we know the roles played by $A^{b}_{o}(x)$ are just a Lagrangian multiplers of the constraints (56), which is arbitrary because of the freedom of choosing any moving frame. In fact, it is always possible to get $A^{b}_{o}=0$ by choosing the moving frames which are parallel along the time direction.

We also can have

$$\frac{\partial q^{A}(\vec{x})}{\partial x^{i}} = \{q^{A}(\vec{x}),\ P_{i}+\rho_{i}+\int A^{b}_{i}(\vec{x}')\mathcal{F}_{b}(\vec{x}')\, d^{3}x'\},$$

$$\frac{\partial \wp^{o}_{A}(\vec{x})}{\partial x^{i}} = \{\wp^{o}_{A}(\vec{x}),\ P_{i}+\rho_{i}+\int A^{b}_{i}(\vec{x})\mathcal{F}_{b}(x')\, d^{3}x'\},$$

$$\frac{\partial A^{a}_{j}(\vec{x})}{\partial x^{i}} = \{A^{a}_{j}(\vec{x}),\ P_{i}+\rho_{i}+\int A^{b}_{i}(\vec{x}')\mathcal{F}_{b}(\vec{x}')\, d^{3}x'\},$$

$$\frac{\partial F^{oj}_{a}(\vec{x})}{\partial x^{i}} = \{F^{oj}_{a}(\vec{x}),\ P_{i}+\rho_{i}+\int A^{b}_{i}(\vec{x}')\mathcal{F}_{b}(\vec{x}')\, d^{3}x'\} + \delta^{j}_{i}\mathcal{F}_{a}(\vec{x})$$

$$\approx \{F^{oj}_{a}(\vec{x}),\ P_{i}+\rho_{i}+\int A^{b}_{i}(x')\mathcal{F}_{b}(\vec{x}')\, d^{3}x'\}. \tag{59}$$

Combining with (58), we finally have

$$\frac{\partial q^{A}(\vec{x})}{\partial x^{\mu}} = \{q^{A}(\vec{x}),\ P_{\mu}+\rho_{\mu}+\int A^{b}_{\mu}(\vec{x}')\mathcal{F}_{b}(\vec{x}')\, d^{3}x'\},$$

$$\frac{\partial \wp^{o}_{A}(x)}{\partial x^{\mu}} = \{\wp^{o}_{A}(\vec{x}),\ P_{\mu}+\rho_{\mu}+\int A^{b}_{\mu}(\vec{x}')\mathcal{F}_{b}(\vec{x}')\, d^{3}x'\},$$

$$\frac{\partial A^{a}_{j}(\vec{x})}{\partial x^{\mu}} = \{A^{a}_{j}(\vec{x}),\ P_{\mu}+\rho_{\mu}+\int A^{b}_{\mu}(\vec{x}')\mathcal{F}_{b}(\vec{x})\, d^{3}x'\},$$

$$\frac{\partial F^{oj}_a(\vec{x})}{\partial x^\mu} = \{ F^{oj}_a(\vec{x}), P_\mu + \rho_\mu + \int A^b_\mu(\vec{x}')\mathcal{F}_b(\vec{x}')\ d^3x' \} \tag{60}$$

We call the quantities $P_\mu + \rho_\mu + \int A^b_\mu(\vec{x}')\mathcal{F}_b(\vec{x}')\ d^3x'$ the effective energy-momentum.

$$P_{eff\,\mu} \equiv P_\mu + \rho_\mu + \int A^b_\mu(\vec{x}')\mathcal{F}_b(\vec{x}')\ d^3x' \ . \tag{61}$$

## IV. Commutation Relation

By using the commutation relations (53) and (54) we then can prove that the four momenta and angular momenta form a Poincare algebra weakly:

$$\{ P^\mu + \rho^\mu\ ,\ P^\nu + \rho^\nu \} \in 0\ ,$$

$$\{ M^{\mu\nu} + m^{\mu\nu},\ P^\sigma + \rho^\sigma \} \in (P^\mu + \rho^\mu)\,\eta^{\nu\sigma} - (P^\nu + \rho^\nu)\,\eta^{\mu\sigma},$$

$$\{ M^{\mu\nu} + m^{\mu\nu},\ M^{\rho\sigma} + m^{\rho\sigma} \}$$

$$\in (M^{\nu\rho} + m^{\nu\rho})\,\eta^{\mu\sigma} + (M^{\mu\sigma} + m^{\mu\sigma})\,\eta^{\nu\rho}$$

$$- (M^{\mu\rho} + m^{\mu\rho})\eta^{\nu\sigma} - (M^{\nu\sigma} + m^{\nu\sigma})\,\eta^{\nu\rho}\ , \tag{62}$$

