The Journal of National Chiao Tung University
Vol. I, April 1976
pp. 91-102

含無法量得之狀態變數之最佳回饋控制系統某些結果之靈敏度分析

Sensitivity Analyses of Certain Results in the Optimal Control with Incomplete State Feedback

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(Received January 15, 1976)

ABSTRACT — Sensitivity study is essential in the design of optimal control system with inaccessible state variables which must be estimated from noisy incomplete data. In this paper, the sensitivities of such systems given in the author's previous results and of the Kalman type optimal state feedback systems are investigated and compared.

I. Introduction

The optimal control law for a linear system with a quadratic performance index requires feedback of each and every state variable. Due to difficulty in practice of measuring every state variable, the practical usefulness of optimal control law is restricted. In these cases, either the method must be abandoned or a reasonable substitute for the state variables must be found.

Several recent papers have discussed the determination of the optimal control of the state system which cannot be measured completely [1-11]. The author [1] has presented two methods of obtaining optimal dynamic controllers when some of the state variables are not available for continuous measurements. Method A is based on the fact that the optimal control law can be differentiated a number of times and combined appropriately to obtain an equivalent control law requiring only those state variables which are measurable. Method B, on the other hand, is based on the idea that a model can be constructed to generate an approximation to the unknown state vector using the available outputs and the control inputs. One can then use a suitable transformation on the states of the model to obtain the optimal control law in terms of the output alone.

The main problem of optimal control with inaccessible state variables is that of estimating the state variables from noisy incomplete data. Therefore, a sensitivity analysis must be performed to account for the possible errors in the change of initial conditions and system parameters.

Recently, it has been demonstrated that for a single input linear time-invariant controllable n-th order system the sensitivity functions (or parameter influence coefficients) of the state with respect to any number of system parameters can be generated by an n-th order sensitivity model in addition to a system mode [12]. In this paper, we use the method of Kokotovic and Rutman [13,14], to generate the sensitivity functions for the optimal systems obtained in one of the author's previous publications. Only the second order systems are considered. The results are compared with that obtained for the Kalman type optimal state feedback systems by means of an example.

II. Sensitivity Analysis for Kalman Type Optimal State Feedback Systems

Consider the standard second-order system and quadratic performance index,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a_1x_1 - a_2x_2 + u \tag{1}$$

$$y = x_1$$

and

$$J = \int_0^\infty \frac{1}{2}(x^TQx+\rho u^2)dt$$

The optimal control is

$$u = -k_1x_1 - k_2x_2$$

By elimination, the system may now be reduced to:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -(a_1+k_1)x_1-(a_2+k_2)x_2 \tag{2}$$

The optimal system response may be found by solving Equation (2) and is given by

$$x_1^*(t)= e^{-\frac{B}{2}t} \{(\cos rt+\frac{B}{2r} \sin rt)x_{10}+\frac{1}{r} \sin rt\ x_{20}\}$$

$$x_2^*(t) = e^{-\frac{B}{2}t}\{-\frac{A}{r}\sin rt\ x_{10} + [\cos rt - \frac{B}{2r}\sin rt]x_{20}\} \quad (3)$$

where

$$A = a_1 + k_1$$

$$B = a_2 + k_2$$

and

$$r = \sqrt{A - \frac{B^2}{4}}$$

The performance index may be written in the form

$$J = \frac{1}{2}\int_0^\infty (\alpha_1 x_1^2 + \alpha_2 x_1 x_2 + \alpha_3 x_2^2)dt \quad (4)$$

where

$$\alpha_1 = Q_{11} + \rho k_1^2$$

$$\alpha_2 = 2Q_{12} + 2\rho k_1 k_2$$

$$\alpha_3 = Q_{22} + \rho k_2^2$$

Substituting Equation (3) into Equation (4) we obtain

$$J = \int_0^\infty \frac{1}{2} e^{-Bt}(\beta_1\cos^2 rt + \beta_2\sin^2 rt + \beta_3\sin 2rt)dt$$

where

$$\beta_1 = \alpha_1 x_{10}^2 + \alpha_2 x_{10}x_{20} + \alpha_3 x_{20}^2$$

$$\beta_2 = \frac{1}{4r^2}[\alpha_1(Bx_{10}+2x_{20})^2 - \alpha_2(2Ax_{10}+Bx_{20})(Bx_{10}+2x_{20}) + \alpha_3(2Ax_{10}+Bx_{20})^2]$$

$$\beta_3 = \frac{1}{2r}[\alpha_1 x_{10}(Bx_{10}+2x_{20}) + \alpha_2(x_{20}^2 - Ax_{10}^2) - \alpha_3 x_{20}(2Ax_{10}+Bx_{20})]$$

