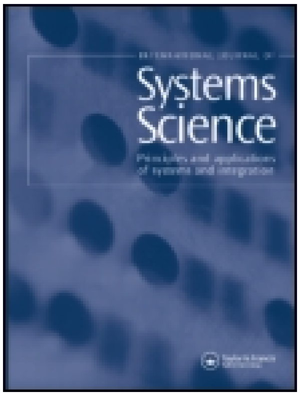

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Discrete fuzzy covariance control for specified decay rate

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This paper addresses the covariance control problem with decay rate for a class of nonlinear discrete stochastic systems using the Takagi-Sugeno (T-S) fuzzy models. A methodology is developed to find the discrete fuzzy controllers for achieving individual state variance constraints of discrete T-S fuzzy stochastic models. The approach developed in this paper is based on the concept of parallel distributed compensation (PDC) and covariance control. For each rule of the discrete T-S fuzzy model, it shows how to parameterize the static linear state feedback control gains to achieve a common covariance matrix and decay rate for each subsystem. Finally, a numerical example is provided to verify the effects of the proposed design method.

1. Introduction

Tanaka and Sugeno (1992) proposed a theorem on the stability analysis for T-S fuzzy models, which are described by fuzzy IF-THEN rules. For a nonlinear system that is successfully transformed to a T-S fuzzy model (Takagi and Sugeno 1985, Tanaka and Sugeno 1992) the stability of an overall nonlinear system still cannot be guaranteed even if the subsystem of the T-S fuzzy model is stable. To overcome this problem, Wang *et al*. (1996) proposed the concept of PDC as a design framework and also modified the Tanaka's stability theorem to include the effect of control. The goal of PDC concept is to design linear feedback gain for each local linear model, and let the overall system input be blended by these linear feedback gains. This method requires finding a common positive definite matrix **P** such that the proposed sufficient stability conditions are satisfied for every IF-THEN rule. The linear matrix inequality (LMI) (Boyd *et al*. 1994) method is a powerful tool in finding this common positive definite matrix **P**. Thus, many approaches are developed to find **P** via the LMI method (Tanaka and Wang 2001). Different from the LMI methodology, the authors have developed a T-S fuzzy controller design approach (Chang and Shing 2003, 2004, Chang 2001, 2003), which is based on the generalised inverse theory (Rao and Mitra 1971, Campbell and Meyer 1991).

For the T-S fuzzy stochastic models, a discrete state feedback fuzzy controller is developed in this paper by using the covariance control methodology (Hsien and Skelton 1990, Chung and Chang 1990, 1991, 1992, Fujioka and Hara 1995). In the covariance control theory, the characterisation of assignable covariance and the parametrization of controllers that assign these covariance matrices are presented. This type of control action has many promising features through control of the state covariance. Therefore, it would be to our great advantage to specify a state covariance matrix. According to the different requirements on the system robustness and performance, covariance control technique can be used to design a controller such that the state covariance of the closed-loop system is equal to a specified covariance matrix. The covariance control methodology has been applied to deal with the controller design problems for the linear and bilinear systems (Hsien and Skelton 1990, Chung and Chang 1990, 1991, 1992, Fujioka and Hara 1995).

In order to achieve individual state variance constraints, the fuzzy covariance control methodology

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is developed in this paper by designing a state feedback gain for each rule of the fuzzy controller. The individual state variance constrained design problem is difficult to solve for the LMI-based design method. Since these state feedback gains cannot be constructed to a standard LMI form, this paper derives common state covariance matrix assignment technique to find the state feedback fuzzy controller for T-S type fuzzy stochastic models. The first step in the proposed approach is assigning a common state covariance matrix to replace the common positive definite matrix **P** for the stability conditions. Subject to this specified common state covariance matrix, the linear feedback control gains are solved by using the theory of generalised inverse. In addition to the specified common state covariance matrix, this paper also considers the decay rate for the speed of response. Applying the proposed design approach, the designers can directly assign the common state covariance matrix and decay rate, simultaneously.

This paper is organised as follows. Section 2 describes the discrete T-S fuzzy model and its stability conditions. In section 3, a fuzzy covariance controller design method is developed to find state feedback gains for the discrete fuzzy controller which can drive the closed-loop system to achieve the specified common state covariance matrix and decay rate. A numerical example is given in section 4. Finally, a conclusion is provided in section 5.

