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Theory of Self-resonant Modes with A Zero-external Magnetic Field in Josephson Tunnelling Junction

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Abstract --The effect of self-resonant modes without any external magnetic field in Josephson junctions, the so-called "Anomalous DC Current Singularities" by Chen et al. in 1971, is first quantitatively investigated by solving the Josephson's equation under the assumption of no external magnetic field but with a dc bias and open-ended cavity boundary conditions. The theory is on the basis of the nonlinear mutual interaction of harmonics of the superconducting current with the electrical and magnetic fields induced by the superconducting current itself in the junction cavity. A series of even-modes of current singularities in the dc current-voltage characteristics appears accordingly. It also allows us to explain the Q-value (quality value in the resonance cavity) dependence of the current singularities, behaviour of the cut-off voltage, and the maximum height of the first even-mode of the current singularities in the experimental observations.

I. Introduction

Self-resonant modes in Josephson tunneling junction have received considerable theoretical and experimental attention. Such self-resonant modes show clearly as current singularities or steps in the dc current-voltage characteristics of a junction. Two groups of the current steps associated with the modes are found in the experiment. The first group occurs on applying an external magnetic field. These are known as Fiske mode [1] ; When an externally applied dc magnetic field He exists in the junction and the junction voltage is not zero, the tunneling current is not uniform. It will have the form as a travelling wave, moving at right angles to He. As a result, an ac electromagnetic field builds up in the barrier. Since impedence of the junction barrier is very low compared to that of free space (a factor 10^5!), we know that the wave will be almost complete reflected at the boundary and at certain frequencies, an intense standing wave is established. The junction barrier has been viewed as a microwave open-ended cavity. Because of the nonlinearity of the system, this ac field interacts with the current density wave to produce a dc supercurrent that shows up as a step structure or current singularities in the current-voltage characteristics of the junction. The first explanation of such resonant modes is given by Dmitrenko and Yanson [2] . Then the theory of the effect was developed by Kulik [3], Eck et al. [4] , and Coon et al. [5] , practically at the same time. All of these methods are too simple to explain some phenomena associated with the effect. So Werthamer and Shapiro [6] used a computer simulation, and Kulik [7] developed a more general perturbation method, to predict some qualitative and quantitative properties of the effect in 1967. These theories seem in general to be quite satisfactory with experimental observations in near-ideal Josephson tunneling junctions [8] .

The other group of current singularities, which was observed and called as "Anomalous DC current singularities" (ADCS) by Chen et al. [9] in 1970, can appear in zero external magnetic field in long junction ($\lambda_J \ll L$), where λ_J is the Josephson penetration depth, and L is a length of the barrier. The singularities can appear only corresponding to the voltages for even numbers instead of all integers in Fiske modes (n=2,4,6,...). This appearently contradicts the above theory of the Fiske modes. One possible explanation for this zero-field phenomena, suggested by Fulton and Dynes [10] in 1972, attributes it to inertial motion of an isolated vortex In 1974, the more experimental results for these signularities in high-Q finite Josephson tunnelling junctions ($\lambda_J \sim L$) reported by Gou et al. [8], such as temperature dependence, the maximum step height to be the same as for Fiske mode, and behavior of non cut-off voltage e.t.c., can not be explained in terms of this vortex model. Probabily the most important reason is that a well-developed theory is needed to provide what kind of physical mechanism gives rise to the current singularties. As will be shown here, all of these Gou's observations can be explained in terms of our theory; they are primarily associated with the nonlinear nature of self-resonance phenomena in tunnelling structures as that of the Fiske mode. We give our theory and formulation in Sec. II. The comparisons of our theory with experimental results will be shown in Sec. III. Finally, the conclusion will be described in Sec. IV.