Setting $\theta = rt$ and $\lambda = \frac{B}{r}$, we deduce immediately that the performance is given by

$$J^* = \frac{1}{2r}\int_0^\infty e^{-\lambda\theta}(\beta_1\cos^2\theta + \beta_2\sin^2\theta + \beta_3\sin 2\theta)d\theta$$

$$= \frac{1}{8AB}\{(B^2+2r^2)\beta_1 + 2r^2\beta_2 + 2rB\beta_3\} \quad (5)$$

Substituting β_i into Equation (5), yields

$$J^*= \frac{1}{4AB} \{[(A+B^2)\alpha_1-AB\alpha_2+A^2\alpha_3]x_{10}^2$$

$$+2B\alpha_1 x_{10}x_{20}+(\alpha_1+A\alpha_3)x_{20}^2\} \tag{6}$$

From the mathematics view-point, $J^*=J(a_1,a_2,x_{10},x_{20})$, the change of performance index, ΔJ, can be written as

$$\Delta J=\frac{\partial J^*}{\partial a_1}\Delta a_1+\frac{\partial J^*}{\partial a_2}\Delta a_2+\frac{\partial J^*}{\partial x_{10}}\Delta x_{10}+\frac{\partial J^*}{\partial x_{20}}\Delta x_{20} \tag{7}$$

Then

$$\frac{\Delta J}{J^*} = \frac{2\Delta x_{10}}{x_{10}} \{[(A+B^2)\alpha_1-AB\alpha_2+A^2\alpha_3]x_{10}^2+B\alpha_1 x_{10}x_{20}\}/\Delta_1$$

$$+ \frac{2\Delta x_{20}}{x_{20}} \{B\alpha_1 x_{10}x_{20}+(\alpha_1+A\alpha_3)x_{20}^2\}/\Delta_1- \frac{\Delta a_1}{A}\{(B^2\alpha_1-A^2\alpha_3)x_{10}^2$$

$$+ 2B\alpha_1 x_{10}x_{20}+\alpha_1 x_{20}^2\}/\Delta_1- \frac{\Delta a_2}{B} \{[(A-B^2)\alpha_1+A^2\alpha_3]x_{10}^2$$

$$+ (\alpha_1+A\alpha_3)x_{20}^2\}/\Delta_1 \tag{8}$$

where

$$\Delta_1= [(A+B^2)\alpha_1-AB\alpha_2+A^2\alpha_3]+A^2\alpha_3]x_{10}^2+B\alpha_1 x_{10}x_{20}$$
$$+(\alpha_1+A\alpha_3)x_{20}^2$$

For the sake of simplicity, consider the second-order system which has been considered by Liou et al. [1]:

$$\frac{d^2c}{dt^2} + 2\xi\frac{dc}{dt} + c = u \tag{9}$$

$$y = c$$

and the performance index

$$J(u)=\frac{1}{2} \int_o^\infty [c^2(t)+\rho u^2(t)]dt$$

with a given initial conditions $c(0)=c_o$, $\frac{dc(o)}{dt} = 0$.

Then

$$\frac{\Delta J}{J^*} = 2\,\frac{\Delta c_o}{c_o} - \frac{2}{B}\cdot\frac{(A-B^2)\alpha_1 + A^2\alpha_3}{(A+B^2)\alpha_1 - AB\alpha_2 + A^2\alpha_3}\,\Delta\xi \tag{10}$$

where

$$A = 1+k_1$$
$$B = 2\xi + k_2$$
$$\alpha_1 = 1+\rho k_1^2$$
$$\alpha_2 = 2\rho k_1 k_2$$
$$\alpha_3 = \rho k_2^2$$

III. Sensitivity Analysis for Liou's Method A Type Optimal Output Feedback Systems

Consider the same system as in Equation (9). The dynamic control law obtained by Liou's Method A [1]is given by

$$\dot{u} + L\,u = M\,\dot{y} + N\,Y \tag{11}$$

where

$$L = k_2 + 2\xi$$
$$M = -k_1$$
$$N = k_2 - 2\xi k_1$$

Take Laplace transform, Equations (9) and (11) yield

$$(s^2+2\xi s+1)\,c(s) = u(s) + (s+2\xi)c_o \tag{12}$$

$$(s+L)\,u(s) = (Ms+N)\,c(s) + u_o - Mc_o$$
$$= (Ms+N)\,c(s) - (M+k_1)\,c_o$$
$$(s+L)\,u(s) = (Ms+N)\,c(s) \tag{13}$$

