



A NEW HIGH-ORDER NON-UNIFORM TIMOSHENKO BEAM FINITE ELEMENT ON VARIABLE TWO-PARAMETER FOUNDATIONS FOR VIBRATION ANALYSIS

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A new finite element model of a Timoshenko beam is developed to analyze the small amplitude, free vibrations of non-uniform beams on variable two-parameter foundations. An important characteristic of the model is that the cross-sectional area, the second moment of area, the Winkler foundation modulus and the shear foundation modulus are all assumed to vary in polynomial forms, implying that the beam element can deal with commonly seen non-uniform beams having different cross-sections such as rectangular, circular, tubular and even complex thin-walled sections as well as the foundation of beams which vary in a general way. Thus this new beam element model enables users to handle vibration analysis of more general beam-like structures. In this paper, by using cubic polynomial expressions for the total deflection and the bending slope of the beam, the mass and stiffness matrices of the element are derived from energy expressions. The element model can accommodate various boundary conditions to represent a Timoshenko beam accurately. Excellent agreement with other investigators' results and a rapid rate of convergence with relatively few elements are demonstrated. This study also brings out the fact that the complicated form of this new beam model is necessary because of its advantage over linear or uniform approximations of the non-uniform foundation and/or geometrical properties of beams. The same accuracy being achieved with fewer elements is the main advantage. Finally, an optimum design problem is illustrated to emphasize the practical application of this element.

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1. INTRODUCTION

Beams resting on elastic foundations are very common in structural systems. Many authors have proposed mathematical models to simulate such structural systems. In these studies [1–4] the foundation is usually modelled on the basis of the well-known Winkler hypothesis which postulates that a foundation behaves like an infinite series of closely spaced, independent, linearly elastic, vertical springs. The limitation of these models is that they assume no interaction between the springs, so the models fail to reproduce the characteristics of a continuous medium. To overcome this problem, two-parameter models (or Pasternak models) were developed to couple the response within the foundation.

Mathematically all these models are equivalent, differing only in their definitions of the foundation parameters. Using the Bernoulli–Euler beam theory, Valsangkar and Pradhanang [5] investigated the influence of foundation continuity (or a partially elastic foundation) on the natural frequencies of beam-columns resting on constant two-parameter models. Eisenberger and Clastornik [6, 7] and Clastornik *et al.* [8] have studied vibrations and buckling of beams on variable Winkler and two-parameter foundations. Other authors [9–12] have solved similar problems in which the effects of shear deformation and rotatory inertia on the natural frequencies of the beam were included (in other words, based on the Timoshenko beam theory).

In many engineering applications, non-uniform beams with cross-sections varying in a continuous or non-continuous manner along their lengths are used in an effort to achieve an optimum distribution of strength and weight. Extensive work has been done on the vibration of a non-uniform beam. With the effects of shear deformation and rotatory inertia included, the finite element approach to this problem has been shown successfully in several published studies [13–15]. Chehil and Jategaonkar [16] used the Galerkin method to estimate the first few natural frequencies of a simple-supported beam with varying section properties. They have also evaluated the natural frequencies of a non-uniform beam with varying cross-section in a continuous or non-continuous manner along its length under other classic boundary conditions [17]. Recently, by using the homogeneous solutions of the governing equations for static deflections as the shape functions, Cleghorn and Tabarrok [18] developed a finite element model for free lateral vibration analysis of linearly tapered Timoshenko beams. However, the mass matrix they derived is only approximate, although the stiffness matrix is exact. Tang [19] derived a second-order finite element formulation of linearly tapered beam-column elements. In their work, different tapering types such as width changing only, height changing only, width and height changing at the same rate and a different rate for various section shapes were considered.

From the above survey, it is found that part of the work has been focused on the problem of a uniform beam resting on a constant or variably elastic foundation. As to the field of vibration of non-uniform beams, the situation where non-uniform beams are fully or partially embedded on elastic foundations is less well-explored. It is well known that the governing equation for uniform foundations has constant coefficients and analytical solutions of it can be obtained. But if the foundations vary along the beams, in most cases the governing equation cannot be solved exactly, so numerical techniques must be applied. Therefore, the purpose of the present paper is to develop a new finite element model to determine the natural frequencies of a non-uniform Timoshenko beam resting on a variable two-parameter foundation. The effects of non-linearly tapered cross-sectional area, variable two-parameter foundations, fully or partially, shear deformation, and rotatory inertia are all taken into account in this new beam element model. The taper type is considered as the cross-sectional area and second moment of area are expressed in polynomial forms. The variable two-parameter foundation also is postulated to be of polynomial form. This element model thus allows the inertia, cross-sectional area and the elastic foundation to vary in a general manner. Cubic polynomials originally suggested by Thomas [20] are employed here for the element model with the total deflection, ψ , total slope, ψ' , bending slope, ϕ , and the first derivative of bending slope, ϕ' as nodal co-ordinates. By using the Lagrange principle, the stiffness and mass matrices for small amplitude, free vibrations of the beam are derived from energy expressions and the governing matrix equation is then obtained after assembling element stiffness and mass matrices. This model is capable of accommodating various boundary conditions such as: (a) free end–zero bending moment and zero shear force; (b) fixed end–zero total deflection

and zero bending slope; (c) hinged end–zero total deflection and zero bending moment, thus representing a Timoshenko beam accurately. Numerical results based on this new beam element model are compared with those presented by various researchers to verify the accuracy of the model. This study also brings out the fact that the complicated form of this new beam model is necessary because of its advantage over linear or uniform approximations of the non-uniform foundation and/or geometrical properties of beams. The use of fewer elements to achieve the same accuracy is the main advantage, as the amount of numerical calculation required is thereby reduced. Finally, an optimum design problem is illustrated to emphasize the practical application of this element.

