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# Chaos synchronization and parameter identification for gyroscope system

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**Abstract**

Synchronization of chaos for a two-degree-of-freedom heavy symmetric gyroscope system are studied in this paper. Because of the nonlinear terms of the system, the system exhibits both regular and chaotic motions. By Lyapunov stability theory with control terms, by adaptive control and by random optimization method, the synchronization of two identical systems and tracking of the parameter of the systems are studied.



*Keywords:* Chaos; Synchronization; Parameter identification

## 1. Introduction

A lot of researches have shown that chaotic phenomena are observed in many physical systems that possess nonlinearity [1,2]. Chaotic motions also occur in many nonlinear control systems.

Most of physical systems in nature are nonlinear and can be described by the nonlinear equations of motion which in general cannot be linearized. So the studies of nonlinear systems spread quickly today. For the nonlinear system, the study of the types of system behavior, the effects to the behavior caused by different parameters and initial conditions, the behavior analysis of the system, consist of the major tasks. Besides, we are also interested in the understanding

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of the complicated phenomena arised from nonlinearity. The central characteristics are that a process like randomization happens in the deterministic system and small differences in the system parameters or initial conditions produce great ones in the final phenomena. The unpredictable and irregular motions of many nonlinear systems have been labeled "chaotic". By applying various numerical results, such as bifurcation, phase portraits, time history analysis, the behavior of the chaotic motion are presented. A large number of studies on the chaotic behavior have been reached up to now. First, the governing equations of motion, the system model and differential equations of motion will be formulated.

Synchronization of chaos for a two-degree-of-freedom heavy symmetric gyroscope system are studied in this paper. By Lyapunov stability theory with control terms, by adaptive control and by random optimization method, the synchronization of two identical systems and tracking of the parameter of the systems are studied.

## 2. Equations of motion

The schematic diagram of a heavy symmetric gyroscope mounted on a vibrating base is shown in Fig. 1. The motion of this physical system can be described by Euler's angles $\theta$, $\phi$ and $\varphi$. The vibration of the base can be de-

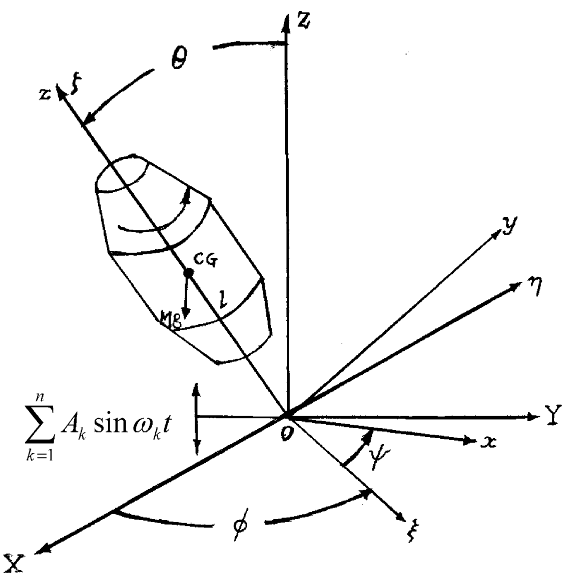


Fig. 1. A schematic diagram of a heavy symmetric gyroscope.

scribed as multiple harmonic motion $\sum_{k=1}^{n} A_k \sin\omega_k t$. Let $x=\theta$, $y=\dot{\theta}$ and $z=\dot{\phi}$, the state equations of the system are described by [3]:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{(\beta_\phi - \beta_\varphi \cos x)(\beta_\varphi - \beta_\phi \cos x)}{I_1^2 \sin^3 x} - \frac{C}{I_1} y + \frac{\mathrm{Mg}l}{I_1}\sin x - \frac{\mathrm{Mg}}{I_1}\sum_{k=1}^{n} A_k \sin\omega_k t \sin x \\ \dot{z} = -\frac{2\cos x}{\sin x} yz + \frac{\beta_\varphi y}{I_1 \sin x} \end{cases}, \tag{2.1}$$

where $I_1$, $I_3$: the polar and equatorial moments of inertia of the symmetric gyroscope, Mg: the gravity force, $l$: the distance between the center of gravity and $O$.

We set the parameters $\beta_\phi = 2$, $\beta_\varphi = 5$, $I_1 = 1$, $\mathrm{Mg} = 4$, $l = 0.25$, $C = 0.5$, $\omega_1 = 1$, $\omega_2 = 2$, $\omega_3 = 3$, $\omega_k = 0 (k > 3)$, $A_1 = A_2 = A_3 = \cdots = A_k = A$.

## 3. Chaos synchronization for systems with unknown parameter

We investigate two third-order heavy symmetric gyroscope systems in this section. Both the systems have the same form and both the parameters are unknown. The drive system is described as Eq. (2.1). And the system (2.1) in which only first term is considered for the Fourier series and we set $A_1 = A$. Eq. (2.1) becomes Eq. (3.1). The response system is described as Eq. (3.2).