II. Formulation of the Theory

According to the Josephson effect, a superconducting current flows across an insulating barrier in a Josephson junction as

$$j=j_s \sin \psi(x,t) \tag{1}$$

Where ψ (x,t) is the phase difference between two superconductors at (x,t), and j_s is the maximum critical current density of the junction. The phase difference ψ acting as a wave function across the junction barrier is a function of magnetic field H and electric field E, which are related to junction coordinates, shown in the Fig. (1),and time ,respectively, as

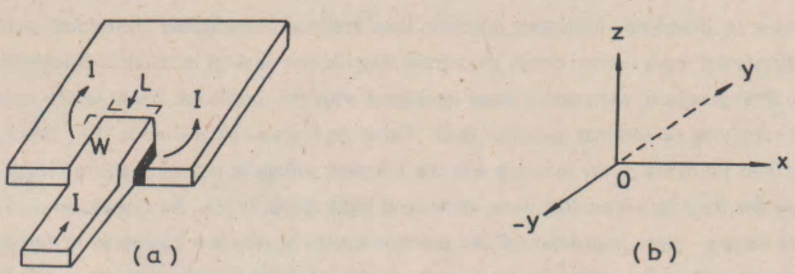

Fig. 1 (a) The geometry and current pattern of the Josephson junction (b) Coordinate system for our problem

$$\frac{\partial \psi}{\partial t} = \frac{2eV}{\hbar} \qquad \frac{\partial \psi}{\partial x} = \frac{2ed}{\hbar c} H \tag{1'}$$

where $d=t+2\lambda_L$, t is the thickness of dielectric (t ~20A), λ_L is the London penetration depth and ν is a potential difference across the barrier and equal to Et.

Since the phase difference ψ is dependent on the stength of E-field and H-field in Eq. (1)', and hence then the EM field alternates interacting with the superconducting current j in Eq. (1) so that the equation describes the phase difference ψ can be obtained by applying Maxwell's equations as follows:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{c_o^2}\left(\frac{\partial^2 \psi}{\partial t^2} + \gamma \frac{\partial \psi}{\partial t}\right) = \frac{1}{\lambda_J^2} \sin \psi \tag{2}$$

This is called Josephson's equation [11], where λ_J is the Josephson penetration depth, c_o is the propagation velocity and γ is the damping factor. The dissipation γ from the normal current resistance and radiation loss is included. Eq. (2) is a highly nonlinear Klein-Gordon equation. Hence we will still use a pertubation method to solve Eq. (2) with no external magnetic field in the Josephson junction. The driving current in the right hand side of Eq. (2)

will be treated by the perturbation.

As same as Fiske modes, wave propagating along the Josephson barrier can be regarded as along a transmission line and the junction looks like a microwave open-ended cavity. The wave, therefore, will be totally reflected from the junction boundary. Thus the boundary condition [3] also uses:

$$\left.\frac{\partial\varphi}{\partial x}\right|_{o} = \left.\frac{\partial\varphi}{\partial x}\right|_{L} = 0 \qquad (3)$$

Now if a dc bias is applied, with no any external magnetic field is applied, we can assume the phase difference φ as

$$\varphi = \omega_{dc}\, t + \int \Omega(x,t)\, dt \qquad (4)$$

where $\hbar\,\omega_{dc}=2$ eV and $\Omega(x,t)$ is viewed as the perturbed term which may be due to the self-magnetic field.

When the open-ended cavity conditions in Eq. (3) are applied, $\Omega(x,t)$ can be expressed as $W(t)\cos\frac{n\pi}{L}x$, and W(t) depends on time only. Differentiating Eq. (2) with respect to time so as to to change the variable into voltage, we get:

$$\left[\frac{\partial^2}{\partial x^2} - \frac{1}{c_o^2}\left(\frac{\partial^2}{\partial t^2}+\gamma\frac{\partial}{\partial t}\right)\right]\Omega(x,t)=\frac{1}{\lambda_J^2}\frac{\partial}{\partial t}\left[\cos\left(\int\Omega(x,t)dt+\omega_{dc}t\right)\right] \qquad (5)$$

with the initial phase difference $\varphi_o = \frac{\pi}{2}$, for convenience. Multiplying Eq. (5) with $\cos\frac{n\pi}{L}x$ and then intergrating over the junction length, we obtain:

$$\left(\frac{\partial^2}{\partial t^2}+\gamma\frac{\partial}{\partial t}+\omega_n^2\right)W(t)=-\omega_J^2\left(\frac{2}{L}\right)\frac{\partial}{\partial t}\int_o^L \cos\left(\int W(t)\cos\frac{n\pi}{L}x\, dx+\omega_{dc}t\right)\cos\frac{n\pi}{L}x\, dx$$

where $k_n=\frac{n\pi}{L}$, $\omega_n=k_n c_o$ and $\omega_J^2=c_o^2/\lambda_J^2$