2. DERIVATION OF ELEMENT MATRICES

The new Timoshenko beam element model with cross-section varying in a continuous manner along its length l is shown in Figure 1. Cross-sectional area $A^e(x)$ and second moment of area $I^e(x)$ are described in the polynomial forms (see Appendix B for nomenclature)

$$A^e(x) = \sum_{i=0}^{n_2} B_i x^i, \quad I^e(x) = \sum_{i=0}^{n_1} A_i x^i, \quad (1, 2)$$

where x represents the co-ordinate along the beam. The variable two-parameter elastic foundation is represented as a general polynomial in x : i.e., the varying elastic foundation,

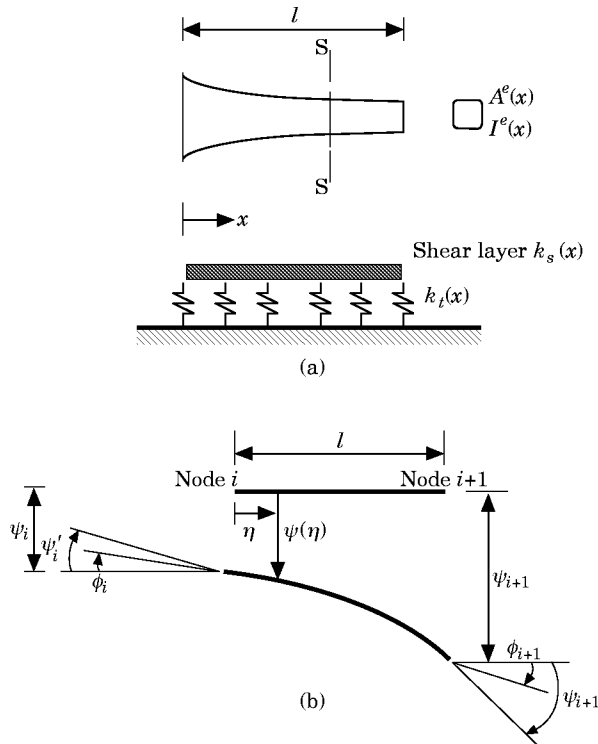


Figure 1. The beam model. (a) Beam element; (b) definition of the nodal co-ordinates.

characterized by two moduli, the Winkler foundation modulus $k_t^e(x)$ and the shear foundation modulus $k_s^e(x)$, are also described in polynomial forms:

$$k_t^e(x) = \sum_{i=0}^{n_3} C_i x^i, \quad k_s^e(x) = \sum_{i=0}^{n_4} D_i x^i, \quad (3, 4)$$

Basis assumptions for the present Timoshenko beam element are as follows: (i) the beam material is isotropic, homogeneous and linearly elastic; (ii) the vibration amplitude of the beam is sufficiently small; (iii) the cross-section initially normal to the neutral axis of the beam remains plane, but no longer normal to that axis after bending; (iv) the damping is negligible.

The potential energy U^e of the beam element of length l including the effects of both shear deformation and elastic foundation is given by

$$\begin{aligned} U^e = & \frac{1}{2} E \int_0^l I^e(x) \left(\frac{\partial \phi}{\partial x} \right)^2 dx + \frac{1}{2} \kappa G \int_0^l A^e(x) \left(\frac{\partial y}{\partial x} - \phi \right)^2 dx + \frac{1}{2} \int_0^l k_t^e(x) y^2 dx \\ & + \frac{1}{2} \int_0^l k_s^e(x) \left(\frac{\partial y}{\partial x} \right)^2 dx. \end{aligned} \quad (5)$$

By substituting $\eta = x/l$ and $\psi = y/l$, U^e can be non-dimensionalized as

$$\begin{aligned} U^e = & \frac{1}{2} \frac{E}{l} \int_0^l I^e(\eta) \left(\frac{\partial \phi}{\partial \eta} \right)^2 d\eta + \frac{1}{2} \kappa G l \int_0^l A^e(\eta) \left(\frac{\partial \psi}{\partial \eta} - \phi \right)^2 d\eta + \frac{1}{2} l^3 \int_0^l k_t^e(\eta) \psi^2 d\eta \\ & + \frac{1}{2} l \int_0^l k_s^e(\eta) \left(\frac{\partial \psi}{\partial \eta} \right)^2 d\eta, \end{aligned} \quad (6)$$

where ψ , ψ' , ϕ and ϕ' representing the total deflection, total slope, bending slope, and the first derivative of bending slope, respectively, are the four-degrees-of-freedom at each node, and $A^e(\eta)$, $I^e(\eta)$, $k_t^e(\eta)$ as well as $k_s^e(\eta)$ take the following forms:

$$A^e(\eta) = \sum_{i=0}^{n_1} b_i \eta^i, \quad I^e(\eta) = \sum_{i=0}^{n_2} a_i \eta^i, \quad k_t^e(\eta) = \sum_{i=0}^{n_3} k_{ti} \eta^i, \quad k_s^e(\eta) = \sum_{i=0}^{n_4} k_{si} \eta^i, \quad (7-10)$$

For a uniform beam, it is worth noting that $n_1 = n_2 = 0$, thus leading to $A^e = b_0$ and $I^e = a_0$.