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{(\beta_\phi - \beta_\varphi \cos x_1)(\beta_\varphi - \beta_\phi \cos x_1)}{I_1^2 \sin^3 x_1} - \frac{C}{I_1} x_2 + \frac{\mathrm{Mg}l}{I_1}\sin x_1 - \frac{\mathrm{Mg}}{I_1} A \sin\omega_1 t \sin x_1 \\ \dot{x}_3 = -\frac{2\cos x_1}{\sin x_1} x_2 x_3 + \frac{\beta_\varphi x_2}{I_1 \sin x_1} \end{cases}, \tag{3.1}$$

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -\frac{(\beta_\phi - \beta_\varphi \cos y_1)(\beta_\varphi - \beta_\phi \cos y_1)}{I_1^2 \sin^3 y_1} - \frac{C}{I_1} y_2 + \frac{\mathrm{Mg}l}{I_1}\sin y_1 - \frac{\mathrm{Mg}}{I_1} A \sin\omega_1 t \sin y_1 \\ \dot{y}_3 = -\frac{2\cos y_1}{\sin y_1} y_2 y_3 + \frac{\beta_\varphi y_2}{I_1 \sin y_1} \end{cases}. \tag{3.2}$$

The true values of the "unknown" parameters are $\beta_\phi = 2$, $\beta_\varphi = 5$, $I_1 = 1$, $\mathrm{Mg} = 4$, $l = 0.25$, $C = 0.5$, $\omega_1 = 1$, $A = 12.1$ in numerical simulation. The initial conditions of the drive and the response systems are $x_1(0) = -0.5$, $x_2(0) = -1.2$, $x_3(0) = 10$ and $y_1(0) = y_2(0) = y_3(0) = 0.1$, respectively. The initial value of estimate for "unknown" parameter is $\frac{\hat{C}}{I_1}(0) = 0.1$.

For synchronizing the two third-order heavy symmetric gyroscope systems, we add three controllers $u_1$, $u_2$, $u_3$, on the first, second and third equation of (3.2), respectively.

$$\begin{cases} \dot{y}_1 = y_2 + u_1 \\ \dot{y}_2 = -\frac{(\beta_\phi - \beta_\varphi \cos y_1)(\beta_\varphi - \beta_\phi \cos y_1)}{I_1^2 \sin^3 y_1} - \frac{C}{I_1} y_2 + \frac{\mathrm{Mg}l}{I_1} \sin y_1 - \frac{\mathrm{Mg}}{I_1} A \sin \omega_1 t \sin y_1 + u_2 \\ \dot{y}_3 = -\frac{2\cos y_1}{\sin y_1} y_2 y_3 + \frac{\beta_\varphi y_2}{I_1 \sin y_1} + u_3 \end{cases}. \tag{3.3}$$

First, subtracting Eq. (3.1) from (3.3), we can obtain the error dynamics as

$$\begin{aligned} \dot{e}_1 &= e_2 + u_1, \\ \dot{e}_2 &= -\frac{(\beta_\phi - \beta_\varphi \cos y_1)(\beta_\varphi - \beta_\phi \cos y_1)}{I_1^2 \sin^3 y_1} + \frac{(\beta_\phi - \beta_\varphi \cos x_1)(\beta_\varphi - \beta_\phi \cos x_1)}{I_1^2 \sin^3 x_1} \\ &\quad - \frac{C}{I_1} e_2 + \frac{\mathrm{Mg}l}{I_1}(\sin y_1 - \sin x_1) - 1\frac{\mathrm{Mg}}{I_1} A \sin \omega_1 t (\sin y_1 - \sin x_1) + u_2, \\ \dot{e}_3 &= -\frac{2\cos y_1}{\sin y_1} y_2 y_3 + \frac{2\cos x_1}{\sin x_1} x_2 x_3 + \frac{\beta_\varphi y_2}{I_1 \sin y_1} - \frac{\beta_\varphi x_2}{I_1 \sin x_1} + u_3, \end{aligned} \tag{3.4}$$

where $e_1 = y_1 - x_1$, $e_2 = y_2 - x_2$, $e_3 = y_3 - x_3$.

Then, choosing a Lyapunov function of the form

$$V\left(e_1, e_2, e_3, \frac{\widetilde{C}}{I_1}\right) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + \left(\frac{\widetilde{C}}{I_1}\right)^2\right), \tag{3.5}$$

where $\frac{\widetilde{C}}{I_1} = \frac{C}{I_1} - \frac{\hat{C}}{I_1}$, and $\frac{\hat{C}}{I_1}$ is estimate value of the unknown parameter $\frac{C}{I_1}$, respectively [4].

Its derivative along the solution of Eq. (3.4) is

$$\begin{aligned} \dot{V}\left(e_1, e_2, e_3, \frac{\widetilde{C}}{I_1}\right) &= e_1(e_2 + u_1) + e_2\Bigg[ -\frac{(\beta_\phi - \beta_\varphi \cos y_1)(\beta_\varphi - \beta_\phi \cos y_1)}{I_1^2 \sin^3 y_1} \\ &\quad + \frac{(\beta_\phi - \beta_\varphi \cos x_1)(\beta_\varphi - \beta_\phi \cos x_1)}{I_1^2 \sin^3 x_1} \\ &\quad - \frac{C}{I_1} e_2 + \frac{\mathrm{Mg}l}{I_1}(\sin y_1 - \sin x_1) - \frac{\mathrm{Mg}}{I_1} A \sin \omega_1 t \\ &\quad \times (\sin y_1 - \sin x_1) + u_2 \Bigg] + e_3\Bigg[ -\frac{2\cos y_1}{\sin y_1} y_2 y_3 \\ &\quad + \frac{2\cos x_1}{\sin x_1} x_2 x_3 + \frac{\beta_\varphi y_2}{I_1 \sin y_1} - \frac{\beta_\varphi x_2}{I_1 \sin x_1} + u_3 \Bigg] \\ &\quad + \left(\frac{\widetilde{C}}{I_1}\right)\left(-\frac{\dot{\hat{C}}}{I_1}\right). \end{aligned} \tag{3.6}$$

