

— Short Notes

對模 n 與零同餘之 K 次根
The k -th Roots of Zero (mod n)

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Abstract — Let $k > 0$ be a positive integer. For each positive n , let $\delta_k(n)$ be the largest integer such that $(\delta_k(n))^k \equiv 0 \pmod{n}$. For any positive real number x , let $D_k(x) = \sum_{n \leq x} \delta_k(n)$. The purpose of this note is to investigate the order of $D_k(x)$. The main result is the following theorem:

$$\frac{D_k(x)}{x} \rightarrow \frac{\zeta(k-1)}{\zeta(k)} \quad \text{for } k > 2$$

where $\zeta(s)$ is known as Riemann zeta function.

There have been long and extensive studies about topics related to the sum and number of divisors of an integer n [1]. The purpose of this note is to study the following problem:

Let $k > 1$ be a positive integer. For each positive n , let $\delta_k(n)$ be the largest integer such that $(\delta_k(n))^k$ divides n . For any positive real number x , let

$$D_k(x) = \sum_{n \leq x} \delta_k(n).$$

We are going to investigate the order of $D_k(x)$.

Lemma 1.

$$\delta_k(n) = d^k \sum_{d|n} \varphi(d)$$

Proof. By the characteristic property of Euler's φ -function, in particular [3, Theorem 262]

$$n = \sum_{d|n} \varphi(d)$$

We obtain

$$\delta_k(n) = \sum_{d|\delta_k(n)} \varphi(d) = \sum_{d^k|n} \varphi(d),$$

since by the fundamental theorem of arithmetic, we have $d^k | n$ if and only if $d | \delta_k(n)$.

Lemma 2. [2, p. 104] For real $s > 2$,

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

Lemma 3. [3]

$$\sum_{n \leq t} \varphi(n) = \frac{3t^2}{\pi^2} + O(t^{2-\delta}).$$

for any $\delta > 1$.

Theorem.

$$D_k(x) = \left(\frac{\zeta(k-1)}{\zeta(k)} \right) x + O(x^{2/k}).$$

Proof. Let $x = x_1^k x_2$, $x_1 \geq 1$, and $x_2 \geq 1$. Then

$$D_k(x) = \sum_{n \leq x} \delta_k(n) = \sum_{\substack{d^k e \leq x \\ d \leq x_1}} \varphi(d) + \sum_{\substack{d^k e \leq x \\ e \leq x_2}} \varphi(d) - \sum_{\substack{d^k e \leq x \\ d \leq x_1, e \leq x_2}} \varphi(d)$$

We denote the three sums by Σ_1 , Σ_2 , Σ_3 , respectively.

$$\Sigma_1 = \sum_{\substack{d^k e \leq x \\ d \leq x_1}} \varphi(d) = \sum_{d \leq x_1} \varphi(d) \left[\frac{x}{d^k} \right]$$

$$= x \sum_{d \leq x_1} \frac{\varphi(d)}{d^k} + O\left(\sum_{d \leq x_1} \varphi(d) \right)$$

$$= x \left(\frac{\zeta(k-1)}{\zeta(k)} \right) + O(x_1^2 x_2).$$

$$\Sigma_2 = \sum_{\substack{d^k e \leq x \\ e \leq x_2}} \varphi(d) \leq x_2 \sum_{d=1}^{\infty} \frac{\varphi(d)}{d^k} = O(x_2).$$

$$\Sigma_3 = \sum_{\substack{d \leq x_1 \\ e \leq x_2}} \varphi(d)$$

$$= \left(\frac{3x_1^2}{\pi^2} + O(x_1^{2-\delta}) \right) [x_2]$$

$$= O(x_1^2 x_2).$$

Collecting Σ_1 , Σ_2 , Σ_3 , it follows that

$$D_k(x) = \left(\frac{\xi(k-1)}{\xi(k)} \right) x + 0(x_1^2 x_2)$$

Now, let $x_1 = x^a$, $x_2 = x^b$, such that $x_1^k x_2 = x$, and $x_1 \geq 1$, $x_2 \geq 1$. Then

$$x_1^2 x_2 = x^{2a+b}$$

Choose $a = 1/k$ and $b = 0$. Then $ka + b = 1$, $a > 0$, $b \geq 0$, and $2a + b = 2/k < 1$. Therefore

$$D_k(x) = \left(\frac{\xi(k-1)}{\xi(k)} \right) x + 0(x^{2/k}).$$

It can be shown that $2/k$ is the smallest number for $2a + b$ with $ka + b = 1$ for $k \geq 3$, $a > 0$, and $b \geq 0$.

The proof of the lemmas will appear somewhere else.

References

1. Dickson, L. E., "History of the Theory of Numbers", Chelsea, 1952.
2. Ayoub, R., "An introduction to the analytic theory of numbers", Amer. Math. Soc., (1963).
3. Hardy, H. H. and Wright, E. M., "The Theory of Numbers", 4th Edition, Oxford, 1960.

I. Introduction

Although $\bar{X}$ -chart is still used widely in industries, because of its simplicity and ease for operation, recently the theory of control charts has been developed to use Cumulative Sum Charts (CUSUM charts) and Moving Average Charts. Many important results have been obtained by Page [4], Chen [1], Lai [3], Reynolds [5] etc. However, from the economical point of view, the consideration of the inspection cost on the design of control charts would improve the efficiency of the control chart. A particular case which is related to the ratio estimation in Sampling Theory has been studied by Huang and Yang [2]. In this paper, we shall generalize the results of Huang and Yang [2] to sample of size k , $k \geq 1$.

II. The production Process and the Sampling Plan

Let the production process produces batches of k items, $k \geq 1$, continuously. Let us sample a batch and, after shipping a certain fixed number of batches, we sample another batch again. We repeat this sampling procedure when the production process continues to run. All k items in each sampled batch are inspected. The number of defectives in each sample is plotted on the control chart and the manager will make a decision, based on this information, to take corrective action. As in [2], we assume the process has two quality levels. Define λ_1 and λ_2 to be the fractions of defectives produced at level 1 and 2 respectively. If the number of defectives in a sample is found to be greater than a properly assigned integer, say L , the corrective action will be taken and the production process is stopped obviously, the probability of taking the corrective action depends on λ_1 , λ_2 , and L .

Define Δ unit time as the time interval in which the production process produces k items. For convenience of simplicity, it is assumed that the process shift (see (1)) will happen only at the end of each unit time interval. In other words, the process shift is not likely to happen within each unit time interval. Let α be the probability of the process shifting from quality level 1 at time n to quality level 2 at time $n + 1$, where n is a positive integer. The value α is independent of time n . Suppose the attribute X of each sampled item is independent of all other items.