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擴充漁業模型之分析

Analysis of General Commercial Fishing Model

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Abstract — We investigate a general commercial fishing model in this paper. The results imply that fishing capital investment is non-explosive and fishery resources are non-extinctive.

I. Introduction

In this paper, we continue the discussion of a general commercial fishing model undertaken by Smith [4], [5], [6], [7] and Leung and Wang [8].

We still restrict our model to non-time-lag investment decision and to the simplest biological equation. The main purposes of this article are again to show that, under such an economic-ecological dynamic model, (1) the fishery resources and fishing capital are non-extinctive, (2) the behavior of fishing capital investment is non-explosive and (3) the solution of the economic-ecological system is globally stable within an invariant region.

In the general model, Smith [4] assumes a quadratic total revenue function,

$$R(Kx)=(\alpha-\beta Kx)Kx$$

and a quadratic total cost function

$$C=K\left[\frac{\gamma x^2}{X}+\hat{\pi}\right]$$

Using the Vessel's profit maximizing rule, we can conclude that x,

$$x=\frac{\alpha X}{2\gamma+2\beta KX},$$

will maximize the profit function π

$$\pi=(\alpha-\beta Kx)Kx-\frac{K\gamma x^2}{X}-K\hat{\pi}.$$

Here X(t) is the population of fish at time t, and K(t) is the amount of fishing capital investment, α, β, γ and $\hat{\pi}$ are

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positive constants, with $(\alpha - \beta Kx)$ representing the price per unit amount of fish and $\hat{\pi}$ the opportunity cost. x is the amount of fish harvested in a unit time per unit investment. Smith's system of equations can now be written as

$$\frac{dX}{dt}=X[a-bX-\frac{\alpha K}{2\gamma+2\beta KX}]$$

$$\frac{dK}{dt}=K\delta\ \{(\alpha-\beta K\frac{\alpha X}{2\gamma+2\beta KX})\frac{\alpha X}{2\gamma+2\beta KX}-[\frac{\gamma}{X}(\frac{\alpha X}{2\gamma+2\beta KX})^2+\hat{\pi}]\}\qquad(1)$$

or

$$\frac{dX}{dt}=X[a-bX-\frac{\alpha K}{2\gamma+2\beta KX}]$$

$$\frac{dK}{dt}=\delta K[\frac{\alpha^2 X}{4(\gamma+\beta KX)}-\hat{\pi}]\qquad(2)$$

where δ is a positive constant and a and b are positive parameters related to the fish population growth rate.

The curve $\frac{dX}{dt}=0$ can be explicitly written down as

$$K=G(X)=\frac{2\gamma(a-bX)}{2\beta bX^2-2\beta aX+\alpha}$$

Its gragh in the first quadrant of the X-K plane depends on the parameter $\xi=\frac{2\beta a^2}{\alpha b}$. Three cases which correspond to (i) $0<\xi<4$, (ii) $\xi=4$, and (iii) $4<\xi$ are shown in Figs. 1, 2 and 3 respectively.

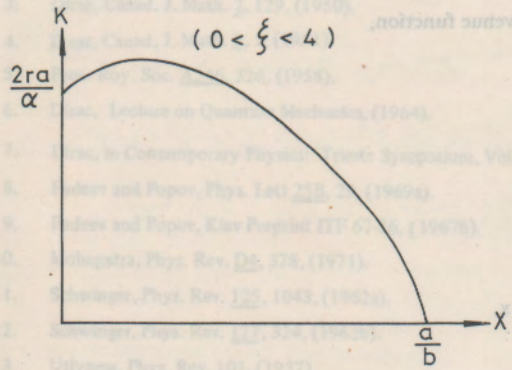


Fig.1

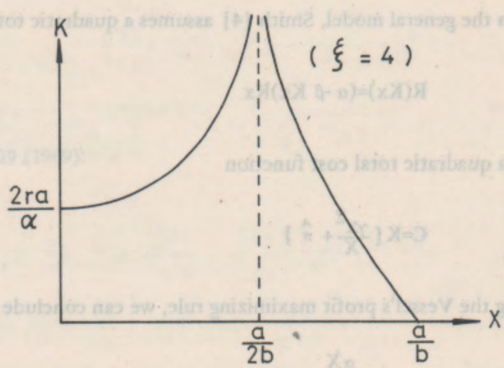


Fig.2

On the other hand, the curve $\frac{dK}{dt}=0$ also can be written down explicitly,

$$K=H(X)=\frac{\alpha^2}{4\beta\hat{\pi}}-\frac{\gamma}{\beta X}$$

Its graph in the first quadrant of the X-K plane is shown in Fig. 4.