They do not form a Poincare algebra strongly. Some explicit formulas are:

$$\{ P_i + \rho_i,\ P_j + \rho_j \} = \int \mathcal{F}_a(\vec{x})\, F^a_{ij}(\vec{x})\ d^3x\ , \tag{63}$$

$$\{ P_i + \rho_i,\ H + h \} = 0\ , \tag{64}$$

$$\{ M^{ij} + m^{ij},\ P^i + \rho^i \} = -\eta^{ii}(P^j + \rho^j) + \int d^3x\ x^i\ F^{ji}_a(\vec{x})\mathcal{F}_a(\vec{x}) \tag{65}$$

$$\{ M^{oi} + m^{oi},\ P^j + \rho^j \} = \eta^{ij}(H + h)\ , \tag{66}$$

$$\{ M^{ij} + m^{ij},\ H + h \} = \int d^3x (x^j\ \eta^{i\ell} - x^i \eta^{j\ell})\ F^a_{o\ell}(\vec{x})\,\mathcal{F}_a(\vec{x})\ d^3x\ . \tag{67}$$

etc. The validation of (62) assures that the choices of our four momenta, angular momenta and commutation relations are self consistent.

Next, we test the commutation relations between momenta and angular momenta with field and guage field. They are:

$$\{ P_\mu + \rho_\mu,\ q^A(\vec{x}) \} = -q^A{}_{|\mu}\ (\vec{x})\ , \tag{68}$$

$$\{ P_\mu + \rho_\mu,\ A^a_i(\vec{x}) \} = F^a_{i\mu}(\vec{x})\ , \tag{69}$$

$$\{ P_\mu + \rho_\mu,\ F^a_{\ell i}(\vec{x}) \} = -F^a_{\ell i|\mu}\ (\vec{x})\ , \tag{70}$$

$$\{ P_\mu + \rho_\mu,\ F^{oi}_a(\vec{x}) \} \in -F^{oi}_{a|\mu}(\vec{x})\ , \tag{71}$$

$$\{ M^{\mu\nu} + \mathcal{M}^{\mu\nu}, q^A(\vec{x}) \} = -x^\mu q^{A|\nu} + x^\nu q^{A|\mu} - \Sigma^{\mu\nu}_{(A)} q^A \tag{72}$$

$$\{ M^{\mu\nu} + \mathcal{M}^{\mu\nu}, A^a_k(\vec{k}) \} = -x^\mu F^a_{\nu k} + x^\nu F^a_{\mu k} , \tag{73}$$

$$\{ M_{\mu\nu} + \mathcal{M}_{\mu\nu}, F^{\lambda\rho}_a(\vec{x}) \} \doteq -x_\mu F^{\lambda\rho}_{a|\nu} + x_\nu F^{\lambda\rho}_{a|\mu} + \Omega_{\mu\nu}{}^\rho{}_\sigma F^{\lambda\sigma}_a + \Omega_{\mu\nu}{}^\lambda{}_\sigma F^{\sigma\rho}_a , \tag{74}$$

Note that the spin terms appear in the right hand side of (74) although there is no explicit spin term in $\mathcal{M}_{\mu\nu}$ . We only prove one of the commutation relations

$$\{ M_{ij} + \mathcal{M}_{ij}, F^{ok}_a(\vec{x}) \} \doteq -x_j F^{ok}_{a|i} + x_i F^{ok}_{a|j} + \Omega_{ji}{}^k{}_\ell F^{o\ell}_a , \tag{75}$$

Proof of (75).