We select

$$
\begin{aligned}
u_1 &= -e_2 - e_1,\\
u_2 &= \frac{(\beta_\phi - \beta_\varphi \cos y_1)(\beta_\varphi - \beta_\phi \cos y_1)}{I_1^2 \sin^3 y_1} - \frac{(\beta_\phi - \beta_\varphi \cos x_1)(\beta_\varphi - \beta_\phi \cos x_1)}{I_1^2 \sin^3 x_1}\\
&\quad - \frac{\mathrm{Mg}l}{I_1}(\sin y_1 - \sin x_1) + \frac{\mathrm{Mg}}{I_1} A \sin \omega_1 t (\sin y_1 - \sin x_1) + \left(\frac{\hat{C}}{I_1} - 1\right) e_2,\\
u_3 &= \frac{2\cos y_1}{\sin y_1} y_2 y_3 - \frac{2\cos x_1}{\sin x_1} x_2 x_3 - \frac{\beta_\varphi y_2}{I_1 \sin y_1} + \frac{\beta_\varphi x_2}{I_1 \sin x_1} - e_3,\\
\frac{\dot{\hat{C}}}{I_1} &= -e_2^2.
\end{aligned}
$$

Then, Eq. (3.6) becomes

$$\dot{V}(e_1, e_2, e_3) = -e_1^2 - e_2^2 - e_3^2 < 0. \tag{3.7}$$

This means that the synchronization of the two third-order heavy symmetric gyroscope systems can be achieved. The results are shown in Figs. 2–4.

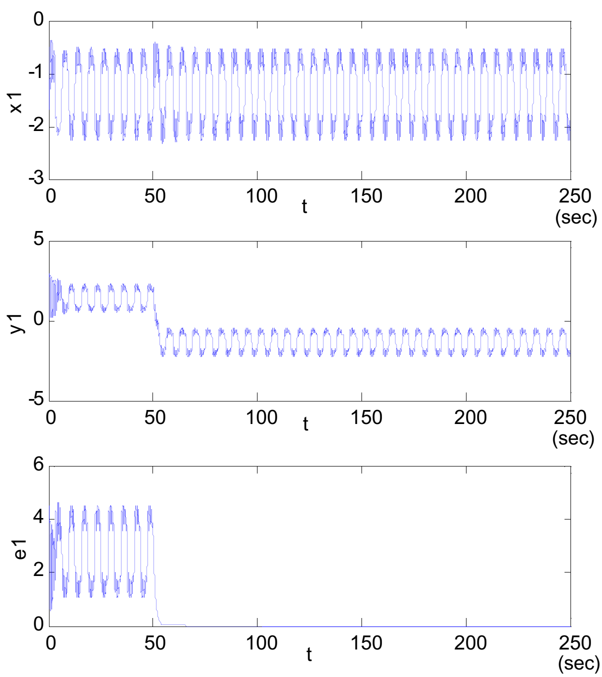


Fig. 2. Time history of $x_1$, $y_1$ and the error between them.

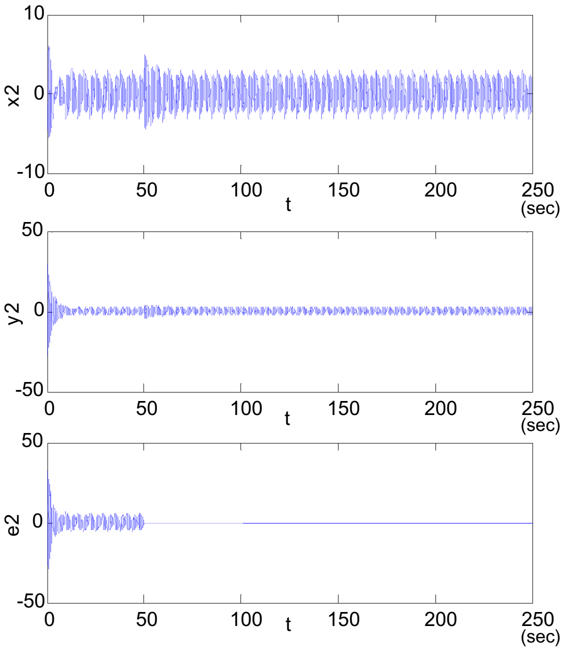


Fig. 3. Time history of $x_2$, $y_2$ and the error between them.

## 4. Parameter identification

### 4.1. Synchronization of uncertain chaotic systems via adaptive control

We investigate two third-order heavy symmetric gyroscope systems in this section. Both the systems have the same form. But the parameter, $\frac{C}{I_1}(t)$ of the response system is time-varying and uncertain.