It is very helpful to introduce some dimensionless variables and constants into Eq. (2) for further mathematical analysis. Let

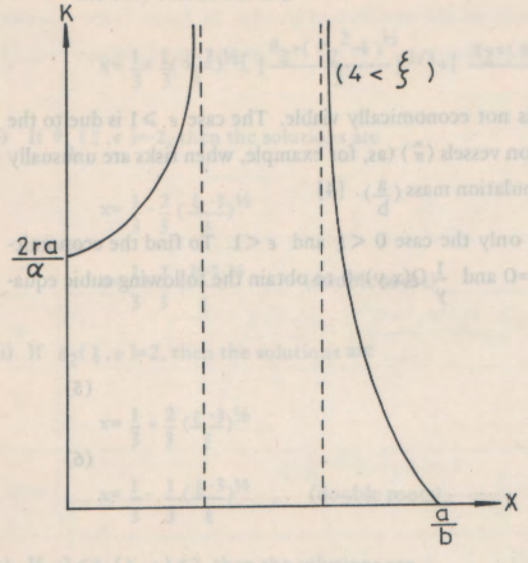


Fig. 3

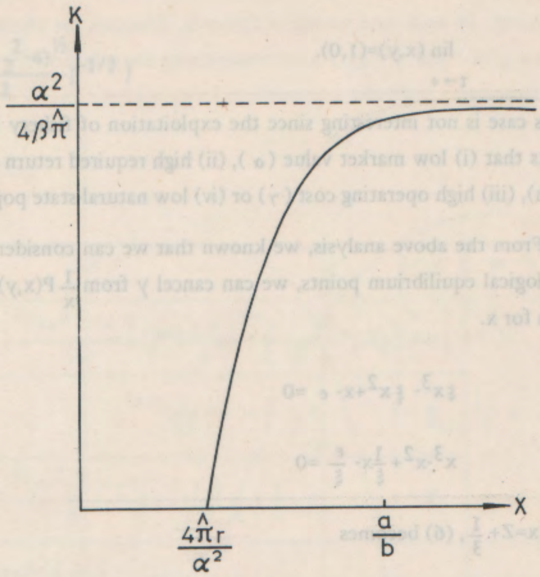


Fig. 4

$$X=\frac{a}{b}x,\ K=\frac{2\gamma a}{\alpha}y,\ \frac{4\hat{\pi}\gamma}{\alpha^2}=\frac{a}{b}\epsilon\ \text{, and}\ \frac{\alpha b}{2\beta a^2}=\frac{1}{\xi}$$

(For reasons, see Figs. 1 to 4.)

Then we arrive at the following version of Eq. (2),

$$\frac{dx}{dt}=ax\left(1-x-\frac{y}{1+\xi xy}\right)$$

$$\frac{dy}{dt}=\frac{\delta\hat{\pi}}{\epsilon}\ y\left(\frac{x}{1+\xi xy}-\epsilon\right) \tag{3}$$

Furtheremore, as it is well known, we can introduce new time variable τ, such that $dt=(1+\xi xy)d\tau$. This change of variable only changes the time scale. It does not change the path in the x-y plane. Now Eq. (3) becomes

$$\frac{dx}{d\tau}=\dot{x}=ax(1-x-y+\xi xy-\xi x^2y)=P(x,y)$$

$$\frac{dy}{d\tau}=\dot{y}=\frac{\delta\hat{\pi}}{\epsilon}\ y(x-\epsilon-\xi\epsilon xy)=Q(x,y) \tag{4}$$

II. Economic-Ecological Equilibrium Points

We can easily see that the case $\xi=0$ is equivalent to the simple BCF model [2,3], which was discussed rather extensively in reference [8]. So we consider only the case $\xi>0$.

From the first equation of Eq. (4), we know that, if $x\geqslant 1$, then $(1-x-y+\xi xy-\xi x^2y)=-(x-1)-y-\xi xy(x-1)<0$, that is, if $x\geqslant 1$, then $x<0$. Thus $x<1$ is a region such that if $x(t_1)<1$, then for all $t>t_1$, $x(t)<1$ and if $x(t_1)>1$, then there exists a time $t_2>t_1$, for all $t>t_2$, $x(t)<1$. Now if $\epsilon\geqslant 1$, we can always wait until $x(t)<1$, at that time and all time later,

$$\dot{y}=\frac{\delta\hat{\pi}}{\epsilon}\ y(x-\epsilon-\xi\epsilon xy)<0.$$

It is easy to see that

$$\lim_{t\to +} (x,y)=(1,0).$$

This case is not interesting since the exploitation of fishery is not economically viable. The case $\epsilon \geqslant 1$ is due to the facts that (i) low market value (α), (ii) high required return on vessels ($\hat{\pi}$) (as, for example, when risks are unusually high), (iii) high operating cost (γ) or (iv) low natural-state population mass ($\frac{a}{b}$). [4]

From the above analysis, we known that we can consider only the case $0 < \xi$ and $\epsilon < 1$. To find the economic-ecological equilibrium points, we can cancel y from $\frac{1}{x}P(x,y)=0$ and $\frac{1}{y}Q(x,y)=0$ to obtain the following cubic equation for x.

$$\xi x^3 - \xi x^2 + x - \epsilon = 0 \tag{5}$$

or

$$x^3 - x^2 + \frac{1}{\xi}x - \frac{\epsilon}{\xi} = 0 \tag{6}$$

Let $x=Z+\frac{1}{3}$, (6) becomes

$$Z^3 - \left(\frac{1}{3} - \frac{1}{\xi}\right)Z = \frac{2}{27} - \frac{1}{3\xi} + \frac{\epsilon}{\xi} \tag{7}$$

Case 1: $0<\xi<3$

Let $Z=\frac{1}{3}\left(\frac{3-\xi}{\xi}\right)^{1/2}u$, then (7) becomes

$$u^3+3u=\left(2-\frac{9}{\xi}+\frac{27\epsilon}{\xi}\right)\left(\frac{\xi}{3-\xi}\right)^{3/2} = \theta_1(\xi,\epsilon)$$