Using the identity of Bessel function:

$$e^{ix\cos\theta} = \sum_{k=-\infty}^{\infty} i^k J_k(x)\, e^{ik\theta} \qquad (6)$$

the right hand side of the above integral can be simplified and then integrate with respect to time in Eq. (5), we get

$$\left(\frac{\partial^2}{\partial t^2}+\gamma\frac{\partial}{\partial t}+\omega_n^2\right)\int W(t)dt=2\,\omega_J^2\, J_1\left(\int W(t)dt\right)\sin\omega_{dc}\,t \qquad (7)$$

Here if weak-coupling between the superconducting current and the EM field is considered in the junction, and a single dominant mode solution [7] will be assumed as that of Kulik's method in Fiske modes, then $\int W(t)dt= Z\cos(\omega t+\alpha)$, where Z is the amplitude, α is the phase angle, and $\omega_{dc}=m\,\omega$, m is any integer.

Substituting into Eq. (7) and simplifying, we get:

$$Z=\frac{2\omega_J^2}{(\omega_n^2-\omega^2+i\gamma\omega)}\;\frac{1}{\pi}\int_o^{2\pi}\sin(m\,\omega t)J_1(z\cos(\omega t+\alpha))\, e^{i(\omega t+\alpha)}\, d(\omega t) \qquad (8)$$

Using another identity in Bessel function:

$$J_n(Z\cos\theta)=\sum_{k=-\infty}^{\infty}(-1)^k J_k\left(\frac{Z}{2}\right)J_{k+n}\left(\frac{Z}{2}\right)e^{-i(2k+n)\alpha} \qquad (9)$$

we can simplify Eq. (8) as follows:

$$Z=\frac{2\omega_J^2}{(\omega_n^2-\omega^2+i\gamma\omega)}\frac{1}{i}[J_{\frac{m}{2}}(\frac{Z}{2})J_{1+\frac{m}{2}}(\frac{Z}{2})(-1)^{\frac{m}{2}}\ell^{-im\alpha}-J_{-\frac{m}{2}}(\frac{Z}{2})J_{1-\frac{m}{2}}(\frac{Z}{2})(-1)^{-\frac{m}{2}}\ell^{im\alpha}] \quad (10)$$

where m's should be chosen as even integers in order to get a self-consisten solution, otherwise, $J_{-\frac{m}{2}}(Z)$ will approach to infinite as z approaching to zero. Therefore, we automatically obtain the even mode solutions when a zero external magnetic field is assumed in Eq. (4).

Now let us set $m=2\ell$,

$$\cos\theta=\frac{\gamma\omega}{\sqrt{(\omega_n^2-\omega^2)^2+\gamma^2\omega^2}} \quad (11a)$$

and

$$\Gamma=\frac{\omega_J2}{\omega\gamma} \quad (11b)$$

where Γ is called a coupling strength which is relative to the superconducting current coupled into the junction. Then Eq. (10) becomes

$$Z=-2\Gamma\cos\theta\,\ell^{i\theta}[J_\ell(\frac{Z}{2})J_{\ell+1}(\frac{Z}{2})\ell^{-2i\alpha\ell}+J_\ell(\frac{Z}{2})J_{\ell+1}(\frac{Z}{2})\ell^{2i\alpha\ell}](-1)^\ell$$

Since Z is the amplitude, and a real, positive number of Z should be taken so Im(z)=0.