The drive system is described as Eq. (3.1). The response system is described by

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -\frac{(\beta_\phi - \beta_\varphi \cos y_1)(\beta_\varphi - \beta_\phi \cos y_1)}{I_1^2 \sin^3 y_1} - \frac{C}{I_1}(t) y_2 + \frac{\mathrm{Mg}l}{I_1} \sin y_1 - \frac{\mathrm{Mg}}{I_1} A \sin \omega_1 t \sin y_1 \\ \dot{y}_3 = -\frac{2\cos y_1}{\sin y_1} y_2 y_3 + \frac{\beta_\varphi y_2}{I_1 \sin y_1} \end{cases} . \tag{4.1}$$

For synchronizing two third-order heavy symmetric gyroscope systems, we add three controllers $u_1$, $u_2$, $u_3$, on the first, second and third equation of Eq. (4.1), respectively.

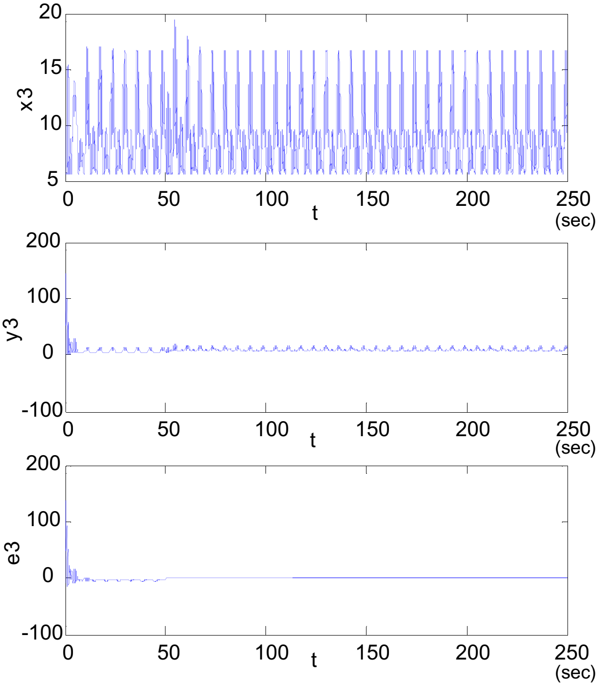


Fig. 4. Time history of $x_3$, $y_3$ and the error between them.

$$\begin{cases} \dot{y}_1 = y_2 + u_1 \\ \dot{y}_2 = -\frac{(\beta_\phi - \beta_\varphi \cos y_1)(\beta_\varphi - \beta_\phi \cos y_1)}{I_1^2 \sin^3 y_1} - \frac{C}{I_1}(t) y_2 + \frac{\mathrm{Mg}l}{I_1} \sin y_1 - \frac{\mathrm{Mg}}{I_1} A \sin \omega_1 t \sin y_1 + u_2 \\ \dot{y}_3 = -\frac{2\cos y_1}{\sin y_1} y_2 y_3 + \frac{\beta_\varphi y_2}{I_1 \sin y_1} + u_3 \end{cases}. \tag{4.2}$$

The drive and the response system can be written as [5,6]

$$\begin{aligned} \dot{x} &= f(x) + F(x)\left(\frac{C}{I_1}\right), \\ \dot{y} &= f(y) + F(y)\left(\frac{C}{I_1}(t)\right) + U, \end{aligned} \tag{4.3}$$

where

$$f(x) = \begin{bmatrix} x_2 \\ -\frac{(\beta_\phi - \beta_\varphi \cos x_1)(\beta_\varphi - \beta_\phi \cos x_1)}{I_1^2 \sin^3 x_1} + \frac{\mathrm{Mg}l}{I_1} \sin x_1 - \frac{\mathrm{Mg}}{I_1} A \sin \omega_1 t \sin x_1 \\ -\frac{2\cos x_1}{\sin x_1} x_2 x_3 + \frac{\beta_\varphi x_2}{I_1 \sin x_1} \end{bmatrix},$$

and

$$F(x)=\begin{bmatrix}0\\-x_2\\0\end{bmatrix}.$$

From Eq. (4.3), we can obtain the error dynamics

$$\dot{e}=f(y)-f(x)+F(y)\left(\frac{C}{I_1}(t)\right)-F(x)\left(\frac{C}{I_1}\right)+U. \tag{4.4}$$

Before solving our problem, we have some work to do first. Considering the special case when the drive and the response systems have the same parameters, which are time invariant. The drive and the response systems can be written as

$$\begin{aligned}\dot{x}&=f(x)+F(x)\left(\frac{C}{I_1}\right),\\ \dot{y}&=f(y)+F(y)\left(\frac{C}{I_1}\right)+U.\end{aligned} \tag{4.5}$$

The error dynamics can be obtained

$$\dot{e}=f(y)-f(x)+(F(y)-F(x))\left(\frac{C}{I}\right)+U. \tag{4.6}$$

Choosing a Lyapunov function of the form

$$V(e)=\frac{1}{2}e^Te=\frac{1}{2}(e_1^2+e_2^2+e_3^2). \tag{4.7}$$

Its derivative along the solution of Eq. (4.6) is

$$\frac{\mathrm{d}V}{\mathrm{d}t}=e^T\dot{e}=e^T\left[f(y)-f(x)+(F(y)-F(x))\left(\frac{C}{I_1}\right)+U\right]. \tag{4.8}$$