The only real solution is

$$x=\frac{1}{3}+\frac{1}{3}\left(\frac{3-\xi}{\xi}\right)^{1/2}\left\{\left[\frac{\theta_1+(\theta_1^2+4)^{1/2}}{2}\right]^{1/3}-\left[\frac{\theta_1+(\theta_1^2+4)^{1/2}}{2}\right]^{-1/3}\right\}$$

Case 2: $\xi=3$

The only real solution is

$$x=\frac{1}{3}+\left(-\frac{1}{27}+\frac{\epsilon}{3}\right)^{1/3}$$

Case 3: $3<\xi$

Let $Z=\frac{1}{3}\left(\frac{\xi-3}{\xi}\right)^{1/2}u$, then (7) becomes

$$u^3-3u=\left(2-\frac{9}{\xi}+\frac{27\epsilon}{\xi}\right)\left(\frac{\xi}{\xi-3}\right)^{3/2}=\theta_2(\xi,\epsilon)$$

In this case, the solution of (7) depend on the value of $\theta_2(\xi,\epsilon)$ in the following way:

(i) If $|\theta_2(\xi,\epsilon)|>2$

the only real solution is

$$x=\frac{1}{3}+\frac{1}{3}\left(\frac{\xi-3}{\xi}\right)^{1/2}\left\{\left[\frac{\theta_2+(\theta_2^{\,2}-4)^{1/2}}{2}\right]^{1/3}+\left[\frac{\theta_2+(\theta_2^{\,2}-4)^{1/2}}{2}\right]^{-1/3}\right\}$$

(ii) If $\theta_2(\xi,\epsilon)=-2$, then the solutions are

$$x=\frac{1}{3}-\frac{2}{3}\left(\frac{\xi-3}{\xi}\right)^{1/2}$$

$$x=\frac{1}{3}+\frac{1}{3}\left(\frac{\xi-3}{\xi}\right)^{1/2} \qquad \text{(double roots)}$$

(iii) If $\theta_2(\xi,\epsilon)=2$, then the solutions are

$$x=\frac{1}{3}+\frac{2}{3}\left(\frac{\xi-3}{\xi}\right)^{1/2}$$

$$x=\frac{1}{3}-\frac{1}{3}\left(\frac{\xi-3}{\xi}\right)^{1/2} \qquad \text{(double roots)}$$

(iv) If $-2<\theta_2(\xi,\epsilon)<2$, then the solutions are

$$x=\frac{1}{3}+\frac{1}{3}\left(\frac{\xi-3}{\xi}\right)^{1/2}\cdot 2\cos\left(\frac{\phi}{3}\right)$$

$$x=\frac{1}{3}+\frac{1}{3}\left(\frac{\xi-3}{\xi}\right)^{1/2}\cdot 2\cos\left(\frac{\phi}{3}+\frac{2\pi}{3}\right)$$

$$x=\frac{1}{3}+\frac{1}{3}\left(\frac{\xi-3}{\xi}\right)^{1/2}\cdot 2\cos\left(\frac{\phi}{3}+\frac{4\pi}{3}\right)$$

where

$$e^{i\phi}=\frac{\theta_2+i(4-\theta_2^{\,2})^{1/2}}{2}, \quad \pi>\phi>0.$$

For $\xi>3$ and $\epsilon>\frac{1}{9}$, we can prove that $\theta_2(\xi,\epsilon)>2$. Now we can divide the ξ - ϵ plane into subregions, in each subregion, we have explicit solutions for the cubic equation (6). (See Fig. 5). We tabulate the subregions of

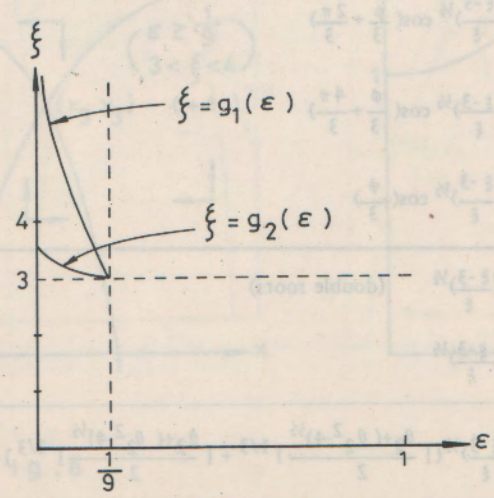


Fig. 5

ξ and ϵ and corresponding solutions in Table 1. We also use a subscript to classify different kind of equilibrium points. In each case we use a figure to illustrate the character of the equilibrium points. In Table 1, the functions $\xi = g_1(\epsilon)$ and $\xi = g_2(\epsilon)$ are respectively the solutions for equations $\theta_2(\xi, \epsilon)=2$ and $\theta_2(\xi, \epsilon)=-2$. When $0<\epsilon<\frac{1}{9}$, it is not very difficult to prove that $3<g_2(\epsilon)<4$.

