A Hierarchy of Importance Indices

F. K. Hwang

***Index Terms*—Birnbaum importance, component importance, cutset, optimal assignment, pathset, structural importance.**

SUMMARY AND CONCLUSIONS

An importance index measures the relative importance of a component in comparison to other components with respect to the system reliability. We survey many importance indices proposed in the literature, and some new ones of our own, to show how they relate. Our results can be used in two situations:

1) When components are functionally interchangeable, then more reliable units should go to the more important components.
2) When there is a fixed budget to improve the system reliability, the more important components should get first attention if everything else is equal.

NOTATION

cutset (pathset)	A subset of components whose collective inoperation (operation) causes the inoperation (operation) of the system.
C	The set of cutsets; $C(d)$ if the total number of inoperative components is d.
$\overline{C}$	If C is minimal (no proper subset is a cutset).
C_i	The set of cutsets containing i; $C_i(d)$ if d is given.
$C_{(i)}$	The set of cutsets not containing i; $C_{(i)}(d)$ if d is given.
$\overline{C}\,(d,k)$	The number of k minimum cutsets whose union contains i and is of size d.
P	The set of pathsets; $P(d)$ if d is given.
$\overline{P}$	If P is minimal.
P_i	The set of pathsets containing i; $P_i(d)$ if d is given.
$P_{(i)}$	The set of pathsets not containing i; $P_{(i)}(d)$ if d is given.
$\overline{S}$	The complement of the subset S.
$\lvert S\rvert$	The cardinality of the set S.
$R(p_1 \ldots p_n)$	The reliability of a system whose component i has reliability p_i.

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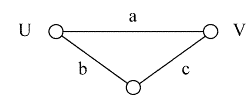


subset	cutset	pathset
a		Y
ab	Y	Y
ac	Y	Y
abc	Y	Y
bc		Y
c		
b		
∅		

A Y in a cell means the corresponding subset is a cutset (or pathset).

Fig. 1. A simple system.

I. INTRODUCTION

Assumption:

1) Components are functionally interchangeable.
2) Both the system and components are bi-state, either operative or not.

An importance index measures the relative importance of a component, in comparison to other components, with respect to the system reliability. The contribution of a component to the system reliability has two sources: 1. the reliability of the physical unit installed at the location of the component; 2. the location itself.

This paper concentrates on Source 2. Because of Assumption 1, the n physical units (with different reliabilities) can be arbitrarily assigned to the n locations in the system. An importance index then serves as a guide as to where the more reliable units should go.

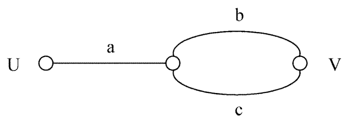
Let N denote the set of n components of a binary system. Most of the commonly used importance indices are defined in terms of either cutsets or pathsets. This paper studies the spectrum of importance indices, including some new ones, based on cutsets and pathsets, their relations, and their relevance to the assignment problem.

II. CUTSETS AND PATHSETS

Two examples demonstrate the concepts of cutsets and pathsets, which also help to illustrate many results in this paper.

Example. 2.1: The system consists of three components a, b, c as shown in Fig. 1. The system is operative if U and V are connected.

Example. 2.2:



subset	cutset	pathset
a	Y	
ab	Y	Y
ac	Y	Y
abc	Y	Y
bc	Y	
c		
b		
∅		

Fig. 2. Another simple system.

Lemma 2.3: $\overline{S}$ is a cutset if and only if $\overline{S}$ is not a pathset.

Proof: Suppose to the contrary that S is a cutest, and $\overline{S}$ a pathset. Consider the case that a component is operative if and only if it is in $\overline{S}$. Then the system would be simultaneously operative, and inoperative: an absurdity. □

Corollary 2.4: $|C_i| + |P_{(i)}| = 2^{n-1}$.

Proof: For any of the 2^{n-1} subsets containing a given component, either it is a cutset or its complement is a pathset. □

Corollary 2.5:

$$|C_i(d)| + \left|P_{(i)}(n-d)\right| = \binom{n-1}{d-1}.$$

In Examples 2.1 and 2.2, the eight subsets are arranged into two halves; those containing component a are in the left half, and their complements are in the right half. The cutset column in one half is complementary to the pathset column of the other half, thus illustrating Lemma 2.3.

III. Importance Indices Based on Cutsets and Pathsets

Component i is ***c*-absolutely more important** than component j if $C_i \supseteq C_j$, and ***p*-absolutely more important** if $P_i \supseteq P_j$. These two concepts are not equivalent, because in Example 2.1, component a is c-absolutely more important than component b, but not p-absolutely, and vice versa in Example 2.2. These two types of dominance cannot coexist.