By using a cubic polynomial distribution for ψ and ϕ , i.e.

$$\psi = \sum_{i=0}^3 \alpha_i \eta^i, \quad \phi = \sum_{i=0}^3 \beta_i \eta^i, \quad (11, 12)$$

and expressing the coefficients α_i and β_i in terms of nodal values ψ , ψ' , ϕ and ϕ' at the nodes i and $i+1$ (the prime denotes differentiation with respect to η), the following equations can be obtained:

$$\psi = [N_1 \ N_2 \ N_3 \ N_4] \begin{bmatrix} \psi_i \\ \psi'_i \\ \psi_{i+1} \\ \psi'_{i+1} \end{bmatrix}, \quad \phi = [N_1 \ N_2 \ N_3 \ N_4] \begin{bmatrix} \phi_i \\ \phi'_i \\ \phi_{i+1} \\ \phi'_{i+1} \end{bmatrix}. \quad (13, 14)$$

N_1 – N_4 are the shape functions and have the following forms

$$N_1 = 1 - 3\eta^2 + 2\eta^3, \quad N_2 = \eta - 2\eta^2 + \eta^3, \quad N_3 = 3\eta^2 - 2\eta^3, \quad N_4 = -\eta^2 + \eta^3. \quad (15-18)$$

Upon substituting equations (13)–(18) into equation (6), the strain energy becomes

$$U^e = (E/2I) \{\xi^e\}^T [K^e] \{\xi^e\}, \quad (19)$$

where $\{\xi^e\}^T = \{\psi_i \phi_i \psi'_i \phi'_i \psi_{i+1} \phi_{i+1} \psi'_{i+1} \phi'_{i+1}\}$. The terms of $[K^e]$ are given in Appendix A.

The kinetic energy T^e of this beam element can be written as

$$T^e = \frac{1}{2} \rho \int_0^l A^e(x) \left(\frac{\partial y}{\partial t} \right)^2 dx + \frac{1}{2} \rho \int_0^l I^e(x) \left(\frac{\partial \phi}{\partial t} \right)^2 dx. \quad (20)$$

After non-dimensionalization, T^e becomes

$$T^e = \frac{1}{2} \rho l^3 \int_0^1 A^e(\eta) \left(\frac{\partial \psi}{\partial t} \right)^2 d\eta + \frac{1}{2} \rho l \int_0^1 I^e(\eta) \left(\frac{\partial \phi}{\partial t} \right)^2 d\eta. \quad (21)$$

With the help of equations (13)–(18), T^e can be written in the form

$$T^e = \frac{1}{2} \rho l^3 \{\dot{\xi}^e\}^T [M^e] \{\dot{\xi}^e\}, \quad (22)$$

in which the dot indicates differentiation with respect to time. The entries of $[M^e]$ are also listed in Appendix A. Furthermore, when the beam is not embedded on elastic foundations, $k_i^e = k_s^e = 0$. In this case, the element matrices $[K^e]$ and $[M^e]$ derived here will reduce to the forms presented by Thomas [20].

Applying Lagrange's principle to the sum of individual element energies over the whole beam gives the dynamic equilibrium equation for small amplitude free vibration of a non-uniform Timoshenko beam on a two-parameter elastic foundation as

$$[K] \{\xi\} + (\rho l^4/E) [M] \{\ddot{\xi}\} = \{0\}, \quad (23)$$

where

$$[K] = \sum_e [K^e], \quad [M] = \sum_e [M^e], \quad \{\xi\} = \sum_e \{\xi^e\}, \quad (24)$$

are the global stiffness matrix, the global consistent mass matrix, and the global displacement vector assembled by adding the contributions from all the elements, respectively. For the operations in equations (24), the matrix or vector on the right must be expanded with zero to make it the same size as the matrix or vector on the left. When $\{\xi\}$ is harmonic in time with circular frequency ω , equation (23) takes the standard eigenvalue problem form

$$([K] - \lambda^4 [M]) \{\bar{\xi}\} = \{0\}, \quad (25)$$

where $\{\xi\} = \{\bar{\xi}\} \exp(i\omega t)$ and $\lambda^4 = \omega^2 \rho l^4/E$. Thus the non-trivial solution of equation (25) gives the natural frequencies and the corresponding mode shapes.

TABLE 1
Basic parameters used in the first two numerical examples

ν	1/3
G/E	3/8
κ	2/3
L	25
b_0	1
$b_i, i=1, \dots, n_2$	0
a_0	1
$a_i, i=1, \dots, n_1$	0
$k_{ii}, i=1, \dots, n_3$	0
K_i	$k_{i0}L^4/Ea_0$
$k_{si}, i=1, \dots, n_4$	0
K_s	$k_{s0}L^2/Ea_0$
C^4	$\omega^2 \rho b_0 L^4 / Ea_0$

3. ILLUSTRATIVE EXAMPLES AND DISCUSSION

In this section, the first three numerical examples are presented to demonstrate the accuracy and convergence rate of the new element model by comparing results with those of other authors. Data used in this example, which were presented by Yokoyama [11], are summarized in Table 1. The structure studied in the first example is a uniform hinged-hinged beam fully supported on a constant two-parameter foundation. It should be noted that, for such a structure, equations (7)–(10) reduce to $A^e = b_0$, $I^e = a_0$, $k_i^e = k_{i0}$, and $k_s^e = k_{s0}$, representing the fact that the cross-sectional area, the second moment of area of the beam and the foundation are all constants. With foundation parameters $K_i = k_{i0}L^4/Ea_0$ and $K_s = k_{s0}L^2/Ea_0$ varying from 1 to 10^4 and 0 to $2.5\pi^2$, respectively, Table 2 compares the first three frequency parameters calculated here with exact solutions computed from the frequency equations reported in reference [9] and results presented by Yokoyama [11]. The exact solutions reported in reference [9] were obtained from the coupling differential equations for transverse vibrations of uniform Timoshenko beams on

