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—— 短文 *Short Notes*

近乎90度角度反射器的光傳播矩陣
Ray Transfer Matrix for Nearly 90 Degree Reflectors

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Abstract — Due to the convenience of using ray transfer matrix to represent an optical component and wide application of 90 degree reflectors (includes totally internal reflectors), we derive the ray transfer matrix for 90 degree reflectors which take the small manufacturing tolerance into account. This derived ray transfer matrix will then be used to discuss the stability condition of a resonantor which consists of mirror and nearly 90 degree reflector. Also some interesting results are obtained by this application.

90 degree reflectors have wide application in optics. It can be used as laser resonators', etc.. Also, in geometrical optics, it is found that the mathematical operation becomes quite convenient if we are able to represent the optical component by ray transfer matrix[2,3]. Hence, we will derive the ray transfer matrix for 90 degree reflectors. In general, manufacturing tolerances and small mis-alignment with other optical components are always exist, thus it will be instructive to study how these non-ideal conditions affect the ray transformation and its effect when the reflector is used as part of a laser resonator.

A) Ray Transfer Matrix for Nearly 90 Degree Reflector

Suppose we have a nearly 90 degree reflector as shown in Fig. 1 and the in-coming paraxial ray hits the upper surface F_1 first. Then the surfaces F_1, F_2 and in-coming ray are represented by Eq. (1),Eq.(2) and Eq.(3) respectively

$$Y = -(x-a) \tan \gamma \tag{1}$$

$$y = (x-a) \tan \xi \tag{2}$$

$$y = y_o + x \tan \mu_o \tag{3}$$

where

γ = angle between upper surface F_1 and the optical axis

ξ = angle between lower surface F_2 and the optical axis

μ_o = angle between lower surface F_2 and the optical axis

a = height of the reflector

Here, we assume $\gamma = \frac{\pi}{4} + \delta$, $\xi = \frac{\pi}{4} + \phi$ with $|\delta| << 1$, and $|\phi| << 1$.

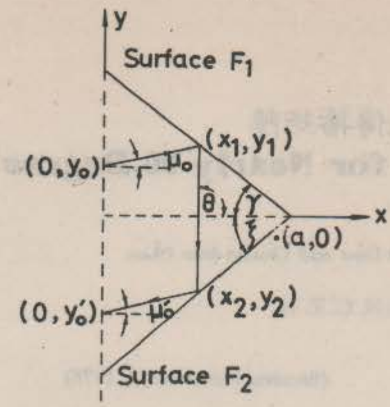


Fig. 1. Schematic diagram used to derive the ray transfer matrix for nearly 90 degree reflector.

In order to find the position y_o' and the slope of the ray, $\tan \mu_o'$, after reflection from the reflector, we have to find point (x_1, y_1) and point (x_2, y_2), which are defined in Fig. 1.

Point (x_1, y_1) can be found by setting Eq. (1) equal to Eq. (3) and we obtain

$$x_1 = \frac{a \tan \gamma - y_o}{\tan \gamma + \tan \mu_o} \tag{4}$$

$$y_1 = \frac{y_o \tan \gamma + a \tan \gamma \tan \mu_o}{\tan \gamma + \tan \mu_o} \tag{5}$$

Also, the angle θ which is the angle between reflected ray (from surface F_1) and the optical axis, can be found by simple geometrical optical rule and is given by

$$\theta = \frac{\pi}{2} - (2\delta + \mu_o)$$

Thus, this reflected ray is represented by Eq. (6)

$$y - y_1 = (x - x_1) \tan \theta \tag{6}$$

By setting Eq. (6) equal to Eq. (2), we find

$$x_2 = \frac{y_1 + a \tan \xi - x_1 \tan \theta}{\tan \xi - \tan \theta} \tag{7}$$

$$y_2 = (x_2 - a) \tan \xi \tag{8}$$

Under the assumption $\delta << 1$, $\phi << 1$ and $\tan \mu_o \simeq \mu_o$, we have $\tan(\frac{\pi}{4} + \delta) \simeq 1 + 2\delta$, $\tan(\frac{\pi}{4} + \phi) \simeq 1 + 2\phi$. With these approximations substituted into Eqs. (4), (5), (7), (8) and neglect higher order terms, then we obtain