Combination of Equations (12) and (13) yields

$$(s^2+2\xi s+1)(s+L)\,c(s) = (Ms+N)\,c(s) + (s+2\xi)(s+L)\,c_o$$

$$c(s)=\frac{[s^2+(2\xi+L)s+2\xi L]c_o}{s^3+(L+2\xi)s^2+(1+2\xi L-M)s+(L-N)}$$

$$u(s)=\frac{[Ms^2+(2\xi M+N)s+2\xi N]c_o}{s^3+(L+2\xi)s^2+(1+2\xi L-M)s+(L-N)}$$

Since

$$\int_o^\infty f^2(t)\,dt=\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}F(s)F(-s)\,ds$$

When F(s) is rational, the value of this complex integral has been tabulated in terms of the polynominal coefficients by Newton et al. [15]

Then

$$J^*=\frac{1}{2}\int_o^\infty [c^2(t)+\rho u^2(t)]dt$$

$$=\frac{c_o^2}{2\phi_1}\{(1+\rho M^2)d_o d_1+[(2\xi-L)^2+\rho(2\xi M-N)^2]d_o$$

$$+\ 4\xi^2 d_2(L^2+\rho N^2)\}$$

where

$$d_2=L+2\xi\ ,\quad d_1=1+2\xi L-M,\ d_o=L-N$$

and

$$\phi_1=2d_o\,(d_1 d_2-d_o)$$

Now consider

$$J^*=J^*(c_o,\ \xi)$$

$$\Delta J=\frac{\partial J^*}{\partial c_o}\,\Delta c_o+\frac{\partial J^*}{\partial \xi}\,\Delta\xi$$

Then

$$\frac{\Delta J}{J^*}=2\frac{\Delta c_o}{c_o}+$$

$$\{\frac{2L(1+\rho M^2)d_o+[8\xi(1+\rho M^2)-4(L+\rho MN)]d_o+8\xi(L^2+\rho N^2)(\xi+d_2)}{(1+\rho M^2)d_o d_1+[4\xi^2(1+\rho M^2)-4\xi(L+\rho MN)+(L^2+\rho N^2)]d_o+4\xi^2(L^2+\rho N^2)d_2}$$

$$-\frac{\phi_1'}{\phi_1}\}\Delta\xi$$

where $\phi_1' = 4d_o(d_1 + Ld_2)$

IV. Sensitivity Analysis for Liou's Method B Type Optimal Output Feedback Systems

Consider the same system as in Equation (9). The dynamic control law obtained by Liou's Method B [1] is given by

$$z + \bar{M}z = \bar{N}y$$

where

$$z = u+Ty$$

$$\bar{M} = 1+k_2$$

$$\bar{N} = \overline{MT}-k_1-k_2$$

$$T = k_1+(1-2\xi)k_2$$

and $z(0)=(T-k_1)c_o$

Define

$$\bar{z} = [c,\ c,\ z]^T$$

Then

$$\bar{z}(s)= \frac{1}{\Delta_3}\begin{bmatrix} s^2+(2\xi+\bar{M})s+2\xi\bar{M} & s+\bar{M} & 1 \\ -(1+T)s-\bar{M}(1+T)+\bar{N} & s^2+\bar{M}s & s \\ \bar{N}(s+2\xi) & \bar{N} & s^2+2\xi s+(1+T) \end{bmatrix}$$

where

$$\Delta_3=s^3+(2\xi+\bar{M})s^2+(1+T+2\xi\bar{M})s+(\bar{M}+\bar{M}T-\bar{N})$$

or

$$c(s)= \frac{c_o}{\Delta_3}\{s^2+(2\xi+\bar{M})s+(2\xi\bar{M}+T-k_1)\}$$

$$u(s)=z(s)-Tc(s)$$

$$=\frac{c_o}{\Delta_3}\{-k_1s^2+(k_2-k_1-2\xi k_1)s+[T-k_1-2\xi(k_1-k_2)]\}$$

Then, $$J^*= \frac{c_o}{2\phi_2}\{\alpha_1d_od_1+(4\alpha_1\xi^2+\delta_1)d_o+(4\xi^2\delta_2+4\xi\delta_3+\delta_4)d_2\}$$

where

$$\alpha_1= 1+\rho k_1^2$$

$$\delta_1=\delta_2-2(T-k_1)(1-\rho k_1)$$

$$\delta_2 = (1+k_2)^2 + \rho(k_1-k_2)^2$$

$$\delta_3 = (T-k_1)[\overline{N}-\rho(k_1-k_2)] = (T-k_1)[(1-\rho k_1)+k_2(1+\rho)]$$

$$\delta_4 = (T-k_1)^2(1+\rho)$$

$$d_2 = \overline{M} + 2\xi$$

$$d_1 = 2\xi\overline{M} + 1 + T$$

$$d_o = \overline{M} + \overline{M}T - \overline{N} = 1 + k_1$$

$$\phi_2 = 2d_o d_1 d_2$$

since

$$\Delta J = \frac{\partial J^*}{\partial c_o}\Delta c_o + \frac{\partial J^*}{\partial \xi}\Delta\xi$$