We find that

$$\tan(2\alpha\ell+\ell\pi)=\tan\theta\frac{J_{\ell-1}(\frac{Z}{2})+J_{\ell+1}(\frac{Z}{2})}{J_{\ell-1}(\frac{Z}{2})-J_{\ell+1}(\frac{Z}{2})}$$

and

$$Z=2\Gamma\,|\,J_\ell(\frac{Z}{2})\,\|\,J_{\ell-1}^2(\frac{Z}{2})-J_{\ell+1}^2(\frac{Z}{2})|\frac{1}{\{[J_{\ell+1}(\frac{Z}{2})-J_{\ell-1}(\frac{Z}{2})]^2+\tan\theta^2[J_{\ell+1}(\frac{Z}{2})+J_{\ell-1}(\frac{Z}{2})]^2\}^{1/2}}$$

Then the dc current, which is usually observe,can be readily obtained by averaging over time and length of the junction. Finally we obtain an important result;

$$\frac{j_{dc}}{j_s}=\frac{1}{2\pi L}\int_0^{2\pi}\int_0^L\cos(m\omega t+Z\cos(\omega t+\alpha))\cos\frac{n\pi}{L})(dx\,d(\omega t)$$

$$=\frac{Z^2}{8\Gamma} \quad (14)$$

Eq. (14) will be used to compare with the experimental observations in the following section.

III. Comparison with Experimental Results and Discussion

Here we will concentrate several important predictions, which can be compared with the experimental observations, in our theory. (1) From Eq. (10), we have already seen that only even modes can appear, (m=2,4,6...or ℓ =1,2, 3....,) when the external magnetic field is not applied. Such even-mode solutions are confirmed with the significant observations for self-resonant modes with a zero external magnetic field in the experiments [8,9,10]. (2) The prediction of j_{dc}/j_s, an important experimentally observable quantity, varying with coupling strength Γ for different modes in Eq. (14) is described. The figures of j_{dc}/j_s via Γ are plotted with tan θ as parameter in Fig. (2,3,4) for ℓ =1,2,3, respectively.

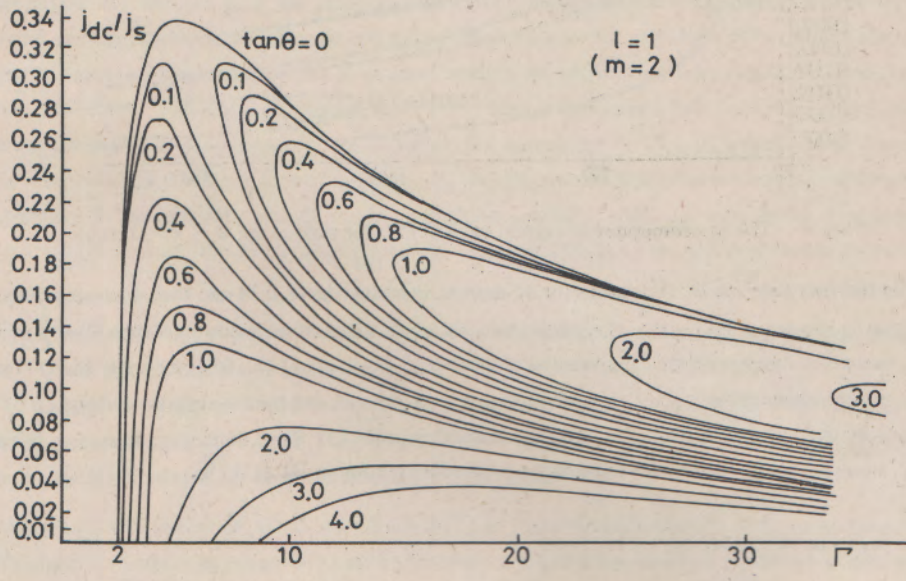

Fig. 2 The dc component of current versus Γ, coupling strength, for various tan θ ; ℓ =1 (m=2). When tan θ is not equal to zero, the curves are double valued. Further, there are many branches of curves when Γ is larger (shown in Fig. (5)). No dc current flows when Γ is below 2. That is corresponding to a cutoff voltage in ADCS. This figure in much like Fig. (13) in Werthamer [14].