TABLE 2
Comparison of the frequency parameter C of the first three modes of a uniform hinged-hinged Timoshenko beam fully supported on a two-parameter foundation

		Mode	Exact [9]	Reference [11] 64 d.o.f.s	Present no. of d.o.f.s		
					8	12	20
$K_i = 1$	$K_s = 0$	1	3.092	3.09	3.094	3.092	3.092
		2	5.881	5.88	5.970	5.888	5.882
		3	8.301	8.31	8.946	8.373	8.305
	$K_s = 2.5\pi^2$	1	4.267	4.27	4.268	4.267	4.267
		2	6.795	6.80	6.862	6.800	6.796
		3	9.085	9.09	9.632	9.148	9.088
$K_i = 10^4$	$K_s = 0$	1	9.984	9.98	9.985	9.985	9.984
		2	10.187	10.19	10.204	10.188	10.187
		3	10.903	10.90	11.220	10.935	10.905
	$K_s = 2.5\pi^2$	1	10.044	10.04	10.044	10.044	10.044
		2	10.400	10.40	10.419	10.401	10.400
		3	11.278	11.28	11.588	11.311	11.280

TABLE 3

Comparison of the fundamental frequency parameter C of a uniform Timoshenko beam fully supported on a two-parameter foundation

	L^2	Exact [9]	Reference [12]	Present no. of d.o.f.s	
				8	16
Hinged-hinged beam					
$K_t = K_s = 0$	50	2.735	2.740	2.735	2.735
	4000	3.134	3.135	3.141	3.134
$K_t = K_s = 25$	50	4.170	4.194	4.170	4.170
	4000	4.378	4.379	4.381	4.379
Clamped-clamped beam					
$K_t = K_s = 0$	50	3.305	3.318	3.306	3.305
	4000	4.682	4.691	4.712	4.684
$K_t = K_s = 25$	50	4.439	4.443	4.440	4.439
	4000	5.324	5.324	5.367	5.326

constant elastic foundations. Yokoyama presented approximate solutions obtained from a beam model developed by himself, in which beams and their foundations (if any) are both restricted to be of uniform forms. As shown in Table 2, by comparing with the exact solutions, to attain the same level of numerical error, only 20 d.o.f.s (degrees-of-freedom), i.e., five elements, are sufficient in the present model but 64 d.o.f.s are used in Yokoyama's model. It should also be noted that the present results with 12 d.o.f.s approach those with 20 d.o.f.s very closely. The above numerical results reveal that the rate of convergence of the present model is more rapid than that of Yokoyama's model without losing accuracy as the number of degrees-of-freedom increases.

Under the conditions used by Filipich [12], the second example again verifies the beam element model developed here. As with the similar structure studied in the first example, data are the same as those listed in Table 1, except $\kappa = 1$ and $G/E = 9/28$. This example considers two types of boundary conditions, hinged-hinged and clamped-clamped, under several different sets of K_t , K_s , and L . Table 3 shows the comparison of results calculated for the fundamental frequency parameters with the corresponding exact solutions [9] and approximate solutions obtained by Filipich [12], who used a variant of Rayleigh's method to determine the natural frequencies. For both hinged-hinged and clamped-clamped boundary conditions with various K_t , K_s , and L , the accuracy of the present beam model is again demonstrated because the results obtained with 16 d.o.f.s in the present model approach the exact solutions and those presented by Filipich very closely. From Table 3, one can observe that the present results based on 8 d.o.f.s and 16 d.o.f.s are almost the same; thus the convergence rate of this new beam model is further demonstrated.

The third example is a case of a hinged-hinged rectangular cross-section beam of linearly varying width and depth, giving rise to non-linear variations in both the cross-sectional area and second moment of area. Basic data and comparison results are shown in Table 4. From Table 4, it is apparent that the discrepancy between the numerical results presented by Jategaonkar [17] and the results based on the proposed model is relatively small. Thus the accuracy of the present model is demonstrated for the case of non-uniform beams.

Usually, while analyzing the dynamic behaviour of non-uniform beams on foundations varying in a general form, cross-sections and foundations are approximated to change linearly or to be distributed uniformly in order to simplify the analysis. If such

approximations work well in most cases, the need for the derivation of this new beam element model is doubtful due to its complex and lengthy form, which users may feel is difficult to use. In this example therefore, the necessity of using the exact forms of variation on foundations and cross-sections instead of using the linear approximation is presented. Consider a non-uniform free-free beam resting on a variable two-parameter foundation. The original forms of $k_t(x)$, $k_s(x)$ are assumed in this example to be

$$k_t(x) = (c_1 - c_2)(1 - x/L)^3 + c_2, \quad k_s(x) = (c_3 - c_4)(1 - x/L)^3 + c_4, \quad (26, 27)$$