Table 1

ϵ	ξ	x_i	Fig.
$\epsilon \geqslant \frac{1}{9}$	$0<\xi<3$	$x_3=\frac{1}{3}+\frac{1}{3}(\frac{3-\xi}{\xi})^{1/2}\{[\frac{\theta_1+(\theta_1{}^2+4)^{1/2}}{2}]^{1/3}-[\frac{\theta_1+(\theta_1{}^2+4)^{1/2}}{2}]^{-1/3}\}$	6
	$\xi=3$	$x_3=\frac{1}{3}+(\frac{\epsilon}{3}-\frac{1}{27})^{1/3}$	7
	$3<\xi$	$x_3=\frac{1}{3}+\frac{1}{3}(\frac{\xi-3}{\xi})^{1/2}\{[\frac{\theta_2+(\theta_2{}^2-4)^{1/2}}{2}]^{1/3}+[\frac{\theta_2+(\theta_2{}^2-4)^{1/2}}{2}]^{-1/3}\}$	8-10
$0<\epsilon<\frac{1}{9}$	$0<\xi<3$	$x_1=\frac{1}{3}+\frac{1}{3}(\frac{3-\xi}{\xi})^{1/2}\{[\frac{\theta_1+(\theta_1{}^2+4)^{1/2}}{2}]^{1/3}-[\frac{\theta_1+\theta_1{}^2+4)^{1/2}}{2}]^{-1/3}\}$	11
	$\xi=3$	$x_1=\frac{1}{3}+(\frac{\epsilon}{3}-\frac{1}{27})^{1/3}$	12
	$3<\xi<g_2(\epsilon)$	$x_1=\frac{1}{3}+\frac{1}{3}(\frac{\xi-3}{\xi})^{1/2}\{[\frac{\theta_2+(\theta_2{}^2-4)^{1/2}}{2}]^{1/3}+[\frac{\theta_2+(\theta_2{}^2-4)^{1/2}}{2}]^{-1/3}\}$	13
	$\xi=g_2(\epsilon)$	$x_1=\frac{1}{3}-\frac{2}{3}(\frac{\xi-3}{\xi})^{1/2}$ $x_2=\frac{1}{3}+\frac{1}{3}(\frac{\xi-3}{\xi})^{1/2}$ (double roots)	14
	$g_2(\epsilon)<\xi$ $<g_1(\epsilon)$	$x_1=\frac{1}{3}+\frac{2}{3}(\frac{\xi-3}{\xi})^{1/2}\cos(\frac{\phi}{3}+\frac{2\pi}{3})$	15
		$x_2=\frac{1}{3}+\frac{2}{3}(\frac{\xi-3}{\xi})^{1/2}\cos(\frac{\phi}{3}+\frac{4\pi}{3})$	16
		$x_3=\frac{1}{3}+\frac{2}{3}(\frac{\xi-3}{\xi})^{1/2}\cos(\frac{\phi}{3})$	17
	$\xi=g_1(\epsilon)$	$x'_2=\frac{1}{3}-\frac{1}{3}(\frac{\xi-3}{\xi})^{1/2}$ (double roots) $x_3=\frac{1}{3}+\frac{2}{3}(\frac{\xi-3}{\xi})^{1/2}$	18-20
$0<\epsilon<\frac{1}{9}$	$g_1(\epsilon)<\xi$	$x_3=\frac{1}{3}+\frac{1}{3}(\frac{\xi-3}{\xi})^{1/2}\{[\frac{\theta_2+(\theta_2{}^2-4)^{1/2}}{2}]^{1/3}+[\frac{\theta_2+(\theta_2{}^2-4)^{1/2}}{2}]^{-1/3}\}$	21-23