2. Descriptions of discrete T-S fuzzy stochastic models

Most physical systems can be expressed in some forms of mathematical models, or as an aggregation of a set of mathematical models. Based on this phenomenon, Takagi and Sugeno (1985) developed a T-S fuzzy model to present a class of nonlinear systems. The state responses can be approximated to the original nonlinear system via the T-S fuzzy modelling. The T-S fuzzy model of a discrete nonlinear stochastic system is described by fuzzy IF-THEN rules, where the i-th rule is of the following form:

Rule^i:

$$\begin{aligned}&\text{IF } z_1(k) \text{ is } M_{i1} \ldots \text{and } z_{n_x}(k) \text{ is } M_{in_x},\\ &\text{THEN } x(k+1) = \mathbf{A}_i x(k) + \mathbf{B}_i u(k) + \mathbf{D}_i v(k),\end{aligned} \tag{1}$$

where $x(k) \in \mathcal{R}^{n_\times}$ is the state vector, $u(k) \in \mathcal{R}^{n_u}$ is the control input vector, $v(k) \in \mathcal{R}^{n_u}$ is a zero-mean white noise with the covariance $\mathbf{V} > 0$, and $v(k)$ and $x(0)$ are independent; $i = 1,2,\ldots,r$ and r is the number of IF-THEN rules. The M_{ij} are fuzzy sets, $(\mathbf{A}_i, \mathbf{B}_i, \mathbf{D}_i)$ is the i-th subsytem of the fuzzy model (1), where $\mathbf{A}_i \in \mathcal{R}^{n_x \times n_x}$, $\mathbf{B}_i \in \mathcal{R}^{n_x \times n_u}$ and $\mathbf{D}_i \in \mathcal{R}^{n_x \times n_v}$. Besides, $z_1(k)$, $z_2(k), \ldots, z_{n_x}(k)$ are the premise variables of the fuzzy model and they are the functions of state variables.

For the fuzzy model (1), the overall state equation can be represented as

$$x(k+1) = \frac{\sum_{i=1}^{r} \omega_i(z(k))\{\mathbf{A}_i x(k) + \mathbf{B}_i u(k) + \mathbf{D}_i v(k)\}}{\sum_{i=1}^{r} \omega_i(z(k))} \tag{2}$$

where

$$\omega_i(z(k)) = \prod_{j=1}^{n_x} M_{ij}(z_j(k)). \tag{3}$$

and $z(k) = [\, z_1(k), z_2(k), \ldots, z_{n_x}(k)]$. In addition, $M_{ij}(z_j(k))$ is the grade of membership of $z_j(k)$ in M_{ij}; $\omega_i(z(k))$ is the weight of the i-th rule.

Using the concept of PDC, a fuzzy control law can be developed for the stabilisation of a class of nonlinear stochastic systems that is represented by the T-S fuzzy model (1). The idea of PDC approach is to design the feedback gains to compensate each rule in the T-S fuzzy models. In other words, one can use linear control design techniques to design these linear controllers for each subsystem. Hence, a nonlinear controller can be blended by linear controllers and it shares the same fuzzy sets with the discrete T-S fuzzy model (1). The state feedback T-S type fuzzy controller formula is represented as follows:

Rule^i:

$$\begin{aligned}&\text{IF } z_1(k) \text{ is } M_{ij} \ldots \text{and } z_{n_x}(k) \text{ is } M_{in_x},\\ &\text{THEN } u(k) = \mathbf{G}_i x(k),\end{aligned} \tag{4}$$

where $i = 1, 2, \ldots, r$ and r is the number of IF-THEN rules. Hence, the state feedback T-S fuzzy controller is of the following form:

$$u(k) = \frac{\sum_{i=1}^{r} \omega_i(z(k))\mathbf{G}_i x(k)}{\sum_{i=1}^{r} \omega_i(z(k))}, \tag{5}$$

where $u(k)$ is the nonlinear feedback controller.

Substituting (5) into (2), one obtains

$$\begin{aligned}x(k+1) = &\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(k))h_j(z(k))[\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j]x(k)\\ &+ \sum_{i=1}^{r} h_i(z(k))\mathbf{D}_i v(k),\end{aligned} \tag{6}$$

where

$$h_i(z(k)) = \frac{\omega_i(z(k))}{\Sigma_{i=1}^{r}\omega_i(z(k))}. \tag{7}$$

The $h_i(z(k))$ can be regarded as the normalised weight of each IF-THEN rule. Rewriting (6), one has

$$\begin{aligned} x(k+1) = &\sum_{i=1}^{r} h_i(z(k))h_i(z(k))(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i)x(k) \\ &+ \sum_{i=1}^{r} h_i(z(k))\mathbf{D}_i v(k) \\ &+ 2\sum_{i<j} h_i(z(k))h_j(z(k))\mathbf{R}_{ij}x(k), \end{aligned} \quad (8)$$

where

$$\mathbf{R}_{ij} = \frac{(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_j) + (\mathbf{A}_i + \mathbf{B}_j\mathbf{G}_i)}{2}, \quad i < j \leq r. \quad (9)$$

Note that $\mathbf{R}_{ij}$ denotes the influence item between each other rule. From Theorem 8 of Tanaka and Wang (2001), the equilibrium of closed-loop fuzzy system (8) is asymptotically stable in the large if there exists a common positive definite matrix $\mathbf{P}$ satisfying

$$(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i)^T\mathbf{P}(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i) - \mathbf{P} < 0, \quad i = 1, 2, \ldots, r. \quad (10)$$

$$\mathbf{R}_{ij}^T\mathbf{P}\mathbf{R}_{ij} - \mathbf{P} \leq 0, \quad i < j \leq r. \quad (11)$$