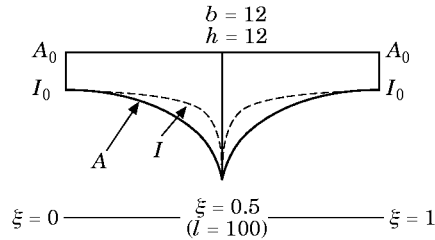
and the cross-section is assumed to be rectangular with width $b(x)$ and depth $h(x)$ varying as

$$b(x) = [(x/L) + c_5]^3, \quad h(x) = [(x/L) + c_6]^3. \quad (28, 29)$$

When using the linear approximation, values of $k_t(x)$, $k_s(x)$, $b(x)$, and $h(x)$ on nodal points are obtained from equations (26)–(29) and set to vary linearly within each element. Based on the values of parameters given in Table 5, Figure 2 shows the first three calculated natural frequencies obtained from the original forms and from the linear approximation of the cross-section and foundation. The results obtained based on the original forms of $k_t(x)$, $k_s(x)$, $b(x)$, and $h(x)$, have nearly converged with four elements, but those with the linear approximation apparently converge very slowly. This shows the superior rate of

TABLE 4
Comparison of the natural frequency ω_n (rad/s) of a Timoshenko beam with both cross-sectional area and second moment of area non-linearly varying

Mode	Reference [17]	Present no. of d.o.f.s	
		24	32
1	6.56	6.259	6.259
2	22.68	21.979	21.975
3	47.09	46.463	46.435
4	73.95	74.222	74.112
5	103.14	105.219	104.811



$$I = \begin{cases} I_0 (1 + \xi)^4, & 0 \leq \xi \leq \frac{1}{2} \\ I_0 (2 + \xi)^4, & \frac{1}{2} \leq \xi \leq 1 \end{cases}, \quad A = \begin{cases} A_0 (1 + \xi)^2, & 0 \leq \xi \leq \frac{1}{2} \\ A_0 (2 + \xi)^2, & \frac{1}{2} \leq \xi \leq 1 \end{cases}$$

$$I_0 = 1000, I_{\xi=0.5} = 5062.5, A_0 = 120, A_{\xi=0.5} = 270, E = 3 \times 10^6, \nu = 0.3, \kappa = 0.85, \rho = 1$$

TABLE 5
Values of parameters used in Figure 2

ν	0.3
E	3×10^6
ρ	1
κ	0.85
L	9
c_1	10^8
c_2	10^1
c_3	10^8
c_4	10^1
c_5	0.9
c_6	0.9

convergence of the natural frequencies based on the exact forms given in equations (26)–(29). Therefore, one can draw the conclusion that using foundation and cross-section forms that exactly match the original ones has an advantage over the linear approximation in that fewer elements are needed to achieve the same accuracy, thus reducing the amount of numerical calculation required.

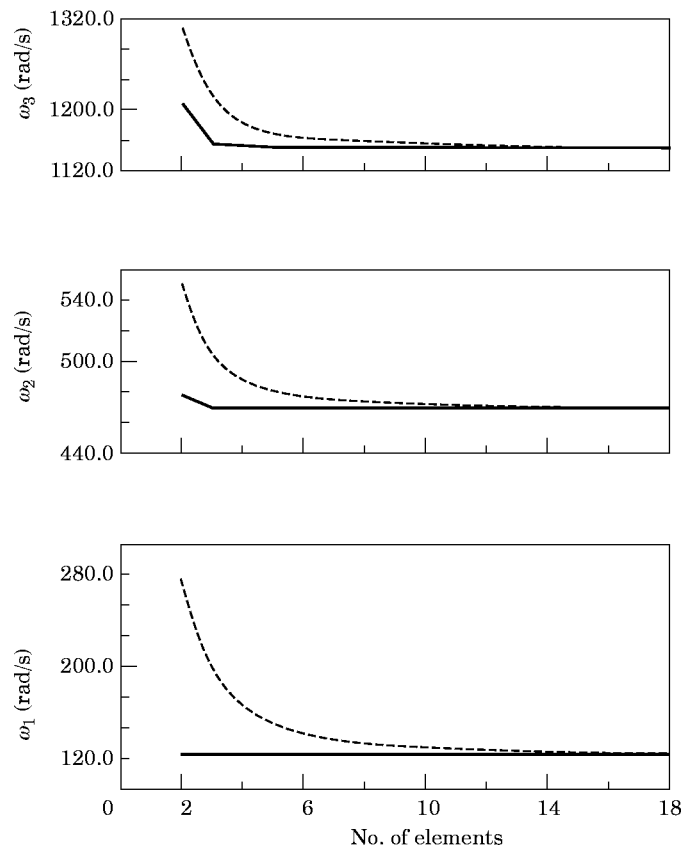


Figure 2. The first three circular frequencies as functions of the number of elements for a free–free non-uniform Timoshenko beam resting on a variable two-parameter foundation. —, Original forms of equations (26)–(29); ---, linear approximation to the original forms.

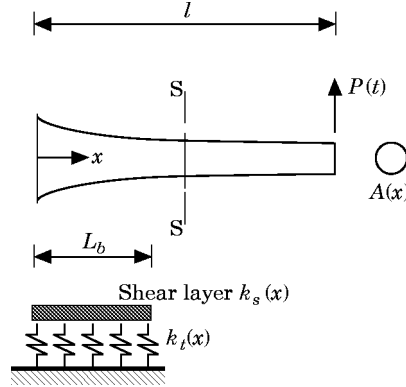


Figure 3. A beam partially embedded in a two-parameter foundation and subjected to external stationary loading $P(t)$.