$$\{ M_{ij} + \mathcal{M}_{ij}, F^{ok}_a(\vec{x}) \}$$

$$= \int d^3x' \{ x'_j (\mathcal{J}^o{}_i(\vec{x}') + t^o{}_i(\vec{x}') - x'_i (\mathcal{J}^o{}_j(\vec{x}') + t^o{}_j(\vec{x}')), F^{ok}_a(\vec{x}') \}$$

$$= -\delta^k_i x_j \mathcal{F}^o_a + \int d^3x' x'_j F^{o\ell}_a(\vec{x}')(-\delta^{kb}_{ia;\ell}(\vec{x}'-\vec{x}) - \delta^{kb}_{\ell a;i}(\vec{x}'-\vec{x})$$

$$- f_c{}^b{}_a \delta^{kc}_{\ell a}(\vec{x}-\vec{x}') A^d_i + f_c{}^b{}_d A^c_\ell \delta^{dk}_{ai}(\vec{x}-\vec{x}'))$$

$$- (i \leftrightarrow j)$$

$$\doteq -x_j F^{ok}_{a|i} + x_i F^{ok}_{a|j} + (\delta^k_i g_{j\ell} - \delta^k_j g_{i\ell}) F^{o\ell}_a$$

$$= -x_j F^{ok}_{a|i} + x_i F^{ok}_{a|j} + \Omega_{ji}{}^k{}_\ell F^{o\ell}_a . \qquad \text{q.e.d.}$$

Commutation relation (75) tells us that gauge field is a spin one field. The reason is: the $F^{ok}_a$, which correspond to the electric field in electromagnetic theory, are fundamental dynamical variables.

The geometrical meaning of (68) — (74) can be understood from the formulas themselves except (69) and (73). Now we will give a geometrical interpretation of them. As we know that $P^\mu$ and $M^{\mu\nu}$ are the infinitesmal translation and rotation operators of space-time coordinate. After transformation we will get a new gauge field $A'^a_\mu$ which possesses the same gauge transformation properties as the old gauge field $A^a_\mu$ . So the difference between these two gauge fields $\delta A^a_\mu = A'^a_\mu - A^a_\mu$ must transform like a gauge vector under the same gauge transformation since the inhomogeneous term in trasformation depends on the gauge transformation matrix function (a) only, but not on the gauge field. This corresponds to the fact that the difference of two Christoffel symbols is a tensor in Riemanian geometry. Now we try to find what is $\delta A^a_\mu$:

Consider a transformation

$$x \longrightarrow y = f(x) ,$$

$$u \longrightarrow ua = u' , \tag{76}$$

where u, u' are two moving frames, and a is a matrix function of f. We have

$$\nabla_{\frac{\partial}{\partial x^\mu}} u_B = -A^a_\mu(x)\; T^{\ A}_{(a)B}\; u_A \,,$$

$$\nabla_{\frac{\partial}{\partial y^\mu}} u'_B = -A'^a_\mu(y)\; T^{\ A}_{(a)B}\, u'_A$$

$$= \frac{\partial x^\nu}{\partial y^\mu} \nabla_{\frac{\partial}{\partial x^\nu}} (u_D\; a^D{}_B)$$

$$= \frac{\partial x^\nu}{\partial y^\mu} \left( \frac{\partial a^D{}_B}{\partial x^\nu} (a^{-1})^A{}_D - a^E{}_B\, A^a_\nu(x)\; T^{\ D}_{(a)E}\; (a^{-1})^A{}_D \right) u'_A \,.$$

Hence

$$-\frac{\partial y^\mu}{\partial x^\nu} A'^a_\mu(y)\; T^{\ A}_{(a)B}$$

$$= \frac{\partial a^D{}_B}{\partial x^\nu} (a^{-1})^A{}_D - a^E{}_B\; A^a_\nu(x)\; T^{\ D}_{(a)E}\; (a^{-1})^A{}_D \,. \tag{77}$$

Now if (76) is infinitesmal:

$$y^\mu = x^\mu + \alpha^\mu(x) \,, \tag{78}$$

where $\alpha^\mu$ (x) is an infinitesmal function of x; and if the transformation we consider is a pure coordinate transformation, then we must parallelly move the frames along the coordinate transformation, that is, a is of the form

$$u'_A = u_B\; a^B{}_A = u_A - \alpha^\mu A^a_\mu\; T^{\ B}_{(a)A}\; u_B \,. \tag{79}$$