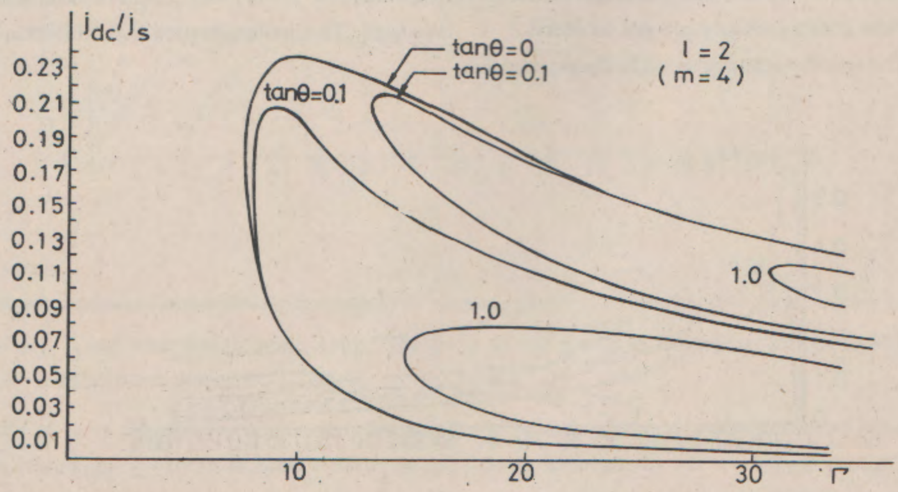

Fig. 3 The dc component of current versus Γ, for various tan θ ; ℓ =2 (m=4). The double value still appears when tan θ is not equal to zero. This figure is much like Fig. (15) in Werthamer [14].

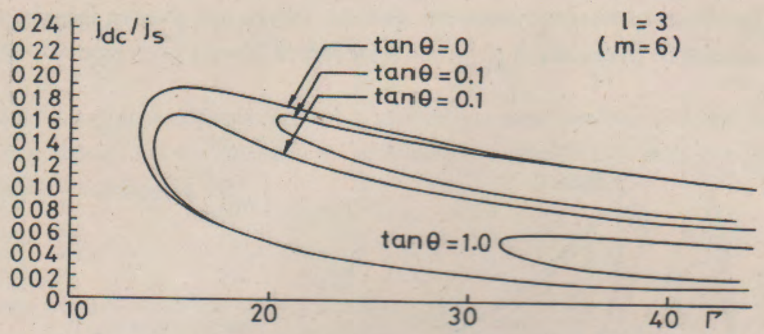

Fig. 4 The dc component of current versus Γ , for various tan θ ; ℓ =3 (m=6)

In Fig. (2), for the first even mode, the maximum dc current increases up to 0.34 and then decreases, when Γ (ω= ω J^2/ γ ω) is increasing. Since the Q value increases while temperature is lowered, then $Q \sim \frac{1}{\Gamma}$, Fig. (2), thus, can be used to compare with the experimental observatons of Gou et al [8]. We justify that the Q-value dependence of the current singularities in the dc current-voltage characteristics and that the maximum height 0.34 j_s of the first even-mode of the current sigularities in the experimental report. (3) Since there is no dc current flowing when Γ is below 2, a cutoff voltage can be obtained in the preceeding section in Eq. (11b) as

$$v_c = \omega_J \ \Phi_o /2\pi\sqrt{\frac{Q}{2}} \tag{15}$$

Fulton and Dynes [9] suggested that the cutoff voltage v_c is in the order of $\omega_J \Phi_o$, which is corresponding to Q ~80 in our theory. It is a rather high value in typical junctions. But the equation also shows that the cutoff voltage v_c will be increasing when Q is oward higher. The case can be obtained in good quality of Josephson junctions. We believe that our theory not only can predict the existence of the cutoff voltage, but also provide an explicit evidence that can be varified by the experiments. (4) If tan θ is not equal to zero, two branches of curves are found in Fig. (2). Additionally, there are still many branches of curves corresponding to Γ =117,485.....(one of them is shown in Fig. (5), but these curves probably can not be found if Γ is so large. The similar situation is also found in Fig. (3) and Fig. (4). The results is still not found in the experiments.