The final example is of an optimum design problem, to illustrate the practical application of this new beam element model. As shown in Figure 3, a beam with circular cross-section is partially embedded on a variable two-parameter foundation and subjected to a stationary load $P(t)$. Data employed in the example are $E=210$ GPa, $\nu=0.3$, $\kappa=0.85$, $\rho=7840$ kg/m³, $L=0.5$ m, $L_b=0.2$ m and $b_0=0.01\pi$ m². The moduli of the foundation are

$$k_t(x) = (10^8 - 10)(1 - x/L_b)^3 + 10 \quad (\text{N/m}^2), \quad k_s(x) = 10^6(x/L_b) \quad (\text{N}). \quad (30, 31)$$

The target of the design is to determine the profile of the beam needed to minimize the weight. Two frequency constraints are specified. The first constraint requires that the first natural frequency be greater than 8 Hz and less than 14 Hz. The second constraint demands the second natural frequency to be greater than 60 Hz and less than 65 Hz. These frequency constraints ensure that the resonance response will not appear because $P(t)$ contains less energies in these frequency ranges. Assume the cross-sectional area varies as

$$A(x) = b_0 + b_1(x/L) + b_2(x/L)^2 + b_3(x/L)^3 \quad (\text{m}^2). \quad (32)$$

Therefore, the second moment of area $I(x)$ is

$$I(x) = [b_0 + b_1(x/L) + b_2(x/L)^2 + b_3(x/L)^3]^2 / 4\pi \quad (\text{m}^4). \quad (33)$$

The optimum design problem can be stated as follows. Minimize an objective function

$$f(x) = \int_0^L \rho A(x) dx \quad (34)$$

subject to

$$8 \text{ Hz} \leq f_1 \leq 14 \text{ Hz}, \quad 60 \text{ Hz} \leq f_2 \leq 65 \text{ Hz}. \quad (35, 36)$$

In this example, three sets of design variables \mathbf{x} are chosen:

Case 1: $\mathbf{x} = \{b_1\}$, i.e., b_1 is the design variable, b_2 and b_3 are assigned to be zero;

Case 2: $\mathbf{x} = \{b_1, b_2\}$, b_1 and b_2 are the design variables, b_3 is set to be zero;

Case 3: $\mathbf{x} = \{b_1, b_2, b_3\}$, b_1 , b_2 and b_3 are all design variables.

In case 1 the cross-section of the beam varies in a linear way. For case 2 one assumes that the cross-section varies as a second order polynomial along the beam. Similarly, the cross-section varies as a third order polynomial in case 3. Five elements are used in the

TABLE 6
Design bounds and initial values of design variables

Case no.	Design variables	Initial values	Lower bound	Upper bound
1	b_1	0	-0.01π	1
2	b_1	0	-0.01π	1
	b_2	0	-0.01π	1
3	b_1	0	-0.01π	1
	b_2	0	-0.01π	1
	b_3	0	-0.01π	1

associated finite element model with two of these elements resting on the foundation. The optimum problem is solved with the help of the (sequential quadratic programming) SQP method [21]. The original design has a uniform beam so that $A(x) = 0.01 \pi \text{ m}^2$, $f_1 = 12.3 \text{ Hz}$, $f_2 = 57.5 \text{ Hz}$, and the mass = 123.2 kg. The designer wishes to adjust the natural frequencies by changing the beam mass so that resonance will not appear. Table 6 lists the design bounds and initial values of the design variables adopted in the optimum numerical algorithm for each design case. The final results of optimum design are shown in Table 7, from which it can be seen that the first two natural frequencies have been adjusted adequately by satisfying the constraints. In each case, f_1 stays finally at its upper bound, which means that for this formulation of the optimum problem, the minimum beam mass is obtained when either f_1 or f_2 reach their upper bound first. By comparing the optimum results obtained in the three design cases, case 3 in which the cross-section is assumed to vary as a third-degree polynomial gives a beam mass of 90 kg, a maximum reduction of mass in the three cases nearly equal to 33 kg. From the standpoint of the optimum design concept, the better design obtained in case 3 implies that one can search for the optimum point more extensively due to the design space being enlarged.

4. CONCLUSIONS

A new Timoshenko beam finite element has been developed for the analysis of small amplitude, free vibration of non-uniform beams on variable two-parameter foundation. The cross-sectional area, the second moment of area of the beam as well as the foundations are all assumed to be polynomial forms, so that the beam model is the only one presented so far which is suitable for general types of beams and foundations. Other advantages of this beam element include (i) the convergence rate of results obtained from this model is

TABLE 7
Optimum results of the design problem in the final example

Case no.	Design variables			$f_1(\text{Hz})$	$f_2(\text{Hz})$	Mass (kg)
	b_1	b_2	b_3			
1	-9.426×10^{-3}	—	—	14.0	60.23	104.7
2	-3.140×10^{-2}	2.763×10^{-2}	—	14.0	62.43	97.7
3	-3.140×10^{-2}	-3.140×10^{-2}	7.084×10^{-2}	14.0	63.95	90.0
				12.3*	57.50*	123.2*

* Denotes the original design (based on a uniform beam).

rapid; (ii) use of a relatively small number of elements can obtain accurate results; (iii) both geometric and free boundary conditions can be applied correctly, according to the comparison results obtained in the first four examples. Therefore a Timoshenko beam can be represented accurately. The final example of an optimum design problem illustrates the practical application of this new beam element model. By using the new beam element model to build up a finite element model as an analysis tool, a precise profile of the beam structure is then determined while the mass of the beam is minimized and frequency constraints are satisfied.