Select

$$\begin{aligned}u_1={}&-e_2-e_1,\\ u_2={}&\frac{(\beta_\phi-\beta_\varphi\cos y_1)(\beta_\varphi-\beta_\phi\cos y_1)}{I_1^2\sin^3 y_1}-\frac{(\beta_\phi-\beta_\varphi\cos x_1)(\beta_\varphi-\beta_\phi\cos x_1)}{I_1^2\sin^3 x_1}\\ &+\left(\frac{C}{I}-1\right)e_2-\frac{\mathrm{Mg}l}{I_1}(\sin y_1-\sin x_1)+\frac{\mathrm{Mg}}{I_1}A\sin\omega_1 t(\sin y_1-\sin x_1),\\ u_3={}&\frac{2\cos y_1}{\sin y_1}y_2y_3-\frac{2\cos x_1}{\sin x_1}x_2x_3-\frac{\beta_\varphi y_2}{I_1\sin y_1}+\frac{\beta_\varphi x_2}{I_1\sin x_1}-\frac{8}{3}e_3.\end{aligned} \tag{4.9}$$

Then, Eq. (4.8) can be rewritten as

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -e_1^2 - e_2^2 - \frac{8}{3}e_3^2 \leqslant 0. \tag{4.10}$$

This means that the synchronization of the two systems is achieved.

Now, we use the results of this special case to solve our problem. Choosing a Lyapunov function for Eq. (4.4)

$$V_1\left(e, \frac{C}{I_1}(t) - \frac{C}{I_1}\right) = V(e) + \frac{1}{2}\left(\frac{C}{I_1}(t) - \frac{C}{I_1}\right)^T\left(\frac{C}{I_1}(t) - \frac{C}{I_1}\right). \tag{4.11}$$

Let

$$\frac{\widetilde{C}}{I_1} = \frac{C}{I_1}(t) - \frac{C}{I_1}.$$

We can rewrite Eq. (4.11) as

$$V_1\left(e, \frac{\widetilde{C}}{I_1}\right) = V(e) + \frac{1}{2}\left(\frac{\widetilde{C}}{I_1}\right)^T\left(\frac{\widetilde{C}}{I_1}\right). \tag{4.12}$$

Its derivative along Eq. (4.4) satisfies

$$\begin{aligned}
\frac{\mathrm{d}V_1}{\mathrm{d}t} &= \frac{\mathrm{d}V}{\mathrm{d}t} + \left(\frac{\dot{C}}{I_1}(t)\right)^T\left(\frac{\widetilde{C}}{I_1}\right) = (\operatorname{grad} V(e), \dot{e}) + \left(\frac{\dot{C}}{I_1}(t)\right)^T\left(\frac{\widetilde{C}}{I_1}\right) \\
&= \left(\operatorname{grad} V(e), f(y) - f(x) + F(y)\left(\frac{C}{I_1}(t)\right)\right. \\
&\qquad \left. - F(x)\left(\frac{C}{I_1}\right) + U\right) + \left(\frac{\dot{C}}{I_1}(t)\right)^T\left(\frac{\widetilde{C}}{I_1}\right) \\
&= \left(\operatorname{grad} V(e), f(y) - f(x) + F(y)\left(\frac{C}{I_1}(t)\right) - F(x)\left(\frac{C}{I_1}(t)\right) + U\right) \\
&\qquad + \left(\operatorname{grad} V(e), F(x)\left(\frac{\widetilde{C}}{I_1}\right)\right) + \left(\frac{\dot{C}}{I_1}(t)\right)^T\left(\frac{\widetilde{C}}{I_1}\right).
\end{aligned} \tag{4.13}$$

Choosing

$$
\begin{aligned}
u_1 =& -e_2 - e_1,\\
u_2 =& \frac{(\beta_\phi - \beta_\varphi \cos y_1)(\beta_\varphi - \beta_\phi \cos y_1)}{I_1^2 \sin^3 y_1} - \frac{(\beta_\phi - \beta_\varphi \cos x_1)(\beta_\varphi - \beta_\phi \cos x_1)}{I_1^2 \sin^3 x_1}\\
&+ \left(\frac{C}{I}(t) - 1\right)e_2 - \frac{\mathrm{Mg}l}{I_1}(\sin y_1 - \sin x_1)\\
&+ \frac{\mathrm{Mg}}{I_1} A \sin \omega_1 t(\sin y_1 - \sin x_1),\\
u_3 =& \frac{2\cos y_1}{\sin y_1} y_2 y_3 - \frac{2\cos x_1}{\sin x_1} x_2 x_3 - \frac{\beta_\varphi y_2}{I_1 \sin y_1} + \frac{\beta_\varphi x_2}{I_1 \sin x_1} - \frac{8}{3} e_3,\\
\frac{\dot{C}}{I_1}(t) =& -F^T(x)(\operatorname{grad} V(e))^T = x_2 e_2.
\end{aligned}
$$

Eq. (4.13) can be rewritten as

$$
\begin{aligned}
\dot{V}_1\left(e, \frac{\widetilde{C}}{I_1}\right) &= \left(\operatorname{grad} V(e), f(y) - f(x) + F(y)\left(\frac{C}{I_1}(t)\right) - F(x)\left(\frac{C}{I_1}\right) + U\right)\\
&= \dot{V}(e) = -e_1^2 - e_2^2 - \frac{8}{3} e_3^2 < 0. \qquad (4.14)
\end{aligned}
$$