So (77) becomes

$$(\delta^\mu_\nu + \alpha^\mu{}_{;\nu})( A'^a_\mu + A'^a{}_{\mu\,;\,\lambda}\alpha^\lambda)\, T^{\ A}_{(a)B}$$

$$= -(\delta^E{}_B - \alpha^\mu A^b_\mu\, T^{\ E}_{(b)B})\, A^a_\nu\, T^{\ D}_{(a)E}\; (\delta^A{}_D + \alpha^\lambda A^\sigma_\lambda\;\; T^{\ A}_{(d)D})$$

$$-(\alpha^\mu\;\; A^a_\mu)\;{}_{;\nu}\; T^{\ D}_{(a)B}\; (\delta^A{}_D + \alpha^\mu A^b_\mu\, T^{\ A}_{(b)D}) \,.$$

Hence $\quad A'^a_\nu - A^a_\nu = F^a_{\nu\mu}\,\alpha^\mu \,.$ (80)

It is a gauge vector as we expect. For translation

$$\alpha^\mu = \text{infinitesimal constant} \,,$$

We have $\quad A'^a_\nu - A^a_\nu = F^a_{\nu\mu}\,\alpha^\mu \,.$

This is the same result as (69). For rotation

$$\alpha^\mu = \alpha^{\mu\lambda}\, x_\lambda \,,\; \alpha^{\mu\lambda} = -\alpha^{\lambda\mu} = \text{infinitesmal constant} \,,$$

we have $\quad A'^a_\nu - A^a_\nu = F^a_{\nu\mu}\alpha^{\mu\lambda}\, x_\lambda = \frac{1}{2}\alpha^{\mu\lambda}(x_\lambda\;\; F^a_{\nu\mu} - x_\mu\;\; F^a_{\nu\lambda}) \,.$

This is the same result as (73), too.

Conclusively, the correct energy-momentum operators should be that their commutation relations with $A_i^a$ equal to $F_{\nu i}^a$ and not equal to $A_{i\,;\nu}^a$, because $A_\mu^a + \epsilon^\nu A_{\mu\,;\nu}^a$ does not transform like a gauge field.

Finally we point out that the constraint operators $\mathcal{F}_a(\vec{x})$ are the infinitesmal local gauge transformation operators, because

$$\{-\int \epsilon^a(x')\mathcal{F}_a(\vec{x}')\, d^3x', A_i^b(\vec{x})\} = f_a{}^b{}_c A_i^c \epsilon^a + \frac{\partial \epsilon^b}{\partial x^i}. \tag{81}$$

## Appendix A More Studies On Energy-Momentum Tensor And Angular Momentum Tensor

If we give up the concept of parallel translation and treat the field and guage field on the ordinary Minkowko space, then, by Nother theorem, we can get a set of energy-momentum tensor and angular momentum tensor. This set of energy-momentum tensor and angular momentum tensor is almost the same as the one we get in the paper. We are going to find their relationships now.

First, we rewrite the Lagrangians of field and gauge field by

$$\mathcal{L}'(q^A, q^A{}_{;\mu}, A_\mu^a) = \mathcal{L}(a^A, q^A{}_{|\mu}), \tag{A.1}$$

and

$$\mathcal{L}'_o(A^a, A^a_{\mu;\nu}) = \mathcal{L}_o(F^a_{\mu\nu}). \tag{A.2}$$

Then the energy-momentum tensor dirived by the usual method for the Lagrangian $\mathcal{L}'+\mathcal{L}'_o$ is

$$\begin{aligned} T^\mu{}_\nu &= \frac{\partial \mathcal{L}'}{\partial q^A{}_{;\mu}} q^A{}_{;\nu} - \delta^\mu_\nu \mathcal{L}' + \frac{\partial \mathcal{L}_o}{\partial A^a{}_{\rho;\mu}} A^a_{\rho;\nu} - \delta^\mu_\nu \mathcal{L}'_o \\ &= \frac{\partial \mathcal{L}'}{\partial q^A{}_{;\mu}} q^A{}_{;\nu} - \delta^\mu_\nu \mathcal{L}' + F^{\rho\mu}_a A^a{}_{\rho;\nu} - \delta^\mu_\nu \mathcal{L}'_o . \end{aligned} \tag{A.3}$$