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APPENDIX B: NOMENCLATURE

- A^e cross-sectional area of the element e
 I^e second moment of area of the element e
 E Young's modulus of beam material
 G shear modulus of beam material
 κ shear coefficient
 ρ mass density of the material
 L entire length of beam
 L_b length of the part of the beam resting on a foundation
 l element length
 T^e kinetic energy of the beam element e
 U^e strain energy of the beam element e
 x co-ordinate along the axis of the beam
 y deflection of the centroid of the beam
 ϕ bending slope
 η x/l , non-dimensional co-ordinate
 ψ y/l , non-dimensional deflection
 ω angular frequency of beam vibration
 k_i^e Winkler foundation modulus of the beam element e
 k_s^e shear foundation modulus of the beam element e
 A_i coefficients representing the variations in $I^e(x)$
 B_i coefficients representing the variations in $A^e(x)$
 C_i coefficients representing the variations in $k_i^e(x)$
 D_i coefficients representing the variations in $k_s^e(x)$
 a_i coefficients representing the variations in $I^e(\eta)$
 b_i coefficients representing the variations in $A^e(\eta)$
 k_{ii} coefficients representing the variations in $k_i^e(\eta)$
 k_{si} coefficients representing the variations in $k_s^e(\eta)$
 n_1 order of the polynomial form which describes I^e
 n_2 order of the polynomial form which describes A^e
 n_3 order of the polynomial form which describes k_i^e
 n_4 order of the polynomial form which describes k_s^e

APPENDIX A

The entries of $[K^e]$ are given as follows:

$$k_{11} = \sum_{i=0}^{n_2} \frac{72b_i S}{C_1} + \frac{72l^4}{E} \sum_{i=0}^{n_3} \frac{k_{ii}(13+3i)}{C_6} + \frac{72l^2}{E} \sum_{i=0}^{n_4} \frac{k_{si}}{C_1}, \quad k_{12} = 36 \sum_{i=0}^{n_2} \frac{b_i(10+3i)S}{C_2},$$

$$k_{13} = 12 \sum_{i=0}^{n_2} \frac{b_i(1+2i)S}{C_3} + \frac{24l^4}{E} \sum_{i=0}^{n_3} \frac{k_{ii}(l+3i)}{C_8} + \frac{12l^2}{E} \sum_{i=0}^{n_4} \frac{k_{si}}{C_3}, \quad k_{14} = 36 \sum_{i=0}^{n_2} \frac{b_i S}{C_4},$$

$$k_{15} = -72 \sum_{i=0}^{n_2} \frac{b_i S}{C_1} + \frac{6l^4}{E} \sum_{i=0}^{n_3} \frac{k_{ii}(54+17i+i^2)}{C_9} - \frac{72l^2}{E} \sum_{i=0}^{n_4} \frac{k_{si}}{C_1}, \quad k_{16} = 6 \sum_{i=0}^{n_2} \frac{b_i(10+i)S}{C_5},$$

$$\begin{aligned}
k_{17} &= 6 \sum_{i=0}^{n_2} \frac{b_i(1-i)S}{C_1} - \frac{6l^4}{E} \sum_{i=0}^{n_3} \frac{k_{ii}(13+3i)}{C_9} + \frac{6l^2}{E} \sum_{i=0}^{n_4} \frac{k_{si}(1-i)}{C_1}, & k_{18} &= -12 \sum_{i=0}^{n_2} \frac{b_i S}{C_5}, \\
k_{22} &= 72 \sum_{i=0}^{n_2} \frac{b_i(13+3i)S}{C_6} + 72 \sum_{i=0}^{n_1} \frac{a_i}{C_1}, & k_{23} &= 12 \sum_{i=0}^{n_2} \frac{b_i(-2+3i)S}{C_7}, \\
k_{24} &= 24 \sum_{i=0}^{n_2} \frac{b_i(11+3i)S}{C_8} + 12 \sum_{i=0}^{n_1} \frac{a_i(1+2i)}{C_3}, & k_{25} &= -36 \sum_{i=0}^{n_2} \frac{b_i(10+3i)S}{C_2}, \\
k_{26} &= 6 \sum_{i=0}^{n_2} \frac{b_i(54+17i+i^2)S}{C_9} - 72 \sum_{i=0}^{n_1} \frac{a_i}{C_1}, & k_{27} &= 6 \sum_{i=0}^{n_2} \frac{b_i(12-i^2)S}{C_2}, \\
k_{28} &= -6 \sum_{i=0}^{n_2} \frac{b_i(13+3i)S}{C_9} + 6 \sum_{i=0}^{n_1} \frac{a_i(1-i)}{C_1}, \\
k_{33} &= 8 \sum_{i=0}^{n_2} \frac{b_i(2+i^2)S}{C_{10}} + \frac{24l^4}{E} \sum_{i=0}^{n_3} \frac{k_{ii}}{C_9} + \frac{8l^2}{E} \sum_{i=0}^{n_4} \frac{k_{si}(2+i^2)}{C_{10}}, & k_{34} &= 12 \sum_{i=0}^{n_2} \frac{b_i i S}{C_2}, \\
k_{35} &= -12 \sum_{i=0}^{n_2} \frac{b_i(1+2i)S}{C_3} + \frac{2l^4}{E} \sum_{i=0}^{n_3} \frac{k_{ii}(13+i)}{C_{11}} - \frac{12l^2}{E} \sum_{i=0}^{n_4} \frac{k_{si}(1+2i)}{C_3}, \\
k_{36} &= 2 \sum_{i=0}^{n_2} \frac{b_i(18+2i+i^2)S}{C_4}, \\
k_{37} &= -2 \sum_{i=0}^{n_2} \frac{b_i(2+i^2)S}{C_3} - \frac{6l^4}{E} \sum_{i=0}^{n_3} \frac{k_{ii}}{C_{11}} - \frac{2l^2}{E} \sum_{i=0}^{n_4} \frac{k_{si}(2+i^2)}{C_3}, \\
k_{38} &= -2 \sum_{i=0}^{n_2} \frac{b_i(3+2i)S}{C_4}, & k_{44} &= 24 \sum_{i=0}^{n_2} \frac{b_i S}{C_9} + 8 \sum_{i=0}^{n_1} \frac{a_i(2+i^2)}{C_{10}}, \\
k_{45} &= -36 \sum_{i=0}^{n_2} \frac{b_i S}{C_4}, & k_{46} &= 2 \sum_{i=0}^{n_2} \frac{b_i(13+i)S}{C_{11}} - 12 \sum_{i=0}^{n_1} \frac{a_i(1+2i)}{C_3}, \\
k_{47} &= 2 \sum_{i=0}^{n_2} \frac{b_i(3-i)S}{C_4}, & k_{48} &= -6 \sum_{i=0}^{n_2} \frac{b_i S}{C_{11}} - 2 \sum_{i=0}^{n_1} \frac{a_i(2+i^2)}{C_3}, \\
k_{55} &= 72 \sum_{i=0}^{n_2} \frac{b_i S}{C_1} + \frac{l^4}{E} \sum_{i=0}^{n_3} \frac{k_{ii}(78+17i+i^2)}{C_{12}} + \frac{72l^2}{E} \sum_{i=0}^{n_4} \frac{k_{si}}{C_1}, & k_{56} &= -6 \sum_{i=0}^{n_2} \frac{b_i(10+i)S}{C_5}, \\
k_{57} &= 6 \sum_{i=0}^{n_2} \frac{b_i(-1+i)S}{C_1} - \frac{l^4}{E} \sum_{i=0}^{n_3} \frac{k_{ii}(11+i)}{C_{12}} + \frac{6l^2}{E} \sum_{i=0}^{n_4} \frac{k_{si}(-1+i)}{C_1},
\end{aligned}$$