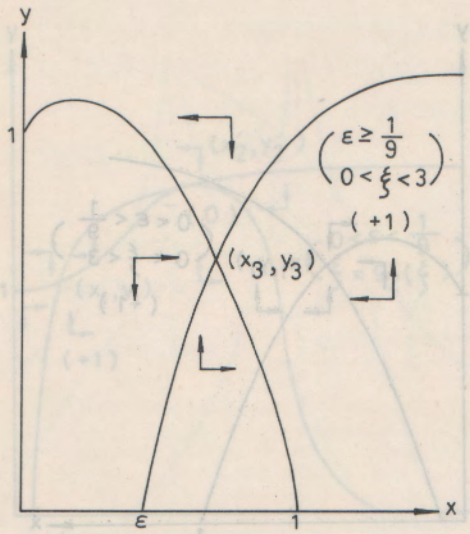


Fig.6

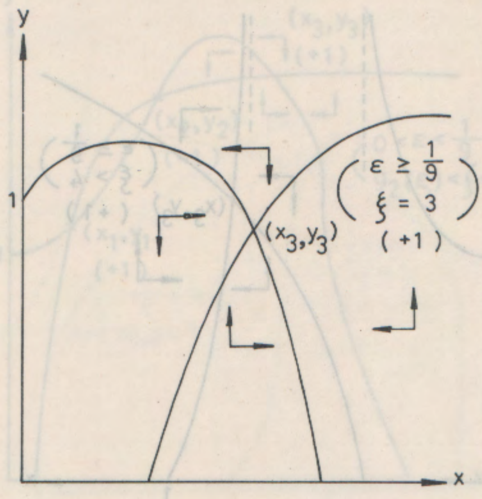


Fig.7

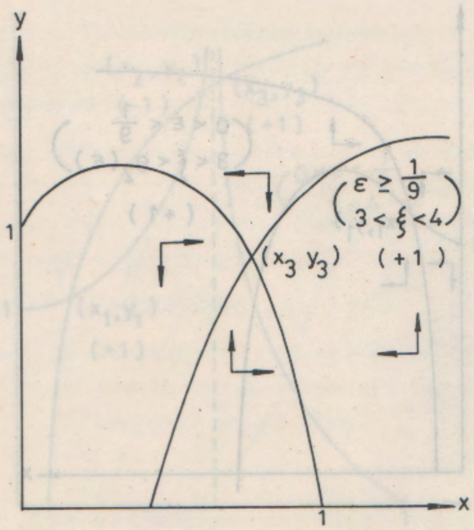


Fig.8

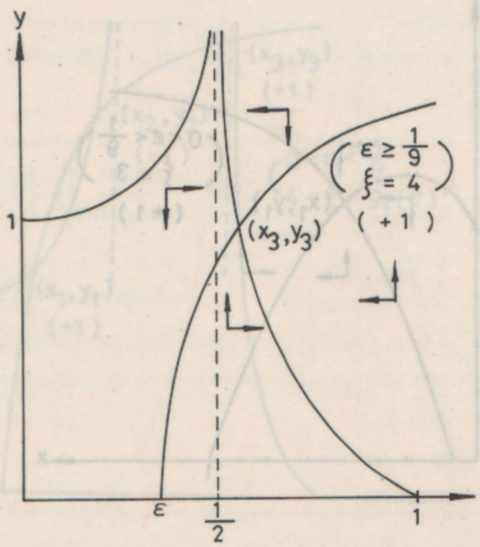


Fig.9

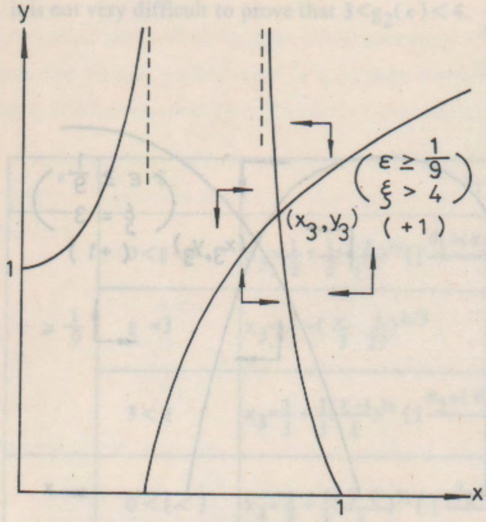
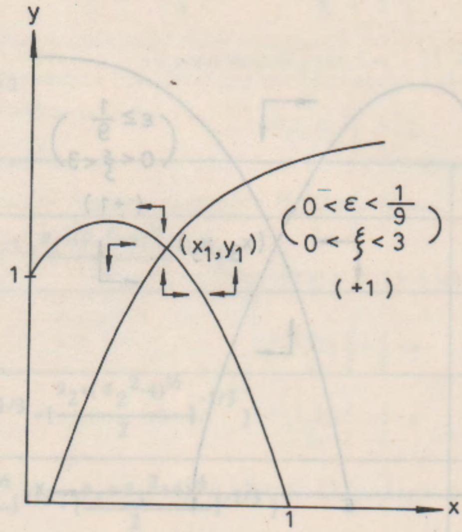


Fig. 10



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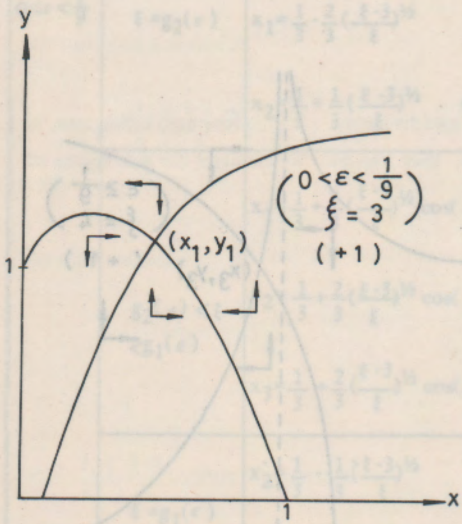