$$x_1 \simeq a(1 - \mu_o) - y_o(1 - 2\delta - \mu_o^2)$$

$$y_1 \simeq y_o(1-\mu_o)+a\,\mu_o$$

$$x_2 \simeq -y_o(1+2\delta+\mu_o)+a(1-\mu_o)$$

$$y_2 \simeq -y_o(1+2\delta+2\phi+\mu_o)-a\,\mu_o$$

Also, by simple geometrical optical rule, we find

$$\mu_o' = 2\phi+2\delta+\mu_o$$

Since

$$y_o' = y_2 - x_2 \tan\ \mu_o \tag{9}$$

$$\because\ y_o' \cong -y_o - 2a\ \mu_o - 2a(\delta+\phi) \tag{10}$$

If we define ray 1 and ray 2 (up-going rays) in Fig. 2 have positive slope, ray 3 and ray 4 (down-going rays) have negative slope, then combine Eq.(9) and Eq. (10), we can write the ray transfer matrix as Eq. (11) for those paraxial rays which hit the upper surface first.

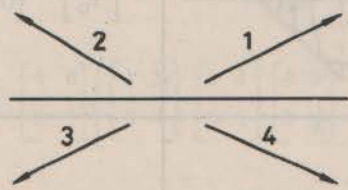


Fig. 2. Schematic diagram used to define the sign of the slope of the rays. Where ray 1 and 2 have positive slope, ray 3 and 4 have negative slope.

$$\begin{bmatrix} r_o \\ r_o' \end{bmatrix} = \begin{bmatrix} -1 & -2a \\ 0 & -1 \end{bmatrix} \begin{bmatrix} r_i \\ r_1' \end{bmatrix} - \begin{bmatrix} 2a(\delta+\phi) \\ 2(\delta+\phi) \end{bmatrix} \tag{11}$$

where

r_i = height of in-coming ray at x=0 (corresponds to y_o in Fig. 1).

r_o = height of out-going ray at x=0 (corresponds to y' in Fig. 1).

r_i' = slope of in-coming ray at x=0 (corresponds to $\tan\mu_o \simeq \mu_o$ for paraxial ray in Fig. 1).

r_o' = slope of the outgoing ray at x=0 (corresponds to $-\tan\mu_o' \simeq -(\mu_o+2\delta+2\phi)$, the minus sing is due to the choice of negative slope defined in Fig. 2).

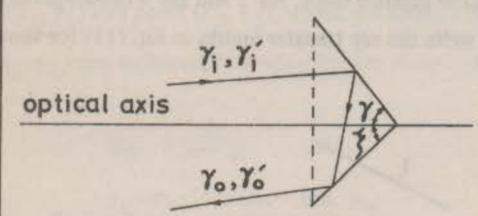
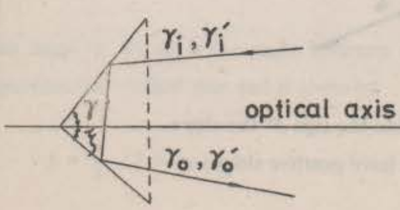
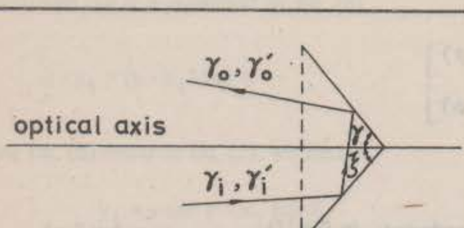
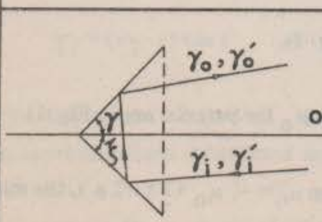
By the same derivation procedure, we find that if a ray hits the lower surface (surface F_2 in Fig. 1) first, then the ray transfer matrix will be of the form shown in Eq. (12)

$$\begin{bmatrix} r_o \\ r'_o \end{bmatrix} = \begin{bmatrix} -1 & -2a \\ 0 & -1 \end{bmatrix} \begin{bmatrix} r_i \\ r'_i \end{bmatrix} + \begin{bmatrix} 2a(\delta+\phi) \\ 2(\delta+\phi) \end{bmatrix} \tag{12}$$

The difference of the sign of the constant term in Eq. (11) and Eq. (12) can be understood by considering the fact that the height and slope of a ray are always inverted after reflection from a nearly 90 degree reflector when the sign of the slope is defined as Fig. 2.