Referring to the results of Konstantinov *et al.* (1986), one can find that the stochastic system is asymptotically stable if and only if the closed-loop state matrix is asymptotically stable. For satisfying stability conditions (10) and (11), one must choose appropriate matrices $\mathbf{P}$ and $\mathbf{G}_i$ ($i = 1, 2, \ldots, r$), simultaneously. Most scholars utilise the LMI method to find the common positive definite matrix $\mathbf{P}$ and linear feedback controllers $\mathbf{G}_i$ for each rule. In this paper, we first assign a common state covariance matrix to replace the common positive definite matrix $\mathbf{P}$. To carry on, the generalised inverse theory is used to develop a methodology to find linear feedback gains $\mathbf{G}_i$.

If $(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i)$ is a stable matrix, the steady state covariance matrix $\mathbf{X}_i$ ($\mathbf{X}_i = \lim_{k\to\infty} E[x(k)x^T(k)]$) of the subsystem has the following form (Kwakernaak and Sivan 1972).

$$\mathbf{X}_i = \mathbf{X}_i^T > 0, \quad (12)$$

where E denotes the expectation operator. Note that $\mathbf{X}_i$ is the unique solution of the following Lyapunov equation for each rule.

$$\mathbf{X}_i = (\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i)\mathbf{X}_i(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i)^T + \mathbf{D}_i\mathbf{V}\mathbf{D}_i^T, \quad i = 1, 2, \ldots, r. \quad (13)$$

Here, we define $\mathbf{X}_i = \mathbf{X}$, $i = 1, 2, \ldots, r$, where $\mathbf{X}$ is called the common state covariance matrix for all rules. According to the common state covariance matrix $\mathbf{X}$, the state variance constraints can be introduced as follows.

2.1 *Individual state variance constraints*

Let σ_ℓ, $\ell = 1, 2, \ldots, n_x$, denote the root mean squared (RMS) constraints for the variances of individual system states. The purpose of the state variance constrained design problem is to find feedback controllers such that the following individual state variance

$$\lim_{k\to\infty} E[x_\ell^2(k)] = [\mathbf{X}]_{\ell\ell} \leq \sigma_\ell^2, \qquad \ell = 1, 2, \ldots, n_x \quad (14)$$

where $[\cdot]_{\ell\ell}$ denotes the ℓth diagonal element of matrix $[\cdot]$ and $\mathbf{X}$ is the common state covariance matrix for all rules. Using the common state covariance matrix $\mathbf{X}$ to replace common positive define matrix $\mathbf{P}$, the stability conditions (10) and (11) result in the following lemma.

Lemma 1: The equilibrium of closed-loop fuzzy system (8) is asymptotically stable in the main if there exists a common state covariance matrix $\mathbf{X} = \mathbf{X}^T > 0$ satisfying

$$\mathbf{X} = (\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i)\mathbf{X}(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i)^T + \mathbf{D}_i\mathbf{V}\mathbf{D}_i^T, \quad i = 1, 2, \ldots, r. \quad (15)$$

$$\mathbf{R}_{ij}\mathbf{X}\mathbf{R}_{ij}^T - \mathbf{X} \leq 0, \quad i < j \leq r. \quad (16)$$

If there is a common state covariance matrix $\mathbf{X}$ satisfying (15) and (16), then the nonlinear stochastic system (8) is asymptotically stable in the main. The problem considered in this paper can be described as follows. When we assign a common state covariance matrix $\mathbf{X}$ of all subsystems in (8), the purpose of this paper is to find the linear feedback gains $\mathbf{G}_i$ such that (15) and (16) are satisfied.

In addition to considering the individual state variance constraints, we consider the control performance for the speed of response. The speed of response is related to decay rate, i.e. the large Lyapunov exponent. Thus, we discuss the stability with decay rate and analyse the stability conditions of T-S fuzzy models by the following lemma.

Lemma 2 (Tanaka and Wang 2001)**:** The equilibrium of closed-loop fuzzy system (8) is asymptotically stable in the main with the decay rate η if there exists a common state covariance matrix $\mathbf{X}_d = \mathbf{X}_d^T > 0$ satisfying

$$\begin{aligned} \eta^2\mathbf{X}_d = &(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i)\mathbf{X}_d(\mathbf{A}_i + \mathbf{B}_i\mathbf{G}_i)^T + \mathbf{D}_i\mathbf{V}\mathbf{D}_i^T, \\ &i = 1, 2, \ldots, r. \end{aligned} \quad (17)$$

$$\mathbf{R}_{ij}\mathbf{X}_d\mathbf{R}_{ij}^T - \eta^2\mathbf{X}_d \leq 0, \quad i < j \leq r, \quad (18)$$

where $\eta \leq 1$.