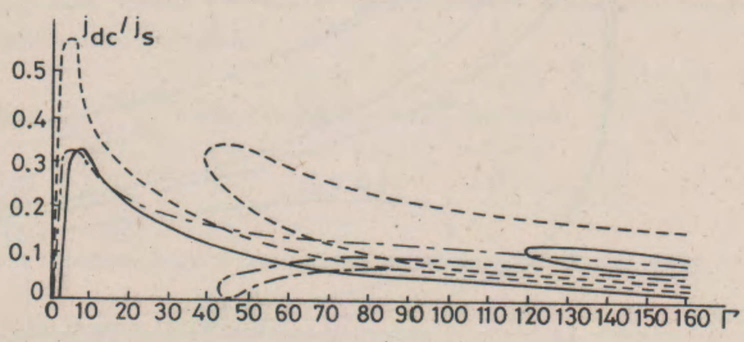

Fig. 5 The same as in Fig. (4), but with range of Γ extended. The multiple branches of these curves can be easily shown, when Γ is larger than 117 in this theory, 44.5 in (6), 37 in (15).

IV. Conclusion

The physical mechanism of this self-resonant phenomenon with no applied external magnetic field in a finite Josephson junction ($\lambda_J \sim L$) does not mean that there is no any magnetic field in the junction at all. If any current flows along the junction, a self-magnetic field will be induced simultaneously. The self-magnetic field, though very small, can propogate and interact with the junction. Therefore, the superconducting current will be very rich in harmonics under the bias building of self-magnetic field. Even though the amplitude of harmonics may be small, this effect can be happened when the junction is in good quality, we expect that even a small defect in the junction will deteriorate this effect. The nonlinear mutual interaction field, self-magnetic field, and proper harmonics of the superconducting current with the junction is the cause of this Anomalous dc Current Singularities. Thus, we conclude that the mechanism is as same as that of Fiske modes. As the external field is increasing, the self-magnetic field losses its importance in the junction, the odd modes will appear gradually while the even modes change a little. If the field is quite large enough, the Fiske modes must be observed with the even and odd modes appearing simultaneously.

Our theory can explain many properties of ADCS in Josephson tunnelling junctions. It is fair to state that we have successfully explained the appearance of even-modes current singularities without applied any magnetic field, the Q-dependence of the current peak, and behavior of the cutoff voltage. Moreover, an explicit expression about the cutoff voltage is predicted. i.e. $V_c = \omega_J \ \Phi_0 / 2\pi\sqrt{\frac{Q}{2}}$. The maximum amplitude of the first even mode 0.34 j_s is also shown in this theory, a remarkable agreement with Gou's experiments.

The comparison of this theory in ADCS with Kulik's [7] in Fiske modes, also with Werthamer-Shapiro-Smith's [6,12] theory about resonances in microwave cavity drived by the Josephson current. These are shown in Fig. (6), in which the range of Γ is extended.

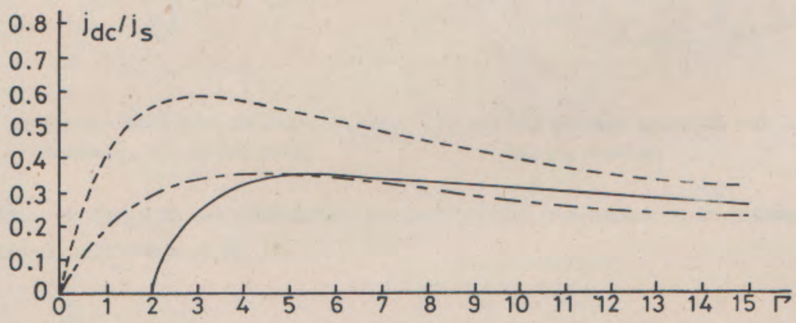

Fig. 6 The comparison of maximum dc component of current versus Γ, for the first mode in this theory, in Kulik [7], and with that in Smith [12]. The curve of this theory is shown in solid line, Kulik's in dashed line, Smith's broken line.

The similarity of these three curves is all due to the nonlinear interaction between superconducting current and the cavity. The maximum dc current in Kulik's theory is also 0.34 j_s conincidently. If Γ is large enough, additional branches of curves are obtained with Γ =44.5 in [7], 37 in [12] and 117 in this theory. Only the minimum value Γ =2 found in ADCS that gives cutoff voltage.

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