Fig. 12

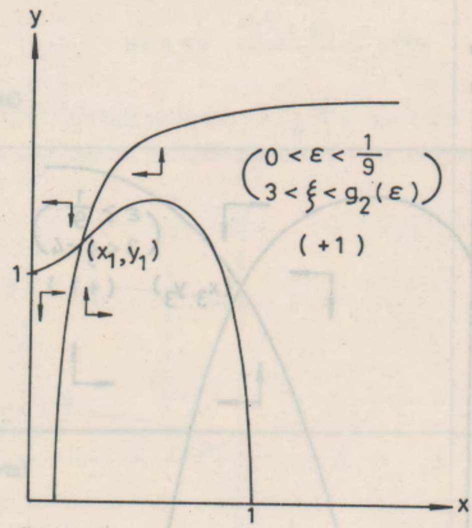


Fig. 13

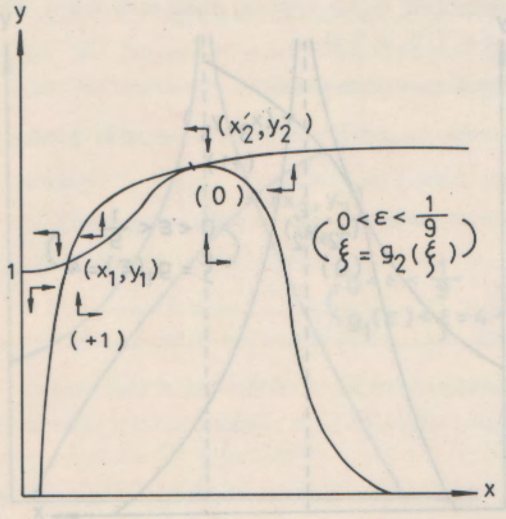


Fig.14

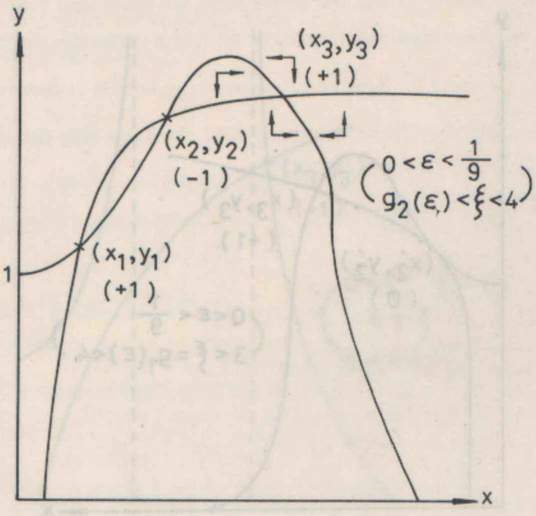
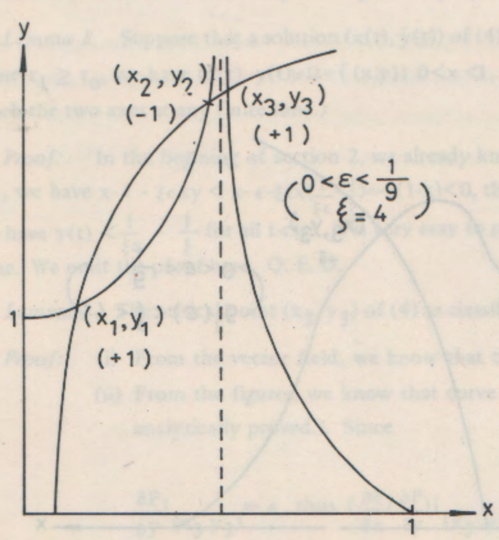


Fig.15



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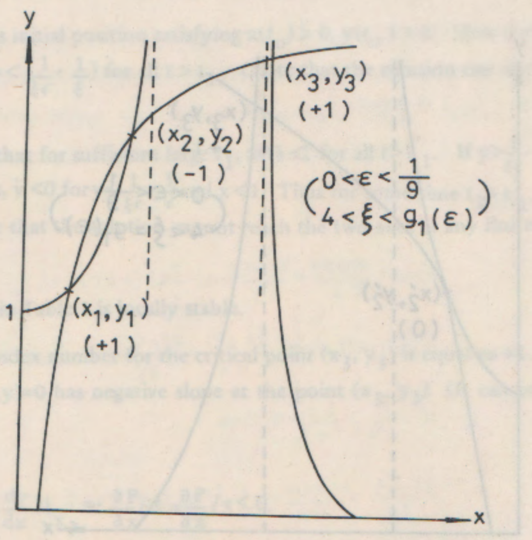


Fig.17

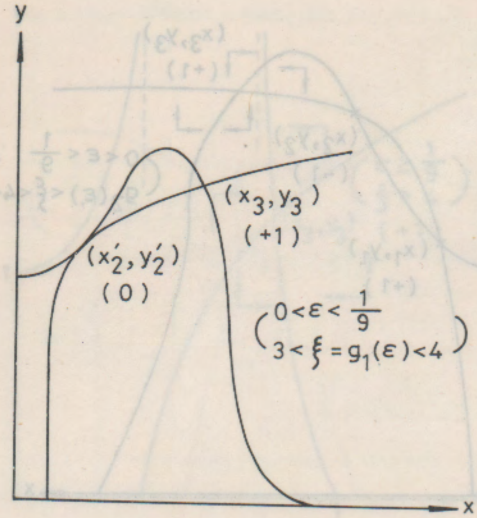
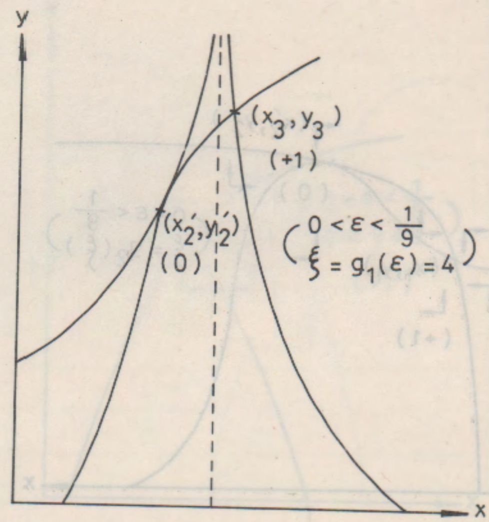


