The Journal of National Chiao Tung University
Vol. 5, July 1978
pp. 65-67

—— *Short Notes*

對模n與零同餘之K次根

The k-th Roots of Zero (mod n)

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(Received March 4, 1978)

Abstract —— Let $k > 0$ be a positive integer. For each positive n, let $\delta_k(n)$ be the largest integer such that $(\delta_k(n))^k \equiv 0 \pmod n$. For any positive real number x, let $D_k(x) = \sum_{n \le x} \delta_k(n)$. The purpose of this note is to investigate the order of $D_k(x)$. The main result is the following theorem:

$$\frac{D_k(x)}{x} \rightarrow \frac{\zeta(k-1)}{\zeta(k)} \qquad \text{for } k > 2$$

where $\zeta(s)$ is known as Riemann zeta function.

There have been long and extensive studies about topics related to the sum and number of divisors of an integer n[1]. The purpose of this note is to study the following problem:

Let $k > 1$ be a positive integer. For each positive n, let $\delta_k(n)$ be the largest integer such that $(\delta_k(n))^k$ divides n. For any positive real number x, let

$$D_k(x) = \sum_{n \le x} \delta_k(n).$$

We are going to investigate the order of $D_k(x)$.

Lemma 1.

$$\delta_k(n) = \sum_{d^k \mid n} \varphi(d)$$

Proof. By the characteristic property of Euler's φ-function, in particular [3, Theorem 262]

$$n = \sum_{d \mid n} \varphi(d)$$

We obtain

$$\delta_k(n) = \sum_{d \mid \delta_k(n)} \varphi(d) = \sum_{d^k \mid n} \varphi(d),$$

since by the fundamental theorem of arithmetic, we have $d^k \mid n$ if and only if $d \mid \delta_k(n)$.

Lemma 2. [2, p. 104] For real $s > 2$,

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)} .$$

Lemma 3. [3]

$$\sum_{n \leqslant t} \varphi(n) = \frac{3t^2}{\pi^2} + 0\,(t^{2-\delta}).$$

for any $\delta > 1$.

Theorem.

$$D_k(x) = \left(\frac{\zeta(k-1)}{\zeta(k)}\right) x + 0(x^{2/k}).$$

Proof. Let $x = x_1^k x_2$, $x_1 \geqslant 1$, and $x_2 \geqslant 1$. Then

$$D_k(x) = \sum_{n \leqslant x} \delta_k(n) = \sum_{\substack{d^k e \leqslant x \\ d \leqslant x_1}} \varphi(d) + \sum_{\substack{d^k e \leqslant x \\ e \leqslant x_2}} \varphi(d) - \sum_{\substack{d^k e \leqslant x \\ d \leqslant x_1,\, e \leqslant x_2}} \varphi(d)$$

We denote the three sums by Σ_1, Σ_2, Σ_3, respectively.

$$\Sigma_1 = \sum_{\substack{d^k e \leqslant x \\ d \leqslant x_1}} \varphi(d) = \sum_{d \leqslant x_1} \varphi(d) \left[\frac{x}{d^k}\right]$$

$$= x \sum_{d \leqslant x_1} \frac{\varphi(d)}{d^k} + 0\left(\sum_{d \leqslant x_1} \varphi(d)\right)$$

$$= x \left(\frac{\zeta(k-1)}{\zeta(k)}\right) + 0\,(x_1^2\, x_2) .$$

$$\Sigma_2 = \sum_{\substack{d^k e \leqslant x \\ e \leqslant x_2}} \varphi(d) \leqslant x_2 \sum_{d=1}^{\infty} \frac{\varphi(d)}{d^k} = 0\,(x_2) .$$

$$\Sigma_3 = \sum_{\substack{d \leqslant x_1 \\ e \leqslant x_2}} \varphi(d)$$

$$= \left(\frac{3x_1^2}{\pi^2} + 0\,(x_1^{2-\delta})\right) [x_2]$$

$$= 0\,(x_1^2 x_2) .$$

Collecting Σ_1, Σ_2, Σ_3, it follows that

$$D_k(x) = \left(\frac{\zeta(k-1)}{\zeta(k)}\right) x + 0\,(x_1^2 x_2)$$

Now, let $x_1 = x^a$, $x_2 = x^b$, such that $x_1^k x_2 = x$, and $x_1 \geqslant 1$, $x_2 \geqslant 1$. Then

$$x_1^2 x_2 = x^{2a+b}$$

Choose $a=1/k$ and $b=0$. Then $ka+b=1$, $a > 0$, $b \geqslant 0$, and $2a+b=2/k < 1$. Therefore

$$D_k(x) = \left(\frac{\zeta(k-1)}{\zeta(k)}\right) x + 0\,(x^{2/k}).$$

It can be shown that $2/k$ is the smallest number for $2a + b$ with $ka + b = 1$ for $k \geqslant 3$, $a > 0$, and $b \geqslant 0$.

The proof of the lemmas will appear somewhere else.

References

1. Dickson, L. E., "History of the Theory of Numbers", Chelsea, 1952.
2. Ayoub, R., "An introduction to the analytic theory of numbers", Amer. Math. Soc., (1963).
3. Hardy, H. H. and Wright, E. M., "The Theory of Numbers", 4th Edition, Oxford, 1960.