In Lemma 2, new stability conditions are provided to instead of (15) and (16) for the decay rate design problem. The method of finding the linear feedback gains $\mathbf{G}_i$ for the stability conditions of Lemma 1 and Lemma 2 is presented in the next section.

3. Fuzzy controller design for specified common state covariance matrix and decay rate

To design a stable state feedback fuzzy controller with specified common state covariance matrix and decay rate for all rules, it is necessary to find suitable matrices $\mathbf{G}_i$ and $\mathbf{X}_d$ to satisfy stability conditions (17) and (18). At first, we assign a common state covariance matrix $\mathbf{X}_d$ to conform to the needs of individual state variance constraints (14). Then, the linear feedback gains $\mathbf{G}_i$ can be solved from equation (17). In order to guarantee the stability for overall closed-loop system, we need to check whether (18) is satisfied. If the stability condition (18) is not satisfied, one needs to reassign the common state covariance matrix $\mathbf{X}_d$ to obtain a feasible fuzzy controller. In the following theorem, we first solve the linear feedback control gain for each rule with no decay rate.

Theorem 1: Given a common state covariance matrix $\mathbf{X}=\mathbf{X}^T>0$ for the discrete T-S fuzzy system (8), there exists state feedback gains $\mathbf{G}_i$ such that $\mathbf{X}$ solves (15) if and only if

$$\mathbf{X}=[(\mathbf{I}-\mathbf{B}_i\mathbf{B}_i^+)\mathbf{A}_i\mathbf{X}+\mathbf{B}_i\mathbf{B}_i^+\hat{\mathbf{Z}}_i]\mathbf{X}^{-1} \times[(\mathbf{I}-\mathbf{B}_i\mathbf{B}_i^+)\mathbf{A}_i\mathbf{X}+\mathbf{B}_i\mathbf{B}_i^+\hat{\mathbf{Z}}_i]^T+\mathbf{D}_i\mathbf{V}\mathbf{D}_i^T, \quad (19)$$

where $\hat{\mathbf{Z}}_i$ is given to satisfy (19).

Moreover, if the condition (19) is satisfied then the feedback gains of each rule are as follows:

$$\mathbf{G}_i=\mathbf{B}_i^+(\hat{\mathbf{Z}}_i-\mathbf{A}_i\mathbf{X})\mathbf{X}^{-1}+(\mathbf{I}-\mathbf{B}_i^+\mathbf{B}_i)\mathbf{Z}_i, \quad (20)$$

where $\mathbf{Z}_i$ is an arbitrary matrix of dimension $n_x\times n_x$.

Proof: Suppose there exists a common state covariance matrix $\mathbf{X}>0$ satisfying the Lyapunov equation (15), then the equation (15) can be rewritten as follows:

$$\mathbf{X}=\mathbf{A}_i\mathbf{X}\mathbf{A}_i^T-\mathbf{S}_i\mathbf{X}^{-1}\mathbf{S}_i^T+\mathbf{D}_i\mathbf{V}\mathbf{D}_i^T+\mathbf{L}_i\mathbf{L}_i^T \quad (21)$$

where

$$\mathbf{L}_i=(\mathbf{B}_i\mathbf{G}_i+\mathbf{S}_i\mathbf{X}^{-1})\boldsymbol{\Gamma}. \quad (22)$$

$$\mathbf{S}_i=\mathbf{A}_i\mathbf{X}. \quad (23)$$

$$\mathbf{X}=\boldsymbol{\Gamma}\mathbf{T}^T. \quad (24)$$

Necessity:
By supposition, the feedback gains $\mathbf{G}_i$ of each rule satisfying (22) exists, and we expand (22) as

$$\mathbf{B}_i\mathbf{G}_i\boldsymbol{\Gamma}=\mathbf{L}_i-\mathbf{S}_i\mathbf{X}^{-1}\boldsymbol{\Gamma}. \quad (25)$$

According to (25), we can get the solution $\mathbf{G}_i$ by the generalized inverse technique (Rao and Mitra 1971, Campbell and Meyer 1991), which is guaranteed if and only if

$$(\mathbf{I}-\mathbf{B}_i\mathbf{B}_i^+)(\mathbf{L}_i-\mathbf{S}_i\boldsymbol{\Gamma}^{-T})=0. \quad (26)$$