In this section, the parameter $\frac{C}{I_1}(t)$ is unknown and the values of the other parameters are given in Section 3. Let $A = 12.1$ and the concerned functions becomes as

$$
f(x) = \begin{bmatrix} x_2 \\ -\dfrac{(2 - 5\cos x_1)(5 - 2\cos x_1)}{\sin^3 x_1} + \sin x_1 - 4A \sin t \sin x_1 \\ -\dfrac{2\cos x_1}{\sin x_1} x_2 x_3 + \dfrac{5x_2}{\sin x_1} \end{bmatrix},
$$

$$
f(y) = \begin{bmatrix} y_2 \\ -\dfrac{(2 - 5\cos y_1)(5 - 2\cos y_1)}{\sin^3 y_1} + \sin y_1 - 4A \sin t \sin y_1 \\ -\dfrac{2\cos y_1}{\sin y_1} y_2 y_3 + \dfrac{5y_2}{\sin y_1} \end{bmatrix}.
$$

The controllers becomes

$$\begin{aligned} u_1 &= -e_2 - e_1, \\ u_2 &= \frac{(2-5\cos y_1)(5-2\cos y_1)}{\sin^3 y_1} - \frac{(2-5\cos x_1)(5-2\cos x_1)}{\sin^3 x_1} \\ &\quad + \left(\frac{C}{I_1}(t) - 1\right)e_2 - (\sin y_1 - \sin x_1) + 4A\sin t(\sin y_1 - \sin x_1), \\ u_3 &= \frac{2\cos y_1}{\sin y_1} y_2 y_3 - \frac{2\cos x_1}{\sin x_1} x_2 x_3 - \frac{5y_2}{\sin y_1} + \frac{5x_2}{\sin x_1} - \frac{8}{3} e_3. \end{aligned}$$

In numerical simulation, the initial conditions of the drive and the response systems are $x_1(0) = -0.5$, $x_2(0) = -1.2$, $x_3(0) = 10$ and $y_1(0) = 1.5$, $y_2(0) = 2.4$, $y_3(0) = 6$, respectively. We find that the synchronization can be achieved, and the results are shown in Figs. 5–7. The identification of

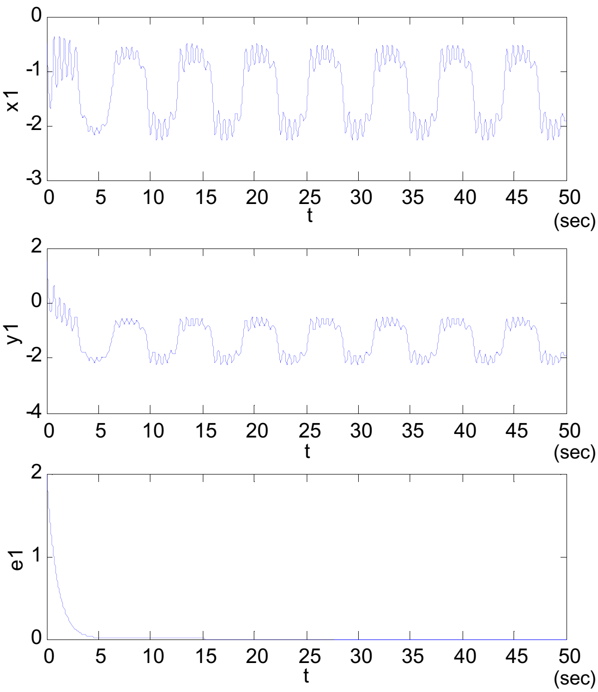


Fig. 5. Time history of $x_1$, $y_1$ and the error between them.

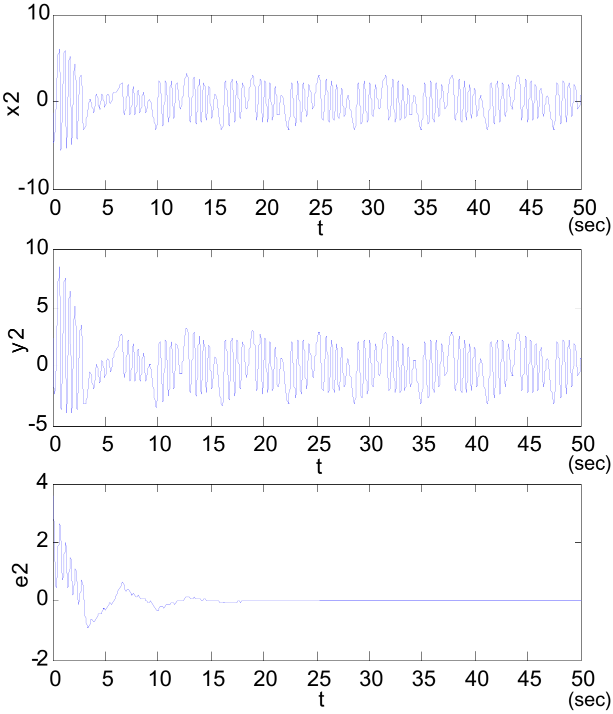


Fig. 6. Time history of $x_2$, $y_2$ and the error between them.

parameter is also achieved. It also means that the value of $\frac{C}{I_1}(t)$ arrives the value of $\frac{C}{I_1} = 0.5$. The result is shown in Fig. 8.