Theorem 3.1: No component can be both p-absolutely, and c-absolutely, more important than another component in a nontrivial system.

Proof: If $\{i\}$ is a cutset, then $\{i\}$ is not a pathset in a nontrivial system. Hence $N \setminus \{i\}$ is a cutset containing j but not i. If $\{i\}$ is not a cutset, then $N \setminus \{i\}$ is a pathset containing j but not i. □

Reference [2] proposed an importance index which is slightly weaker than the absoluteness index, but stronger than other indices proposed in the literature.

Component i is **more critical** than component j if for any subset S not containing i or j, $j \cup S \in C_j \Rightarrow i \cup S \in C_i$. This section shows that c-criticality, and p-criticality, are the same thing.

Theorem 3.2: Component i is more critical than component j if and only if $j \cup S \in P_j \Rightarrow i \cup S \in P_i$ for any S not containing i and j.

Proof: Suppose $j \cup S \in P_j$. Then $\overline{j \cup S} = i \cup S' \notin C_i$, where $S' = \overline{i \cup j \cup S}$. Note that i being critical than j implies $j \cup S' \notin C_i$. Hence $i \cup S \in P_i$. The converse can be similarly proved. □

Theorem 3.3: Both c-absolutely, and p-absolutely more important imply more critical.

Proof: i being c-absolutely more important than j implies the nonexistence of a set S, containing neither i nor j, such that $j \cup S$ is a cutest. Hence the requirement of being more critical is trivially met.

The part involving p-absoluteness can be similarly proved. □

Reference [8] considered "more important" in its strict sense and proved that "c-absolutely more important" implies more critical.

Component i is more **H-important** than component j if $|C_i(d)| \geq |C_j(d)|$ for all d. Again, using cutsets or pathsets in the definition is immaterial.

Theorem 3.4: Component i is more H-important than component j if and only if $|P_i(d)| \geq |P_j(d)|$ for all d.

Proof: By Corollary 2.5,

$$\begin{aligned} |C_i(d)| - |C_j(d)| &= \binom{n-1}{d-1} - \left|P_{(i)}(n-d)\right| \\ &\quad - \left[\binom{n-1}{d-1} - \left|P_{(j)}(n-d)\right|\right] \\ &= \left|P_{(j)}(n-d)\right| - \left|P_{(i)}(n-d)\right| \\ &= |P_i(n-d)| - |P_j(n-d)|, \end{aligned}$$

where the last equality is obtained by subtracting these $(n-d)$-pathsets containing neither j nor i, and adding these containing both. □

Theorem 3.5: More critical implies more H-important.

Proof: Theorem 3.5 follows immediately from the fact that every C_j not containing i is coupled to a distinct C_i. □

The Birnbaum importance [1] was originally defined in the larger framework where component reliabilities, and component locations, are both considered. To be comparable to the other importance indices discussed here, the reliability part is stripped off by assuming each component has the constant reliability p. Then component i is **Birnbaum more important** than component j if

$$\begin{aligned} \frac{\partial R(p_1,\ldots,p_n)}{\partial p_i} &= R(1_i) - R(0_i) \\ &= \sum_d |P_i(d)|\, p^{n-d} q^d \\ &\quad - \sum_d \left|P_{(i)}(d)\right| p^{n-d} q^d \\ &= \sum_d \left(|P_i(d)| - \left|P_{(i)}(d)\right|\right) p^{n-d} q^d. \end{aligned}$$

This is defined as ***p*-Birnbaum importance**; and ***c*-Birnbaum importance** is defined as

$$\sum_d \left(|C_i(d)| - \left|C_{(i)}(d)\right|\right) p^{n-d} q^d. \qquad (1)$$

We strengthen a result of [7] to prove.

Theorem 3.6: H-more-important implies both types of Birnbaum more important.

Proof:

$$\sum_d \left(|C_i(d)| - \left|C_{(i)}(d)\right|\right) p^{n-d} q^d$$
$$= \sum_d \left(|C_i(d)| - [|C(d)| - |C_i(d)|]\right) p^{n-d} q^d$$
$$= \sum_d \left(2|C_i(d)| - |C(d)|\right) p^{n-d} q^d.$$

Hence i is c-Birnbaum more important than j if and only if

$$\sum_d \left(|C_i(d)| - |C_j(d)|\right) p^{n-d} q^d \geq 0, \quad (2)$$

which, on the other hand, is clearly implied by the condition

$$|C_i(d)| \geq |C_j(d)| \text{ for all } d.$$

Similarly, use the alternative condition of H-more-important, i.e.,

$$|P_i(d)| \geq |P_j(d)| \text{ for all } d$$

to prove p-Birnbaum importance. □

Because the Birnbaum importance is very hard to compare, [4] proposed the half-line importance, which is much easier to handle mathematically, and also practical in applications. Component i is half-line more important than component j if (2) holds for all $p \geq 1/2$, a condition easily met in practice. Similar to Birnbaum importance, we can split the half-line importance into the p-type, and the c-type.