Then

$$\frac{\Delta J}{J^*} = 2\frac{\Delta c_o}{c_o} + \left\{\frac{2\overline{M}\alpha_1 d_o + 8\alpha_1\xi d_o + (8\xi\delta_2+4\delta_3)d_2 + 2(4\xi^2\delta_2+4\xi\delta_3+\delta_4)}{\alpha_1 d_o d_1 + (4\alpha_1\xi^2+\delta_1)d_o + (4\xi^2\delta_2+4\xi\delta_3+\delta_4)d_2} - \frac{\phi_2'}{\phi_2}\right\}\Delta\xi$$

where $\quad \phi_2' = 4d_o(d_1+\overline{M}d_2)$

From above analysis, we get $\frac{\Delta J}{J^*} = 2\frac{\Delta c_o}{c_o}$ for all $\Delta\xi = 0$. Now we take a simple example to investigate the effect of $\frac{\Delta J}{J^*}$ by small change of $\Delta\xi$.

Example

Consider the control system which has been considered by Liou et al. [1].

$$\frac{d^2c}{dt^2} + \frac{3}{2}\frac{dc}{dt} + c = u$$

and the performance index

$$J(u) = \frac{1}{2}\int_o^\infty [c^2(t) + \frac{1}{8}u^2(t)]dt$$

Then

$$\rho = \frac{1}{8}, \quad \xi = \frac{3}{4}$$

and $\quad k_1 = 2, \quad k_2 = 1$

(1) For Kalman type system:

$$A=1+k_1=3$$
$$B=2\xi+k_2=\frac{5}{2}$$
$$\alpha_1=1+\rho k_1^2=\frac{3}{2}$$
$$\alpha_2=2\rho k_1 k_2=\frac{1}{2}$$
$$\alpha_3=\rho k_2=\frac{1}{8}$$

Then $\frac{\Delta J}{J^*}=2\frac{\Delta c_o}{c_o}+0.2\frac{\Delta\xi}{\xi}$.

(2) For Method-A system:

$$L=\frac{5}{2}, \quad M=-2, \quad N=-2$$
$$d_2=4, \quad d_1=\frac{27}{4}, \quad d_o=\frac{9}{2}$$
$$\phi_1=\frac{405}{2}, \quad \phi_1'=-\frac{603}{2}$$

Then $\frac{\Delta J}{J^*}=2\frac{\Delta c_o}{c_o}+2.15\frac{\Delta\xi}{\xi}$.

(3) For Method-B system:

$$\bar{M}=2, \quad \bar{N}=2, \quad T=\frac{3}{2}$$
$$d_2=\frac{7}{2}, \quad d_1=\frac{11}{2}, \quad d_o=3$$
$$\alpha_1=\frac{3}{2}, \quad \phi_2=\frac{231}{2} \quad \phi_2'=150$$
$$\delta_1=\frac{39}{8}, \quad \delta_2=\frac{33}{8}, \quad \delta_3=-\frac{15}{16}, \quad \delta_4=\frac{9}{32}$$

Then $\frac{\Delta J}{J^*}=2\frac{\Delta c_o}{c_o}+0.38\frac{\Delta\xi}{\xi}$.

V. Conclusion

The sensitivites of the Kalman type optimal state feedback control systems and the optimal control systems with inaccessible state feedback designed via the author's previously published methods are investigated and compared. For the particular example considered, the Kalman type system is the least sensitive and the Liou's method-B type system is less sensitive than the Liou's method-A type system, to the change of system parameters. However, all three types of systems have exactly the same sensitivity with respect to the change in the initial output. The incremental sensitivity with respect to the initial output, $\frac{\Delta J}{J^*}/\frac{\Delta C_o}{C_o}$, is indopendent of the initial output and the system parameters, and is equal to 2 when the initial rate of output change is zero. That is

$$\frac{\Delta J}{J^*} = 2\,\frac{\Delta C_o}{C_o} \quad \text{for} \quad \Delta\xi=0 \quad \text{and} \quad \dot{C}(0)=0$$

Acknowledgment

The author wishes to express the appreciation to the financial support to this work from the National Science Council of the Republic of China.

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