### 4.2. Parameters identification by random optimization

We investigate two identical third-order heavy symmetric gyroscope systems in this section. Both systems have the same parameters, but the parameter $\alpha$ of the response system is unknown. Our work is to identify the unknown parameter.

The drive system is described by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{(\beta_\phi - \beta_\varphi \cos x_1)(\beta_\varphi - \beta_\phi \cos x_1)}{I_1^2 \sin^3 x_1} - \frac{C}{I_1} x_2 + \frac{\mathrm{Mg}l}{I_1} \sin x_1 - \frac{\mathrm{Mg}}{I_1} A \sin \omega_1 t \sin x_1 \\ \dot{x}_3 = -\frac{2\cos x_1}{\sin x_1} x_2 x_3 + \frac{\beta_\varphi x_2}{I_1 \sin x_1} \end{cases}. \tag{4.15}$$

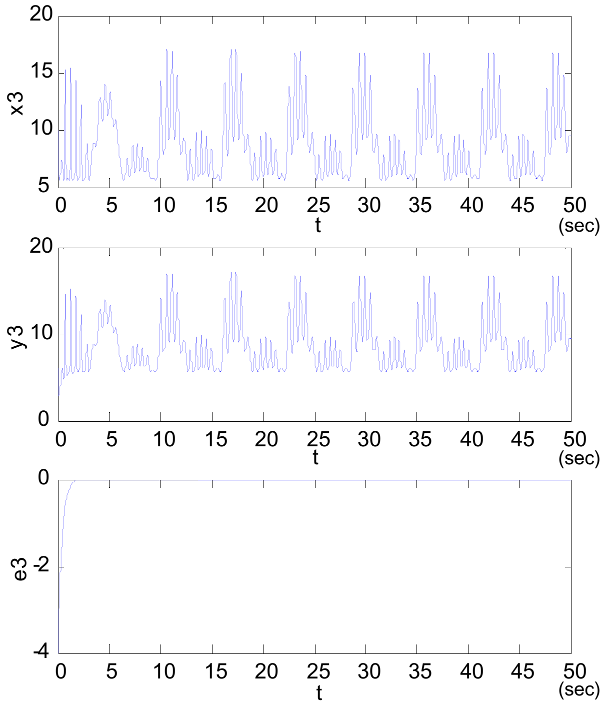


Fig. 7. Time history of $x_3$, $y_3$ and the error between them.

The response system is described by

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -\frac{(\beta_\phi - \beta_\varphi \cos y_1)(\beta_\varphi - \beta_\phi \cos y_1)}{I_1^2 \sin^3 y_1} - \alpha y_2 + \frac{\mathrm{M}gl}{I_1} \sin y_1 - \frac{\mathrm{Mg}}{I_1} A \sin \omega_1 t \sin y_1 \\ \dot{y}_3 = -\frac{2\cos y_1}{\sin y_1} y_2 y_3 + \frac{\beta_\varphi y_2}{I_1 \sin y_1} \end{cases}. \tag{4.16}$$

To synchronize two identical third-order heavy symmetric gyroscope systems, we add one coupling term, $k(x_1 - y_1)$ on the first equation of (4.16).

$$\begin{cases} \dot{y}_1 = y_2 + k(x_1 - y_1) \\ \dot{y}_2 = -\frac{(\beta_\phi - \beta_\varphi \cos y_1)(\beta_\varphi - \beta_\phi \cos y_1)}{I_1^2 \sin^3 y_1} - \alpha y_2 + \frac{\mathrm{M}gl}{I_1} \sin y_1 - \frac{\mathrm{Mg}}{I_1} A \sin \omega_1 t \sin y_1 \\ \dot{y}_3 = -\frac{2\cos y_1}{\sin y_1} y_2 y_3 + \frac{\beta_\varphi y_2}{I_1 \sin y_1} \end{cases}. \tag{4.17}$$

Define the difference by

$$U = \int_{0.9T}^{T} |x_1 - y_1|^2 \, \mathrm{d}t, \tag{4.18}$$

where $T$ is the simulation time.

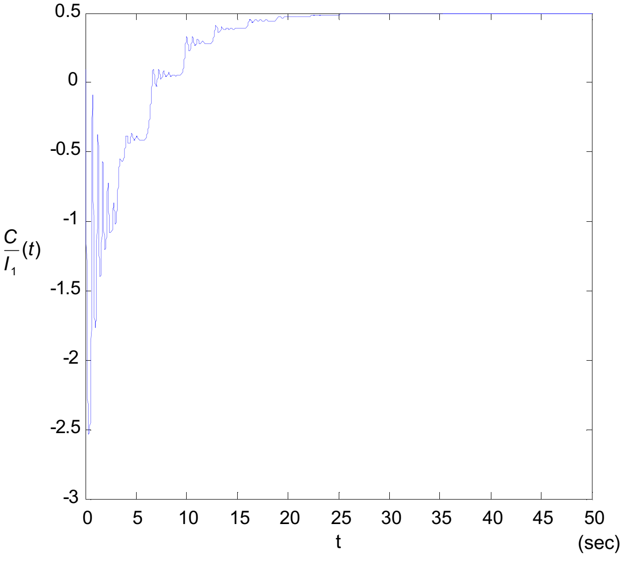


Fig. 8. Time history of $\frac{C}{I_1}(t)$.