Fig.18



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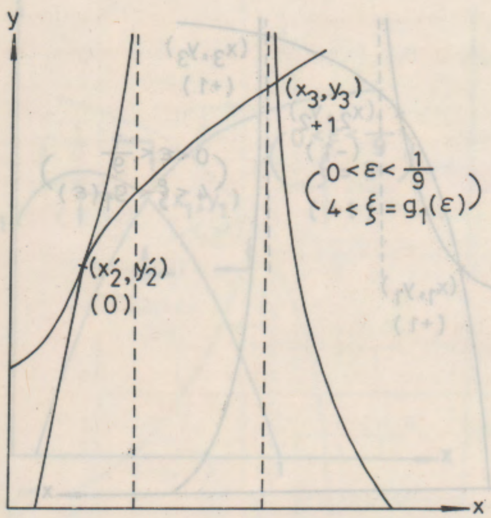
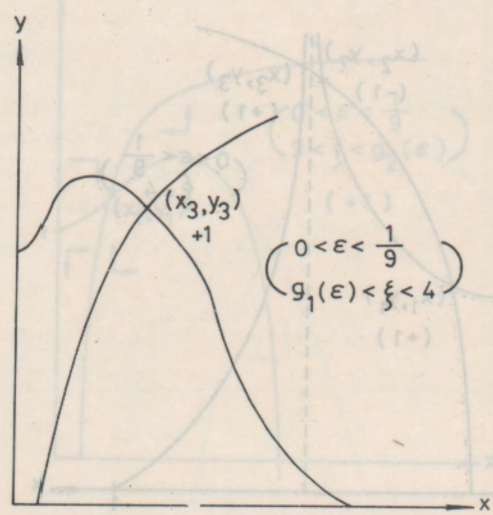


Fig.20



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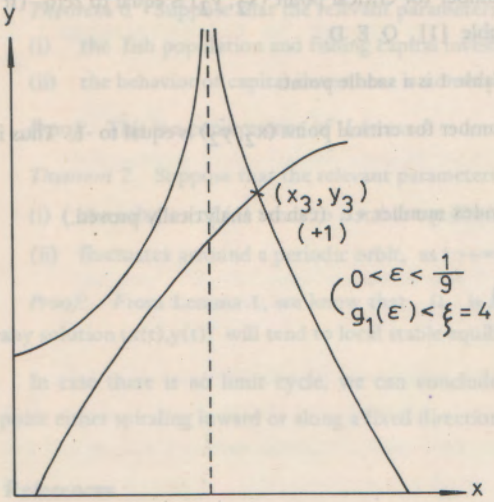

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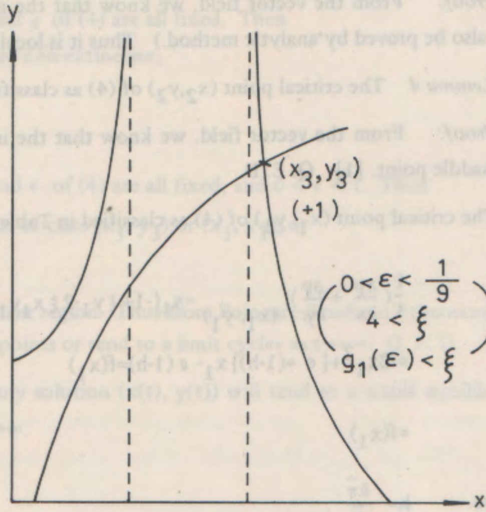

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III. Properties of the Solutions of the General Model

In this section we will carefully investigate the properties of the solutions of equation (4).

Lemma 1 Suppose that a solution $(x(t), \dot{y}(t))$ of (4) has initial position satisfying $x(t_o) > 0$, $y(t_o) > 0$. Then for some $t_1 \geq t_o$, we have $(x(t), y(t)) \in \Omega = \{ (x,y) \mid 0<x<1, 0<y< \frac{1}{\xi\epsilon} - \frac{1}{\xi}\}$ for all $t > t_1$. (Note that the solution can not reach the two axes at any finite time.)

Proof: In the begining of section 2, we already know that for sufficient large t_1', $x(t) < 1$ for all $t > t_1'$. If $y > \frac{1}{\xi\epsilon} - \frac{1}{\xi}$, we have $x - \epsilon - \xi\epsilon xy < x - \epsilon - \xi\epsilon x(\frac{1}{\xi\epsilon} - \frac{1}{\xi}) = -\epsilon(1-x) < 0$, that is, $\dot{y} < 0$ for $y > \frac{1}{\xi\epsilon} - \frac{1}{\epsilon}$ and $x < 1$. Thus for some time $t_1 \geq t_1'$, we have $y(t) < \frac{1}{\xi\epsilon} - \frac{1}{\xi}$ for all $t < t_1$. It is very easy to prove that the solution cannot reach the two axes at any finite time. We omit the proof here. Q. E. D.

Lemma 2 The critical point (x_3, y_3) of (4) as classified in Table 1 is locally stable.

Proof: (i) From the vector field, we know that the index number for the critical point (x_3, y_3) is equal to +1.

(ii) From the figures, we know that curve $P(x,y)=0$ has negative slope at the point (x_3, y_3) (It can be analytically proved.). Since

$$\frac{\partial P}{\partial y}\Big|_{(x_3,y_3)} = -\epsilon, \text{ thus } \left(-\frac{\partial P}{\partial x} \Big/ \frac{\partial P}{\partial y}\right)\Big|_{(x_3,y_3)} = \left(\frac{dy}{dx}\right)\Big|_{x_3} = -\frac{\partial P}{\partial x}\Big/-\epsilon = \frac{\partial P}{\partial x}\Big/\epsilon < 0.$$

Therefore

$$\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\Big|_{(x_3,y_3)} = \left(\frac{\partial P}{\partial x}\right)\Big|_{(x_3,y_3)} - \delta\pi\xi x_3 y_3 < 0.$$

From (i) and (ii), we have proved the *lemma [1]*. Q. E. D.

Lemma 3 The critical point (x_2', y_2') of (4) as classified in Table 1 is locally unstable.