Since

$$\begin{aligned} F^{\rho\mu}_a F^a_{\nu\rho} &= F^{\rho\mu}_a \left(\frac{\partial A^a_\rho}{\partial x^\nu} - \frac{\partial A^a_\nu}{\partial x^\rho} - f_b{}^a{}_c A^b_\nu A^c_\rho\right) \\ &= F^{\rho\mu}_a \frac{A^a_\rho}{\partial x^\nu} - A^a_\nu F^{\mu\rho}_{a|\rho} - \frac{\partial}{\partial x^\rho}(F^{\rho\mu}_a A^a_\nu) \\ &= F^{\rho\mu}_a \frac{\partial A^a_\rho}{\partial x^\nu} - A^a_\nu \mathcal{J}^\mu_a - A^a_\nu (F^{\mu\rho}_{a|\rho} - \mathcal{J}^\mu_a) \\ &\quad - \frac{\partial}{\partial x^\rho}(F^{\rho\mu}_a A^a_\nu). \end{aligned}$$

so we have

$$\begin{aligned} T^\mu{}_\nu &= \mathcal{T}^\mu_\nu + t^\mu{}_\nu + A^a_\nu \mathcal{F}_a + \frac{\partial}{\partial x^\rho}(F^{\rho\mu}_a A^a_\nu) \\ &\in \mathcal{T}^\mu{}_\nu + t^\mu{}_\nu + \frac{\partial}{\partial x^\rho}(F^{\rho\mu}_a A^a_\nu). \end{aligned} \tag{A.4}$$

So we know $T^\mu{}_\nu$ are equal to $\mathcal{T}^\mu{}_\nu + t^\mu{}_\nu$ weakly apart from a total derivative. But as we can see in (A.3) the energy-momentum tensor of field and gauge field are not gauge scalar separately in this formula, only the rearrangement (A.4) can have gauge invariant property.

Similarly, the angular momentum tensor for $\mathcal{L}' + \mathcal{L}'_o$ is

$$M^{\mu\sigma\lambda} = (x^\lambda \eta^{\rho\sigma} - x^\sigma \eta^{\rho\lambda})\left(\frac{\partial \mathcal{L}'}{\partial q^A{}_{|\mu}} q^A{}_{;\rho} - \delta^\mu_\rho \mathcal{L}'\right)$$

$$+ \frac{\partial \mathcal{L}'}{\partial q^A{}_{;\mu}} \Sigma^{\lambda\sigma}_{(A)}\ q^A + (x^\lambda \eta^{\rho\sigma} - x^\sigma \eta^{\rho\lambda}).$$

$$(F^{\nu\mu}_a\ A^a{}_{\nu;\rho} + \delta^\mu_\rho\ \mathcal{L}'_o) + F^{\nu\mu}_a \Omega^{\lambda\sigma\alpha}{}_\nu A^a_\alpha. \tag{A.5}$$

Since the orbit angular momentum of field and gauge field and the spin part of gauge field are not gauge invariant, so this is not a good form. But with a rearrangement, we can show that

$$M^{\mu\sigma\lambda} = \mathcal{M}^{\mu\sigma\lambda} + \eta^{\mu\sigma\lambda} + (x^\lambda \eta^{\rho\sigma} - x^\sigma \eta^{\rho\lambda})\, A^a_\rho \mathcal{F}_a$$

$$+ \frac{\partial}{\partial x^\nu}((x^\lambda \eta^{\rho\sigma} - x^\sigma \eta^{\rho\lambda}) A^a_\rho\ F^{\nu\mu}_a)$$

$$\in \mathcal{M}^{\mu\sigma\lambda} + \eta^{\mu\sigma\lambda} + \frac{\partial}{\partial x^\nu}((x^\lambda \eta^{\rho\sigma} - x^\sigma \eta^{\rho\lambda}) A^a_\rho\ F^{\nu\mu}_a). \tag{A.6}$$

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