The Journal of National Chiao Tung University
Vol. 4, April 1978
pp. 29-32

區位和生產理論
A Note on Location and Production Theory

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(Received March 1, 1978)

Abstract — We point out that the results obtained by Khalili, Mathur and Bodenhorn about the location and production theory are partially incorrect. We give the precise results and rigorous proofs.

I. Introduction

The theories of plant location are one segment of economic theory. The pioneering work in the theory of least-cost location was carried out by Weber [7]. In 1958, Moses [5] presented a method to integrate the theory of location with the theory of production. A few years latter, Bradfield [1] pointed out that Moses confused fixed coefficient production functions with linear production functions and nonlinear production function with non-homogeneous ones. More recently, Khalili, Mathur and Bodenhorn [4] provided a mathematical model to show that the theory as expounded by Moses and the improvement made by Bradfield on Moses' theory are special cases of their theory.

The "Location problem" of the firm can be posed as follows. Assume a one-plant firm which is interested in finding the optimal production location $\vec{x}$ uses two transportable inputs M_1 at $\vec{x}_1$ and M_2 and $\vec{x}_2$ and supplies its single final product to a consumption center at $\vec{x}_3$. (See Fig. 1)

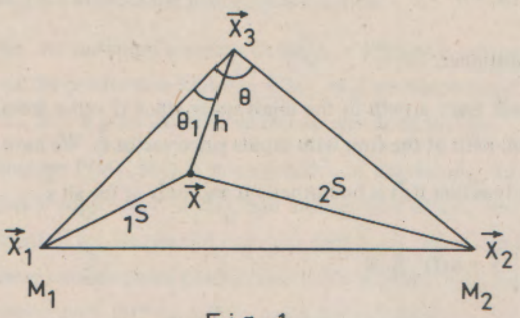

Fig. 1

In the figure, $\vec{x}_1$, $\vec{x}_2$ and $\vec{x}_3$ are Known. The problem is to find that location in the triangle which minimizes total cost given a level of output and to find the effect of changes in the level of output on the optimum location of the firm.

The results obtained by Khalili, Mathur and Bodenhorn [4] are:

(1) When the production location is constrained to remain at specified distance h from the market $\vec{x}_3$, an assumption used by Moses and Bradfield, and it is within the location triangle, they show that the necessary and sufficient condition for the existence of a single optimum production location is that the firms expansion path is linear as output expands.

(2) When h is a variable, they show that the necessary and sufficient condition for a single optimum location as output varies is that the production function is homogeneous of degree one.

(3) Assuming a homogeneous production function, they show that under increasing returns to scale the firm will move towards the market site and under decreasing returns to scale the firm will move away from the market as output rises.

The "if" parts of the results (1) and (2) are correct. But the "only if" parts of the results (1) and (2) are not obvious at all. It is one of the purposes of this note to point out that the "only if" parts are in general not true. In fact, we will give a counter-example to the "only if" part of result (2) a moment later. The underlying secret of this inaccuracy is the confusion between local and global properties of the production function. It is precisely the same confusion which leads Vaughn and Pfouts incorrectly to the conclusion "The production stages of a production function are symmetric if and only if the production function is linear homogeneous". [6].

II. Production Function and Expansion Path

Let inputs price vector $\vec{P} = (P_1, P_2, \ldots P_n)$ be given, then the firm's long-run expansion path $\vec{x}(Q,\vec{P})$ is the solution of the following minimizing problem

$$\min_{\{\vec{x}\}} \vec{P}\cdot\vec{x} \qquad \text{subject to } Q = f(\vec{x}) \tag{1}$$

of course, we assume that for a given Q and $\vec{P}$, the solution is unique. Based on this assumption, the firm's long-run expansion path $\vec{x}(Q, \vec{P})$ is also the solution of the following equations

$$\begin{cases} \vec{\nabla} f(\vec{x}(Q,\vec{P})) = \lambda \vec{P} \\ Q = f(\vec{x}(Q, \vec{P})) \end{cases} \tag{2}$$

where λ is the Lagrangian multiplier.

When $\vec{P}$ is fixed, $\vec{x}(Q,\vec{P})$ will trace a path in the input space when Q varies from zero to infinite. This path in fact is the usual long-run expansion path of the firm with inputs price vector $\vec{P}$. We have the following two theorems.

Theorem 1. A production function $f(\vec{x})$ is homothetic if and only if for all $\vec{x}$

$$\vec{\nabla} f(\vec{x}) \cdot \vec{x} = K(f) > 0$$

Theorem 2. A production function $f(\vec{x})$ is homothetic if and only if for every inputs price vector $\vec{P}$, the expansion path $\vec{x}(Q, \vec{P})$ is a ray through the origin, that is,

$$\vec{x}(Q, \vec{P}) = K(Q)\,\vec{x}(\vec{P}),$$

where K(Q) is a monotonic increasing function of Q with K(O) = 0.