All solutions $\mathbf{G}_i$ are given by (Rao and Mitra 1971, Campbell and Meyer 1991)

$$\mathbf{G}_i=\mathbf{B}_i^+(\mathbf{L}_i\boldsymbol{\Gamma}^{-1}-\mathbf{S}_i\mathbf{X}^{-1})+(\mathbf{I}-\mathbf{B}_i^+\mathbf{B}_i)\mathbf{Z}_i, \quad (27)$$

where $\mathbf{Z}_i$ is an arbitrary matrix of dimension $n_x\times n_x$. Solving (26) for $\mathbf{L}_i$ yields

$$\mathbf{L}_i=(\mathbf{I}-\mathbf{B}_i\mathbf{B}_i^+)\mathbf{S}_i\boldsymbol{\Gamma}^{-T}+\mathbf{B}_i\mathbf{B}_i^+\tilde{\mathbf{Z}}_i, \quad (28)$$

where $\tilde{\mathbf{Z}}_i$ is arbitrary. Substituting (28) into (21) gives

$$\mathbf{X}=\mathbf{A}_i\mathbf{X}\mathbf{A}_i^T+\mathbf{D}_i\mathbf{V}\mathbf{D}_i^T-\mathbf{S}_i\mathbf{X}^{-1}\mathbf{S}_i^T +\left[(\mathbf{I}-\mathbf{B}_i\mathbf{B}_i^+)\mathbf{S}_i+\mathbf{B}_i\mathbf{B}_i^+\hat{\mathbf{Z}}_i\right]\mathbf{X}^{-1}\left[(\mathbf{I}-\mathbf{B}_i\mathbf{B}_i^+)\mathbf{S}_i+\mathbf{B}_i\mathbf{B}_i^+\hat{\mathbf{Z}}_i\right]^T, \quad (29)$$

where $\hat{\mathbf{Z}}_i\equiv\tilde{\mathbf{Z}}_i\boldsymbol{\Gamma}^T$ is arbitrary.

Substituting (28) into (27) yields

$$\mathbf{G}_i=\mathbf{B}_i^+(\hat{\mathbf{Z}}_i-\mathbf{S}_i)\mathbf{X}^{-1}+(\mathbf{I}-\mathbf{B}_i^+\mathbf{B}_i)\mathbf{Z}_i. \quad (30)$$

This completes the necessity.

Sufficiency:
Substituting (20) into (21), we can get

$$\mathbf{L}_i=[(\mathbf{I}-\mathbf{B}_i\mathbf{B}_i^+)\mathbf{S}_i+\mathbf{B}_i\mathbf{B}_i^+\hat{\mathbf{Z}}_i]\boldsymbol{\Gamma}^{-T}. \quad (31)$$

Putting (31) into (21) yields

$$\mathbf{X}-\mathbf{A}_i\mathbf{X}\mathbf{A}_i^T+\mathbf{S}_i\mathbf{X}^{-1}\mathbf{S}_i^T-\mathbf{D}_i\mathbf{V}\mathbf{D}_i^T =[(\mathbf{I}-\mathbf{B}_i\mathbf{B}_i^+)\mathbf{S}_i+\mathbf{B}_i\mathbf{B}_i^+\hat{\mathbf{Z}}_i]\mathbf{X}^{-1}[(\mathbf{I}-\mathbf{B}_i\mathbf{B}_i^+)\mathbf{S}_i+\mathbf{B}_i\mathbf{B}_i^+\hat{\mathbf{Z}}_i]^T. \quad (32)$$

Note that equation (32) equals to equation (19). Since (19) holds, equation (21) can be solved by $\mathbf{G}_i$ given in (20). The sufficiency of this proof is completed.

Based on the above theorem, one can find that if condition (19) holds, then the feedback gains $\mathbf{G}_i$ can be obtained from (20). Theorem 1 considered the linear feedback gain design for each rule of the discrete T-S fuzzy controllers. The theorem is derived based on the stability conditions of Lemma 1, which does not deal with the decay rate constraint. In the following theorem, the common state covariance matrix and decay rate assignment problem is considered. It is shown that the development of Theorem 2 is based on the stability conditions of Lemma 2.

Theorem 2: Given a common state covariance matrix $\mathbf{X}_d = \mathbf{X}_d^T > 0$ and a decay rate $\eta(|\eta| < 1)$ for all rules of the discrete T-S fuzzy system (8), there exists feedback gains $\mathbf{G}_i$ such that $\mathbf{X}_d$ and η solve (17) if and only if

$$\eta^2\mathbf{X}_d = [(\mathbf{I}-\mathbf{B}_i\mathbf{B}_i^+)\mathbf{A}_i\mathbf{X}_d + \mathbf{B}_i\mathbf{B}_i^+\hat{\mathbf{Y}}_i]\mathbf{X}_d^{-1}[(\mathbf{I}-\mathbf{B}_i\mathbf{B}_i^+)\mathbf{A}_i\mathbf{X}_d + \mathbf{B}_i\mathbf{B}_i^+\hat{\mathbf{Y}}_i]^T + \mathbf{D}_i\mathbf{V}\mathbf{D}_i^T, \quad (33)$$

where $\hat{\mathbf{Y}}_i$ is given to satisfy (33).