The p-Birnbaum importance, and the p-halfline importance, reduce to the so-called **combinatorial importance** when $p = q = 1/2$, which can also be defined as

$$2^{-n}\left(|P_i| - \left|P_{(i)}\right|\right).$$

Theorem 3.7: Component i is combinatorial more important than component j if and only if $|C_i| \geq |C_j|$.

Proof: By Corollary 2.4

$$|P_i| - \left|P_{(i)}\right| = (2^{n-1} - \left|C_{(i)}\right|) - (2^{n-1} - |C_i|)$$
$$= |C_i| - \left|C_{(i)}\right|.$$

□

The implication relations among the importance indices discussed in this section is shown in Fig. 3.

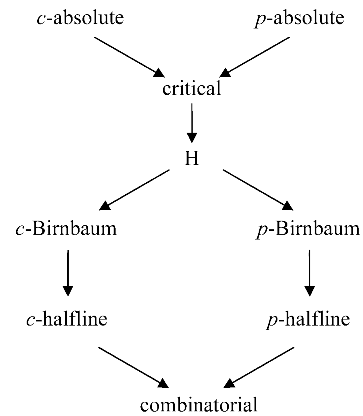


Fig. 3. A hierarchy of importance indices.

IV. Importance Indices Based on Minimal Cutsets or Pathsets

By substituting $(1-q)$ for p in (1), the c-Birnbaum importance can be written as a function of q,

$$B_i(q) = \sum_{d=1}^{n} b_{id} q^{d-1}.$$

Reference [3] called component i to be more **cut-important** than component j if the vector $(b_{i1}, b_{i2}, \ldots, b_{in})$ is lexicographically larger than the vector $(b_{j1}, b_{j2}, \ldots, b_{jn})$. [3] also proved that

$$b_{id} = \sum_{k=1}^{d} (-1)^{k-1} \left|\tilde{C}_i(d,k)\right|.$$

Meng [8] proved that strictly more critical implies strictly more cut-important. This section shows that his proof works under a weaker condition. Component i is said to be more $\tilde{H}$-important than component j if $\tilde{C}_i(d) \geq \tilde{C}_j(d)$ for all d. Clearly, more critical implies more $\tilde{H}$-important by the coupling argument.

Theorem 4.1: More $\tilde{H}$-important implies more cut-important.

Proof: Suppose that component i is more $\tilde{H}$-important than component j. Then

$$\left|\tilde{C}_i(d,1)\right| = \left|\tilde{C}_i(d)\right| \geq \left|\tilde{C}_j(d)\right|$$
$$= \left|\tilde{C}_j(d,1)\right| \text{ for all } 1 \leq d \leq n.$$

If the equality holds throughout, then

$$\tilde{C}_i = \tilde{C}_j$$

and i and j are equivalent. Otherwise, there exists an m, $2 \leq m \leq n$, such that

$$\left|\tilde{C}_i(d,1)\right| = \left|\tilde{C}_j(d,1)\right| \text{ for all } d < m,$$
$$\left|\tilde{C}_i(m,1)\right| > \left|\tilde{C}_j(m,1)\right|.$$

Then

$$\left|\tilde{C}_i(d,k)\right| = \left|\tilde{C}_j(d,k)\right| \text{ for all } d \leq m \text{ and all } k,$$

because each minimal cutset contributing to the k-sum is of size $< m$. It follows that

$$b_{id} = b_{jd} \text{ for } d < m \text{ and } b_{im} > b_{jm}.$$

Hence i is more cut-important than j. □

Similarly, path-importance is defined by using minimal pathsets instead of minimal cutsets. By Theorem 3.2, the notion of criticality can be expressed in terms of pathsets. Therefore, mimic the proof of Theorem 4.1 to obtain the following.

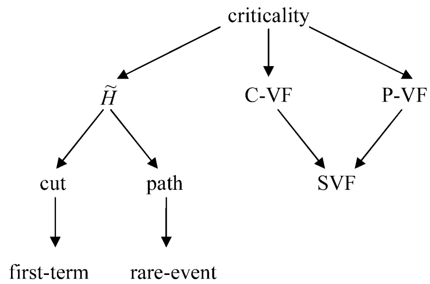


Fig. 4. Relations among importance indices based on minimal cutsets or pathsets.