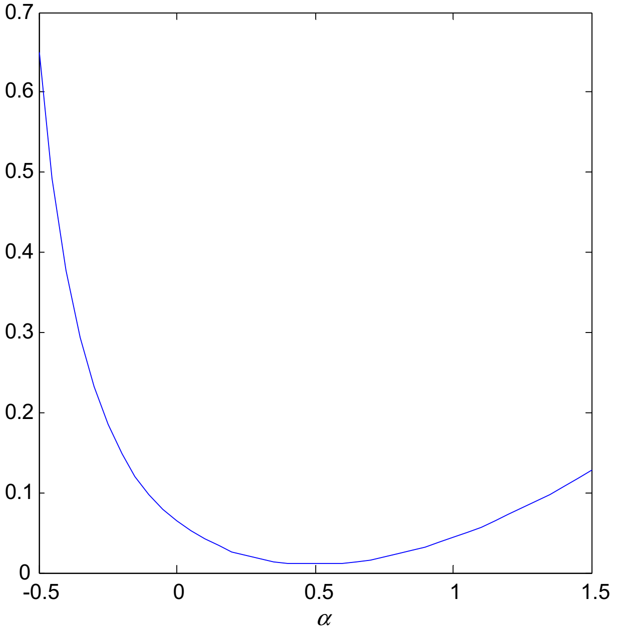


Fig. 9. Difference with respect to the parameter $\alpha$ for $k = 200$.

The difference $U$ can be considered as a function of $\alpha$ and $k$. If $k$ is sufficiently large and $\alpha$ is close to $\frac{C}{I_1}$, the difference $U$ would tend to zero. In other words, with sufficiently large value of $k$, if $U$ is small, $\alpha$ would be close to $\frac{C}{I_1}$. The result is shown in Fig. 9.

To identify the unknown parameter of the response system, we use the random optimization method [7]. The algorithm is as follows:

First, choose a sufficiently large value of $k$. In our case, we choose $k = 200$. By estimating initial value of $\alpha$, we can calculate the difference $U$.

The parameter $\alpha$ is randomly modified as

$$\alpha_{\mathrm{m}} = \alpha + r, \tag{4.19}$$

where $r$ is a random number which obeys the Gaussian distribution with variance $\sigma = 0.0025$.

Substituting the modified parameter $\alpha_{\mathrm{m}}$ into Eq. (4.17), we can obtain $y_1'$. The difference between two systems is

$$U' = \int_{0.9T}^{T} |x_1 - y_1'|^2 \, \mathrm{d}t. \tag{4.20}$$

If the difference $U'$ is smaller than $U$, the parameter is changed from $\alpha$ to $\alpha_{\mathrm{m}}$. On the other hand, if the difference $U'$ is larger than $U$, the parameter is

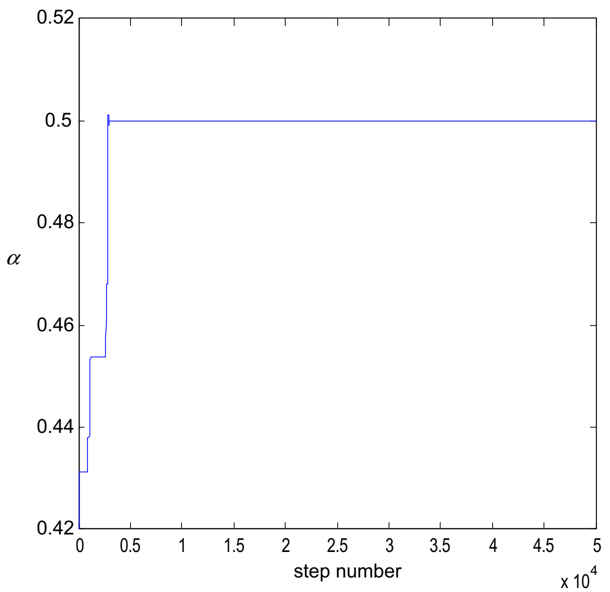


Fig. 10. Time evolution of $\alpha$ by random optimization process.

unchanged and kept to be $\alpha$. The processes are repeated until the difference $U$ tends to zero.

In numerical simulation, we assume that only one parameter, $\alpha$ is unknown. Parameter identification can be achieved. The result is shown in Fig. 10.

## 5. Conclusions

The main studies in this paper is the study of synchronization of two systems. In this paper, both analytical and computational methods have been used to study the dynamical behaviors of the nonlinear system.

We investigate two third-order heavy symmetric gyroscope systems in Section 3. Both the systems have the same form and both the parameters are unknown. The true values of the "unknown" parameters are selected. We choose three suitable controllers, and add them into the slave when $t = 50$. By using Lyapunov stability theory, the synchronization of the two third-order heavy symmetric gyroscope systems can be achieved successfully.

In Section 4, we investigate two third-order heavy symmetric gyroscope systems, while the parameter, $\frac{C}{I_1}(t)$ of the response system is time varying and uncertain. There are two purposes in this Section, one is to synchronize the two identical systems, by the Lyapunov function and controllers which are different from that of Section 3. The another is to track the parameter $\frac{C}{I_1}$ of the drive system. In numerical simulation, the unknown parameter $\frac{C}{I_1}(t)$ and $\alpha$ tracks the known parameter $\frac{C}{I_1}$ successfully.

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