Moreover, if the condition (33) is satisfied then the feedback gains of each rule are given as follows:

$$\mathbf{G}_i = \mathbf{B}_i^+(\hat{\mathbf{Y}}_i - \mathbf{A}_i\mathbf{X}_d)\mathbf{X}_d^{-1} + (\mathbf{I}-\mathbf{B}_i^+\mathbf{B}_i)\mathbf{Y}_i, \quad (34)$$

where $\mathbf{Y}_i$ is an arbitrary matrix of dimension $n_x \times n_x$.

Proof: Comparing the stability condition (17) of Lemma 2 with the stability condition (15) of Lemma 1, the proof of this theorem can be obtained from the proof of Theorem 1 by using $\mathbf{X}_d$ to replace $\mathbf{X}$ and using $\eta^2\mathbf{X}_d$ to replace the left-hand side of the equation (21).

Theorem 2 gives the conditions and solutions for the feedback gains of discrete T-S type fuzzy controllers, which can simultaneously achieve the specified common state covariance matrix $\mathbf{X}_d$ and the decay rate η. If the common state covariance matrix $\mathbf{X}_d$ and the decay rate η can be appropriately assigned to satisfy condition (33), then the linear feedback gains $\mathbf{G}_i$ can be obtained from (34) for each rule. The main contribution of the proposed approach is that one can design the discrete T-S fuzzy controller to achieve the system performance constraints in addition to guaranteeing the stability of the nonlinear stochastic systems. The performance constraints include individual state variance constraints and speed response constraints. To demonstrate the usefulness of the proposed design approach, a simple design procedure is provided as follows. It can be used to find suitable decay rate η and common state covariance matrix $\mathbf{X}_d$ for the fuzzy covariance controllers.

Step 1: Assign the diagonal elements of common state covariance matrix $\mathbf{X}_d$ subject to the state variance constraints (14).

Step 2: Choose $\eta = \varepsilon$, where $\varepsilon < 1$ is a small positive real number.

Step 3: Substitute η and the diagonal elements of $\mathbf{X}_d$ into (33) to solve the off-diagonal elements of $\mathbf{X}_d$ and $\hat{\mathbf{Y}}_i$.

Step 4: If the solution of Step 3 is feasible, then go the Step 6, otherwise adjust $\eta = \eta + \varepsilon$ and go to Step 5.

Step 5: If $\eta > 1$, then go to Step 1 to reassign the diagonal elements of common state covariance matrix $\mathbf{X}_d$; otherwise, go to Step 3 to solve the feasible solutions of $\mathbf{X}_d$ and $\hat{\mathbf{Y}}_i$.

Step 6: Calculate the feedback gains $\mathbf{G}_i$ for each rule from equation (34).

Step 7: Substituting η, $\mathbf{X}_d$ and $\mathbf{G}_i$ into (18) to check whether (18) is satisfied. If (18) is not satisfied, then to go to Step 1 to reassign the common state covariance matrix $\mathbf{X}_d$.

The flowchart of the above design procedure is shown in figure 1. The proposed fuzzy controller design procedure demonstrates the way to choose η and $\mathbf{X}_d$ for satisfying condition (33). Though the present design procedure needs fewer trial-and-error processes to pick out parameters η and $\mathbf{X}_d$, it allows designers to directly assign appropriate decay rate η and common state covariance matrix $\mathbf{X}_d$ such that the speed response

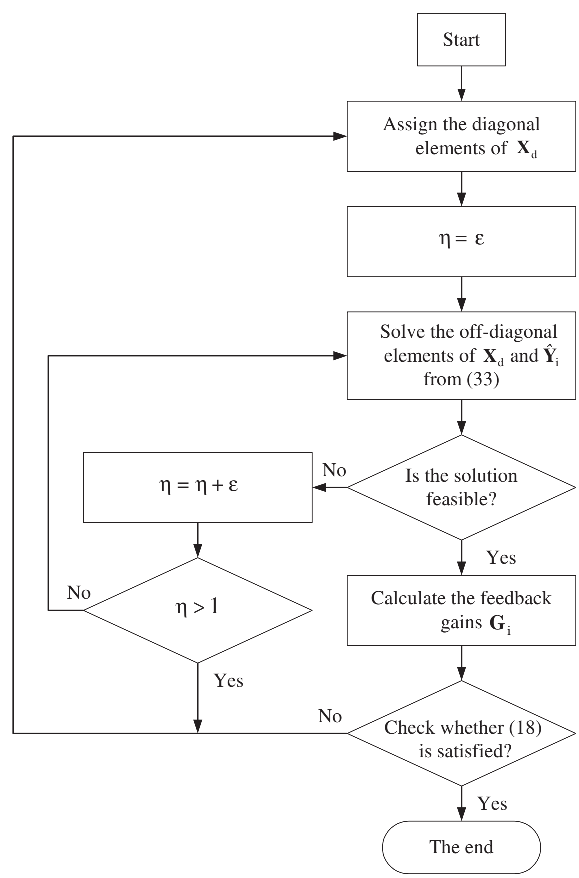

Figure 1. A flowchart of design procedure.

constraints and individual state variance constraints are achieved. In the next section, a simple numerical example is provided to verify the effect of the proposed design method.