Theorem 4.2: More critical implies more path-important.

A specification of the n component-states will be called an **inoperative sample point** if it causes a system inoperativeness. [11] and [6] proposed the **VF importance** of location i as the probability that a random inoperative sample point contains a minimum cutset in $\tilde{C}_i$. Again, to compare locations, only the case $p_i = p$ for all i is considered. In the special case $p = 1/2$, the VF-importance is called structural VF-importance, and abbreviated by SVF. [9] gave a quite complicated proof that "strictly more critical" implies "strictly more VF-important." Ignoring strictness, then the result easily follows from the coupling property of comparing criticality. [9] also defined a pathset version of VF-importance. Thus Theorem 4.3 follows.

Theorem 4.3: More critical implies more c-VF-important, and more p-VF-important.

Computing the vector $(b_{i1}, \ldots, b_{in})$, or its pathset counterpart, can be a tedious task. [10] introduced the notion of "first-term invariance" by comparing the number of minimum cutsets of two assignments. Thus the notion of **first-term** importance, which compares two components i and j only by comparing $\left|\tilde{C}_i\right|$ with $\left|\tilde{C}_j\right|$, is derived."

Recently, [5] introduced the notion of **rare-event** importance, which is the counterpart of first-term importance, by comparing $\left|\tilde{P}_i\right|$ with $\left|\tilde{P}_j\right|$. Rare-event importance is relevant when $p \to 0$. Although $p \to 0$ is not a practical assumption for most real-world problems, the rare-event importance serves as a useful tool to disprove a Birnbaum importance inequality because all we need is a counterexample under the rare-event importance. Furthermore, if an inequality is proved under the half-line importance, or for both the first-term importance and the combinatorial importance, then proving it further under the rare-event importance is a strong indication that the inequality might hold under the Birnbaum importance because p is now covered from both ends.

The relations discussed in this section are summarized by Fig. 4.

V. Applications to the Assignment Problem

Due to the coupling, if component i is more critical than component j, then clearly a more reliable unit should go to location i than location j. [3] also pointed out that for $p \to 1$, then i more cut-important than j dictates a more reliable unit goes to i rather than j. The first-term, and the rare-event importance, apply in the two extreme cases $p \to 1$, and $p \to 0$, respectively. The dependence of assignment on the importance indices from H-importance down to combinatorial importance is more intuitive than exact. They are mostly used in devising heuristics for optimal assignments. Namely, the more important component gets a more reliable unit, at least in the first try.

References

[1] Z. W. Birnbaum, "On the importance of different components in a multicomponent system," in *Multivariate Analysis II*, P. R. Krishnaiah, Ed, N. Y: Academic, 1969.

[2] P. J. Boland, F. Proschan, and Y. L. Tong, "Optimal arrangement of components via pairwise rearrangements," *Nav. Res. Logist. Quart.*, vol. 36, pp. 807–815, 1989.

[3] D. A. Butler, "A complete importance ranking for components of binary coherent systems with extensions to multi-state systems," *Nav. Res. Logist. Quart.*, vol. 4, pp. 565–578, 1979.

[4] H. W. Chang and F. K. Hwang, "Rare-event component importance for the consecutive-k system," *Nav. Res. Logist. Quart.*, vol. 49, pp. 159–166, 2002.

[5] H. W. Chang, R. J. Chen, and F. K. Hwang, "The structural birnbaum importance of consecutive-k systems," *J. Combinatorial Optimization*, vol. 6, pp. 183–197, 2002.

[6] J. B. Fussell, "How to hand-calculate system reliability and safety characteristics," *IEEE Trans. Reliab.*, vol. R-24, pp. 169–174, 1975.

[7] F. K. Hwang, "A new index of component importance," *Oper. Res. Lett.*, vol. 28, pp. 25–29, 2001.

[8] F. C. Meng, "Comparing criticality of nodes via minimal cut (path) sets for coherent systems," *Probabilities in Engineering and Information Sciences*, vol. 8, pp. 79–87, 1994.

[9] ——, "Comparing the importance of system components by some structural characteristics," *IEEE Trans. Reliab.*, vol. 45, pp. 59–65, 1996.

[10] M. Santha and Y. Zhang, "Consecutive-2 systems on trees," *Probabilities in Engineering and Information Sciences*, vol. 1, pp. 441–456, 1987.

[11] W. E. Vesely, "A time dependent methodology for fault tree evaluation," *Nuclear Engineering Designs*, vol. 13, pp. 337–360, 1970.

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