Proof: From the vector field, we know that the index number for critical point (x_2', y_2') is equal to zero. (It can also be proved by analytic method.) Thus it is locally unstable [1]. Q. E. D.

Lemma 4 The critical point (x_2,y_2) of (4) as classified in Table 1 is a saddle point.

Proof: From the vector field, we know that the index number for critical point (x_2, y_2) is equal to -1. Thus it is a saddle point. [1]. Q. E. D.

The critical point (x_1, y_1) of (4) as classified in Table 1 has index number +1. (can be analytically proved.)

But $$\frac{1}{a}\left(\frac{\partial P}{\partial x}+\frac{\partial P}{\partial y}\right)\Big|_{(x_1,y_1)} = x_1(-1+\xi y_1-2\xi x_1 y_1-h\xi y_1)$$

$$=-2x_1^2+[\epsilon+(1-h)]x_1-\epsilon(1-h)=f(x_1)$$

$$=f(x_1)$$

where $h=\dfrac{\delta\hat{\pi}}{a}$.

Hence, we can conclude that

(i) $(1-h)<0 \Rightarrow f(x_1)<0 \Rightarrow (x_1, y_1)$ is locally stable,

(ii) $(1-h)>0$, if $f(x_1)<0 \Rightarrow (x_1, y_1)$ is locally stable.

(iii) $(1-h)>0$, if $f(x_1)=0 \Rightarrow (x_1,y_1)$ is locally stable (center),

(iv) $(1-h)>0$, if $f(x_1)>0 \Rightarrow (x_1,y_1)$ is locally unstable.

We cannot expect that ϵ is very small. Because if that is the case, then $\dot{y}>0$ up to very small x value and there is a tendency to fish extinction. Besides, this model is no longer valid at small x. Thus, we do not have to consider the case $\epsilon << 1$ in detail.

Lemma 5 There is no limit cycle of (4) in the first quadrant in case of $(1-h)\leqslant 0$ or $\frac{1}{1-h}>\frac{1-\epsilon}{\epsilon}>0$.

Proof: Let $\rho(x,y)=\dfrac{1}{xy}$,

Now $$\frac{\partial(\rho P)}{\partial x}+\frac{\partial(\rho Q)}{\partial y}=a[-\frac{1}{y}+\xi(1-h)-2\xi x]$$

If $(1-h)\leqslant 0$, then $\dfrac{\partial(\rho P)}{\partial x}+\dfrac{\partial(\rho Q)}{\partial y}<0$, Q. E. D.

If $\dfrac{1}{1-h}>\dfrac{1-\epsilon}{\epsilon}>0$, then

$$\frac{\partial(\rho P)}{\partial x}+\frac{\partial(\rho Q)}{\partial y}<a\{-\frac{\xi\epsilon}{1-\epsilon}+\xi(1-h)-2\xi x\}$$

$$=a\{-\xi[\frac{\epsilon}{1-\epsilon}-(1-h)]-2\xi x\}<0,$$ Q. E. D.

(See reference [1] or any other book on differential equation.)

In case of $0<\frac{1}{1-h}<\frac{1-\epsilon}{\epsilon}$, it requires additional investigation and is too lengthy for this article.

We arrive at the following main theorem :

Theorem 6. Suppose that the relevant parameters h, ξ and ϵ of (4) are all fixed. Then

(i) the fish population and fishing capital investment are non-extinctive;

(ii) the behavior of capital investment is non-explosive.

Proof: This is a consequence of *Lemma 1.*

Theorem 7. Suppose that the relevant parameters h, ξ and ϵ of (4) are all fixed, and $0 < \epsilon < 1$. Then

(i) the solution of (4) tends to a stable equilibrium point of class (x_3, y_3) or (x_1, y_1), or

(ii) fluctuates arround a periodic orbit, as $t \to +\infty$.

Proof: From Lemma 1, we know that Ω is an invariant region. Thus from Poincare-Bendison's theorem [1], any solution (x(t),y(t)) will tend to local stable equilibrium points or tend to a limit cycles as $t \to +\infty$. Q. E. D.

In case there is no limit cycle, we can conclude that any solution (x(t), y(t)) will tend to a stable equilibrium point either spiraling inward or along a fixed direction as $t \to +\infty$.

References

1. Coddington, E., and Levinson, N., Theory of Ordinary Differential Equations, New York: McGraw-Hill, 1955.
2. Fullenbaum, R. F., E. W. Carlson, and F. W. Bell, "Economics of production from natural resources: Comment" American Economic Review, 61, 483-487(1971).
3. Fullenbaum, R. F., E. W. Carlson, and F. W. Bell, "On models of commercial fishing: A defence of the tranditional literature," Jol. of Political Economy, 80, 761-768 (1972).
4. Smith, V. L., "On models of commercial fishing," Jol. of Political Economy, 77, 181-198 (1969).
5. Smith, V. L., "On models of commercial fishing: Traditional literature no need defence," Jol. of Politiacal Economy, 80, 769-772 (1972).
6. Smith, V. L., "Economics of production from natural resources" American Economic Review, 58, 409-431 (1968).
7. Smith, V. L., "Economics of production from natural resources; Reply" American Economic Review, 61, 488-491, (1971).
8. Leung, A., and Wang, A. Y., "Analysis of models for commercial fishing: Mathematical and econimical aspects," Econometrica, 44, 295-303 (1976).