4. An example

In this section, a nonlinear discrete stochastic system is considered as follows:

$$x_1(k+1) = 0.25x_2(k) + 0.5v(k) \tag{35a}$$

$$x_2(k+1) = 0.25x_2(k) + \frac{1}{\sqrt{5}}x_3(k) + 0.5v(k) \tag{35b}$$

$$x_3(k+1) = 0.8\sin x_1(k) + 0.12x_2(k) + 0.3x_3(k)\cos x_1(k) - \frac{u(k)}{1-0.01\cos x_1(k)}. \tag{35c}$$

It is assumed that the range of $x_1(k)$ is $x_1(k) \in (-\pi/2, \pi/2)$. First, we need to transfer the original nonlinear stochastic system (35) to a T-S type fuzzy model. The detail of the T-S fuzzy modelling process can be referred to Chapter 2 of Tanaka and Wang (2001). In this example, the membership functions are simply defined by using triangular type functions. The number of fuzzy rules influences the approximation of the original nonlinear systems. Indeed, the computational complexity is increased when the rules increase. The choice of the rule number and membership function shape do not influence the proposed fuzzy controller design process and the stability conditions. To minimise the design effort and complexity, we try to use as few rules as possible. Hence, we approximate the nonlinear system (35) by the following two-rule T-S fuzzy model.

Rule^1:

IF $x_1(k)$ is about 0
THEN $x(k+1) = \mathbf{A}_1x(k) + \mathbf{B}_1u(k) + \mathbf{D}_1v(k)$, (36a)

Rule^2:

IF $x_1(k)$ is about $\pm\pi/2$ $(|x_1| < \pi/2)$
THEN $x(k+1) = \mathbf{A}_2x(k) + \mathbf{B}_2u(k) + \mathbf{D}_2v(k)$. (36b)

where

$$\mathbf{A_1} = \begin{bmatrix} 0 & 0.25 & 0 \\ 0 & 0.25 & 1/\sqrt{5} \\ 0.8 & 0.12 & 0.3 \end{bmatrix}, \quad \mathbf{B_1} = \begin{bmatrix} 0 \\ 0 \\ -1.0101 \end{bmatrix},$$
$$\mathbf{D_1} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}, \tag{37a}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0.25 & 0 \\ 0 & 0.25 & 1/\sqrt{5} \\ 1.6/\pi & 0.12 & 0.3\beta \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 \\ 0 \\ -1/(1-0.01\beta) \end{bmatrix},$$
$$\mathbf{D}_2 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}, \tag{37b}$$

and $\beta \equiv \cos(88^\circ)$. The covariance matrix $\mathbf{V}$ of $v(k)$ is given as follows:

$$\mathbf{V} = 0.1. \tag{38}$$

The membership functions of $x_1(k)$ are shown in figure 2. Considering the decay rate constraint, we assign $\eta = 0.5$ in this example. Besides, it is assumed that the state variance constraints for the system (36) have the following forms:

$$[\mathbf{X}_d]_{11} \le 0.25, \quad [\mathbf{X}_d]_{22} \le 0.5 \quad \text{and} \quad [\mathbf{X}_d]_{33} \le 0.3. \tag{39}$$

For the state variance constraints (39), we assign the common state covariance matrix as

$$\mathbf{X}_d = \begin{bmatrix} 0.2 & 0.2 & 0 \\ 0.2 & 0.4 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}. \tag{40}$$

Let us define $\hat{\mathbf{Y}}_i$ as follows:

$$\hat{\mathbf{Y}}_\mathbf{1} = \begin{bmatrix} y1_{11} & y1_{12} & y1_{13} \\ y1_{21} & y1_{22} & y1_{23} \\ y1_{31} & y1_{32} & y1_{33} \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{Y}}_2 = \begin{bmatrix} y2_{11} & y2_{12} & y2_{13} \\ y2_{21} & y2_{22} & y2_{23} \\ y2_{31} & y2_{32} & y2_{33} \end{bmatrix}, \tag{41}$$

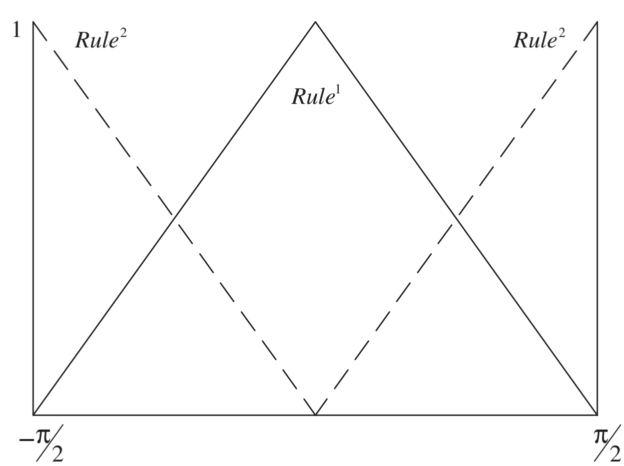

Figure 2. The membership function of $x_1(k)$.