[We omit the proofs here].

III. Location and Production Theory

The mathematical formulation of the location problem is (we use the same notation as in reference [4])

$$\begin{aligned} &\min C = (P_1 + r_1\ {}_1S)\, M_1 + (P_2 + r_2\ {}_2S)\, M_2 + r_3 h\, q_0 \\ &\text{Subject to} \\ &q_0 = F(M_1, M_2) \end{aligned} \tag{3}$$

where $\underline{P}_1$ and $\underline{P}_2$ are base prices, r_1 and r_2 are constant transport rates of M_1 and M_2 respectively and r_3 is the constant transport rate of the final product to $\vec{x}_3$. We assume the unique solution to this location problem is h^*, θ_1^*, M_1^* and M_2^*, then

$$P_1 + r_1\ {}_1S^* - \lambda^* F_1^* = 0 \tag{4}$$

$$P_2 + r_2\ {}_2S^* - \lambda^* F_2^* = 0 \tag{5}$$

$$r_1 M_1^*\ {}_1S_{\theta_1}^* + r_2 M_2^*\ {}_2S_{\theta_1}^* = 0 \tag{6}$$

$$r_1 M_1\ {}_1S_h^* + r_2 M_2^*\ {}_2S_h^* + r_3 q_0 = 0 \tag{7}$$

$$q_0 = F(M_1^*, M_2^*) = F^* \tag{8}$$

Now we can write the correct versions of the results obtained by Khalili, Mathur and Bodenhorn as follows [8].

Proposition 1. Holding relative prices constant, if h is constant and greater than zero and θ_1 is a variable where $0 < \theta_1 < \theta$, the optimum plant location $\theta_1^*(q_0)$ is independent of the output q_0, if and only if the expansion path of the production function $F(M_1, M_2)$ with respect to the given price vector $(P_1^*, P_2^*) = (P_1 + {}_1S^* r_1, P_2 + {}_2S^* r_2)$ is a ray through the origin.

Remark: From Theorem 2, we know that if $F(M_1, M_2)$ is homothetic, then the expansion path must be a ray through the origin for every price vector. Thus in this case $\theta_1^*(q_0)$ is independent of q_0. Conversely, if $\theta_1^*(q_0)$ is independent of q_0, then the expansion path for that particular price vector (P_1^*, P_2^*) is a ray through the origin. We can not conclude that $F(M_1, M_2)$ is a homothetic production function.

Proposition 2. If h is a variable, the optimum location $(h^*(q_0), \theta_1^*(q_0))$ is independent of the output level q_0, if and only if, the expansion path of the production function $F(M_1, M_2)$ passing through the point $(M_1^*(q_0), M_2^*(q_0))$ is a ray through the origin and $F(M_1, M_2)$ is homogeneous of degree one along this ray.

Remark: If the production function $F(M_1, M_2)$ is homogeneous of degree one, then its expansion path through every point is certainly a ray. Thus $h^*(q_0)$ and $\theta_1^*(q_0)$ are independent of q_0. But conversely, if we know that $h^*(q_0)$ and $\theta_1^*(q_0)$ are independent of q_0, we can not conclude that $F(M_1, M_2)$ is linear homogeneous. Let us give an example. Suppose we have a linear homogeneous production function $F(M_1, M_2)$, then surely $h^*(q_0)$ and $\theta_1^*(q_0)$ are independent of q_0. The expansion path $(M_1^*(q_0), M_2^*(q_0))$ for q_0 varies is a ray through the origin. Now we add a function $G(M_1, M_2)$ to $F(M_1, M_2)$, such that

$$G(M_1, M_2) = [M_2 - \frac{M_2^*}{M_1^*} M_1]^4 \cdot K(M_1, M_2)$$

where $K(M_1, M_2)$ is some small positive function of M_1 and M_2. It is easy to see that the optimum location for the production function $F(M_1, M_2) + G(M_1, M_2)$ is the same as the optimum location for the production function $F(M_1, M_2)$.

Proposition 3. **Assuming that the production function is homothetic and both $0 < \theta_1 < \theta$ and $h > 0$ are variables, then $(dh^*(q_0))/(dq_0) \gtreqless 0$, if and only if, $(\epsilon - 1 + \epsilon' q_0) \lesseqgtr 0$, where $\epsilon(q_0)$ is the elasticity of production and $\epsilon' = (d\epsilon(q_0))/(dq_0)$.**

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8. We omit the proof here. More detailed version of this note will be published elsewhere.