For clearity, the ray transfer matrix for different combination of in-coming paraxial rays and the reflector surface are shown in Table 1. For those rays which hit the upper surface (surface above the optical axis) first, the sign of the constant term (such as Eq. 18) is negative and for those rays which hit the lower surface (below the optical axis) first is positive.

Table 1: Ray Transfer Matrices for Possible Combination of Nearly 90 Degree Roof-Prisms and incoming rays, $\gamma = \frac{\pi}{4} + \delta$, $\xi = \frac{\pi}{4} + \phi$; $\delta \ll 1$, $\phi \ll 1$.

	$\begin{bmatrix} \gamma_o \\ \gamma'_o \end{bmatrix} = \begin{bmatrix} -1 & -2a \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \gamma_i \\ \gamma'_i \end{bmatrix} - \begin{bmatrix} 2a(\delta+\phi) \\ 2(\delta+\phi) \end{bmatrix}$
	$\begin{bmatrix} \gamma_o \\ \gamma'_o \end{bmatrix} = \begin{bmatrix} -1 & -2a \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \gamma_i \\ \gamma'_i \end{bmatrix} - \begin{bmatrix} 2a(\delta+\phi) \\ 2(\delta+\phi) \end{bmatrix}$
	$\begin{bmatrix} \gamma_o \\ \gamma'_o \end{bmatrix} = \begin{bmatrix} -1 & -2a \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \gamma_i \\ \gamma'_i \end{bmatrix} + \begin{bmatrix} 2a(\delta+\phi) \\ 2(\delta+\phi) \end{bmatrix}$
	$\begin{bmatrix} \gamma_o \\ \gamma'_o \end{bmatrix} = \begin{bmatrix} -1 & -2a \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \gamma_i \\ \gamma'_i \end{bmatrix} + \begin{bmatrix} 2a(\delta+\phi) \\ 2(\delta+\phi) \end{bmatrix}$

B) Laser Resonator Consists of Mirror and Nearly 90 Degree Reflector

The structure of the resonator under consideration is of the form shown in Fig. 3. Where the left is a solid-state laser rod (such as N_d: Yag laser rods) with one end polished as nearly 90 degree prism reflector and the other end is polished as a flat surface. And the mirror has a radius R.

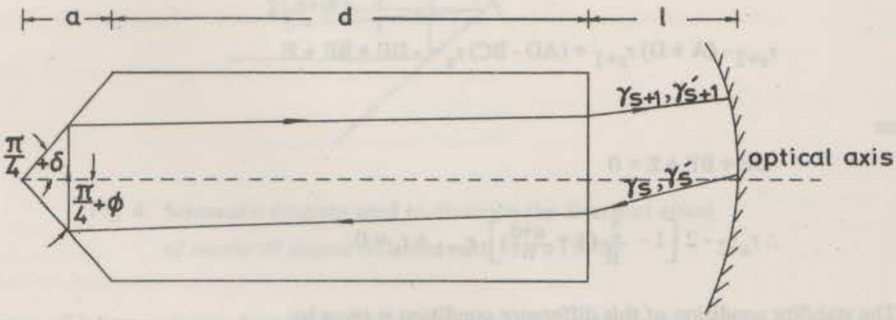


Fig. 3. Schemic diagram used to derive the stability condition of prismmirror resonator.

In order to study how δ and ϕ affects the stability of such a structure, we write down matrices which describes the propagation of paraxial rays as Eq. (13)[2]

$$\begin{bmatrix} r_{s+1} \\ r'_{s+1}\end{bmatrix} = \begin{bmatrix}1 & \ell \\ 0 & 1\end{bmatrix}\begin{bmatrix}1&0\\0&n\end{bmatrix}\begin{bmatrix}1&d\\0&1\end{bmatrix}\left\{\begin{bmatrix}-1&-2a\\0&-1\end{bmatrix}\begin{bmatrix}1&d\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&\frac{1}{n}\end{bmatrix}\begin{bmatrix}1&\ell\\0&1\end{bmatrix}\begin{bmatrix}1&0\\-\frac{2}{R}&1\end{bmatrix}\begin{bmatrix}r_s\\r'_s\end{bmatrix}+\begin{bmatrix}2a(\delta+\phi)\\2(\delta+\phi)\end{bmatrix}\right\}$$

$$= \begin{bmatrix}-1+\frac{4}{R}(\ell+\frac{a+d}{n}) & -2(\ell+\frac{a+d}{n}) \\ \frac{2}{R} & -1\end{bmatrix}\begin{bmatrix}r_s\\r'_s\end{bmatrix} + \begin{bmatrix}2(n\ell+a+d)(\delta+\phi)\\2n(\delta+\phi)\end{bmatrix} \qquad (13)$$