where $y1_{ij}$ and $y2_{ij}$ are the desired parameters. Substituting (40) and (41) into (33), one has

$$\frac{1}{4}y1_{32} = 0, \quad (42a)$$

$$\frac{1}{4}y1_{32} + \frac{1}{\sqrt{5}}y1_{33} = 0, \quad (42b)$$

$$10y1_{31}^2 - 10y1_{31}y1_{32} + 5y1_{32}^2 + 4y1_{33}^2 - \frac{1}{16} = 0, \quad (42c)$$

$$\frac{1}{4}y2_{32} = 0, \quad (42d)$$

$$\frac{1}{4}y2_{32} + \frac{1}{\sqrt{5}}y2_{33} = 0, \quad (42e)$$

$$10y2_{31}^2 - 10y2_{31}y_{32}^2 + 5y2_{32}^2 + 4y2_{33}^2 - \frac{1}{16} = 0. \quad (42f)$$

Solving (42), $\hat{\mathbf{Y}}_1$ and $\hat{\mathbf{Y}}_2$ can be respectively obtained as follows:

$$\hat{\mathbf{Y}}_1 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ \sqrt{10}/40 & 0 & 0 \end{bmatrix} \text{ and } \hat{\mathbf{Y}}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -\sqrt{10}/40 & 0 & 0 \end{bmatrix}. \quad (43)$$

Putting $\mathbf{X}_d$, $\hat{\mathbf{Y}}_1$ and $\hat{\mathbf{Y}}_2$ into (34) the state feedback gains are solved as follows:

$$\mathbf{G}_1 = [0.0093 \quad 0.5101 \quad 0.297] \quad \text{and}$$
$$\mathbf{G}_2 = [1.2998 \quad -0.2753 \quad 0.0105] \quad (44)$$

Substituting the feedback gains $\mathbf{G}_1$ and $\mathbf{G}_2$ into (18), we can find that the condition (18) is satisfied. By Lemma 2, it can be concluded that the fuzzy control system is asymptotically stable in the main. Applying the state feedback gains (44), the simulated results are obtained with the initial conditions $x(0) = [1 \quad -1 \quad -2]^T$. Besides, the simulation range is defined as $k = 0, 1, 2, \ldots, 2000$. The simulated results are shown in table 1, where var$(\cdot)$ denotes the variance value of $(\cdot)$.

In order to demonstrate the usefulness of the present approach, the above results are compared with the LMI-based design method (Tanaka and Wang 2001). Assigning the same decay rate $\eta = 0.5$ and applying the LMI-based design method, one can solve $\mathbf{G}_i$ and $\mathbf{P}$ from (3.34) and (3.35) of Tanaka and Wang (2001). The solutions of feedback gains $\mathbf{G}_i$ can be obtained via MATLAB LMI-toolbox as follows:

$$\mathbf{G}_{L1} = \begin{bmatrix} -0.7925 & -0.1188 & -0.2975 \end{bmatrix} \quad \text{and}$$
$$\mathbf{G}_{L2} = \begin{bmatrix} -0.5098 & -0.12 & -0.011 \end{bmatrix} \quad (45)$$

The simulated results controlled by the above fuzzy control gains (45) are shown in table 1 with the same initial conditions $x(0) = [1 \quad -1 \quad -2]^T$ and the same simulation range $k = 0, 1, 2, \ldots, 2000$. From the results of table 1, one can find that the state variances of the proposed fuzzy control approach are smaller than that of LMI-based fuzzy control method. The fourth table column shows the percentage improvement of the proposed approach when compared to the LMI approach. Hence, the proposed approach provides a more efficient design method for the designers while dealing with the individual state variance constrained control problems.

Table 1. The variances of system states.

	LMI-based design method	Proposed design method	Percentage improvement
var (x_1)	0.0328	0.0302	7.93%
var (x_2)	0.0746	0.0353	52.68%
var (x_3)	0.1292	0.0156	87.93%

5. Conclusions

In this paper, a fuzzy covariance controller has been developed for a class of nonlinear discrete stochastic systems. The T-S fuzzy model was utilised to represent this class of nonlinear systems. Using the covariance control concept, one can directly design a fuzzy covariance controller for the discrete T-S fuzzy stochastic models. In general, most scholars solved the stability control problem of T-S fuzzy models by using the LMI design method. But it is difficult to directly assign a common state covariance matrix for the designers when they applied the LMI design method. This paper provided a methodology to solve state feedback gains for the T-S type fuzzy controllers such that the common state covariance matrix and decay rate can be directly assigned.

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