Where n is the index of refraction of the laser rod, r_s and r'_s are the height and slope of the ray at the immediate left of the mirror. And

$$A = -1+\frac{4}{R}(\ell+\frac{a+d}{n})$$

$$B = -2(\ell+\frac{a+d}{n})$$

$$C = \frac{2}{R}$$

$$D = -1$$

$$E = 2(n\ell+a+d)(\delta+\phi)$$

$$F = 2n(\delta+\phi)$$

Thus

$$r_{s+1} = Ar_s + Br'_s + E$$

$$r'_{s+1} = Cr_s + Dr'_s + F$$

or

$$r_{s+2} - (A + D)\, r_{s+1} + (AD - BC)\, r_s = - DE + BF + E$$

Since

$$-DE + BF + E = 0$$

$$\therefore r_{s+2} - 2\left[1 - \frac{2}{R}(\ell + \frac{a+d}{n})\right] r_{s+1} + r_s = 0$$

The stability condition of this difference condition is given by

$$\left|1 - \frac{1}{R}(\ell + \frac{a+d}{n})\right| \leq 1$$

or

$$0 \leq 1 - \frac{1}{R}(\ell + \frac{a+d}{n}) \leq 1 \qquad (14)$$

Similar results are obtained for those rays which hit the upper prism surface first. Note that in the above derivation, we assume that total internal reflection exist in the prism surface. This condition will be satisfied when the length of the resonator is long enough.

In conclusion from the ray transfer matrices shown in table 1, we find

1. For perfect 90 degree reflector (i.e., $\delta + \phi = 0$), the ray transfer matrix will be of the form

$$\begin{bmatrix} r_o \\ r'_o \end{bmatrix} = \begin{bmatrix} -1 & -2a \\ 0 & -1 \end{bmatrix} \begin{bmatrix} r_i \\ r'_i \end{bmatrix}$$

and the in-coming and out-going rays are in opposite direction. Also, the height is reversed and displacement from the optical axis is proportional to the product of the slope of in-coming ray and the height of the reflector.

2. If the deviation, $\delta + \phi$, is larger than zero, then the nearly 90 degree reflector will have divergent effect. This can be understood by considering Fig. 4, where we have ray 1 and ray 2 parallel to the optical axis, if $(\delta + \phi) = 0$, then the reflected ray will parallel to the optical axis too. However, if $(\delta + \phi) > 0$, then the reflected rays tend to move away from from the optical axis, i.e., we have divergent effect.

3. If the deviation, $\delta + \phi$, is smaller than zero, then the reflector will have convergent effect, i.e., the reflected ray tends to move toward the optical axis.

Further from Eq. (14), it is found that the stability condition of this particallar resonator shown in Fig. 3 is independent of δ and ϕ. Thus, we expect this structure will be particular useful for field and military operation where vibration problem is quite serious and will cause slightly mis-alignment.

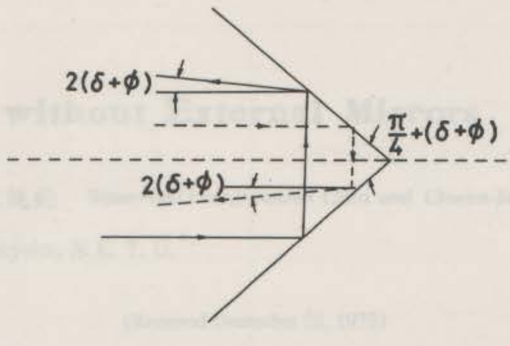


Fig. 4. Schematic diagram used to illustrate the divergent effect of nearly 90 degree reflector with $(\delta + \phi) > 0$.

Due to the existence of inhomogeneous broadening, which is order of magnitude larger than the mode spacing, we can always find modes, which satisfy the phase relationship required for laser action, lie within the gain spectrum. This is especially true when the laser rod is long enough.

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