$$k_{58} = 12 \sum_{i=0}^{n_2} \frac{b_i S}{C_5}, \quad k_{66} = \sum_{i=0}^{n_2} \frac{b_i(78+17i+i^2)S}{C_{12}} + 72 \sum_{i=0}^{n_1} \frac{a_i}{C_1},$$

$$k_{67} = \sum_{i=0}^{n_2} \left(\frac{6}{4+i} - \frac{13}{5+i} + \frac{6}{6+i} \right) b_i S, \quad k_{68} = - \sum_{i=0}^{n_2} \frac{b_i(11+i)S}{C_{12}} + 6 \sum_{i=0}^{n_1} \frac{a_i(-1+i)}{C_1},$$

$$k_{77} = \sum_{i=0}^{n_2} \frac{b_i(8+3i+i^2)S}{C_1} + \frac{2I^4}{E} \sum_{i=0}^{n_3} \frac{k_{ii}}{C_{12}} + \frac{I^2}{E} \sum_{i=0}^{n_4} \frac{k_{si}(8+3i+i^2)}{C_1},$$

$$k_{78} = \sum_{i=0}^{n_2} \frac{b_i i S}{C_5}, \quad k_{88} = 2 \sum_{i=0}^{n_2} \frac{b_i S}{C_{12}} + \sum_{i=0}^{n_1} \frac{a_i(8+3i+i^2)}{C_1}.$$

Each entry of $[M_e]$ is as follows:

$$m_{11} = 72 \sum_{i=0}^{n_2} \frac{b_i(13+3i)}{C_6}, \quad m_{12} = 0, \quad m_{13} = 24 \sum_{i=0}^{n_2} \frac{b_i(11+3i)}{C_8}, \quad m_{14} = 0,$$

$$m_{15} = 6 \sum_{i=0}^{n_2} \frac{b_i(54+17i+i^2)}{C_9}, \quad m_{16} = 0, \quad m_{17} = -6 \sum_{i=0}^{n_2} \frac{b_i(13+3i)}{C_9}, \quad m_{18} = 0,$$

$$m_{22} = 72 \sum_{i=0}^{n_1} \frac{a_i(13+3i)R}{C_6}, \quad m_{23} = 0, \quad m_{24} = 24 \sum_{i=0}^{n_1} \frac{a_i(11+3i)R}{C_8}, \quad m_{25} = 0,$$

$$m_{26} = 6 \sum_{i=0}^{n_1} \frac{a_i(54+17i+i^2)R}{C_9}, \quad m_{27} = 0, \quad m_{28} = -6 \sum_{i=0}^{n_1} \frac{a_i(13+3i)R}{C_9},$$

$$m_{33} = 24 \sum_{i=0}^{n_2} \frac{b_i}{C_9}, \quad m_{34} = 0, \quad m_{35} = 2 \sum_{i=0}^{n_2} \frac{b_i(13+i)}{C_{11}}, \quad m_{36} = 0,$$

$$m_{37} = -6 \sum_{i=0}^{n_2} \frac{b_i}{C_{11}}, \quad m_{38} = 0,$$

$$m_{44} = 24 \sum_{i=0}^{n_1} \frac{a_i R}{C_9}, \quad m_{45} = 0, \quad m_{46} = 2 \sum_{i=0}^{n_1} \frac{a_i(13+i)R}{C_{11}}, \quad m_{47} = 0,$$

$$m_{48} = -6 \sum_{i=0}^{n_1} \frac{a_i R}{C_{11}}, \quad m_{55} = \sum_{i=0}^{n_2} \frac{b_i(78+17i+i^2)}{C_{12}}, \quad m_{56} = 0,$$

$$m_{57} = - \sum_{i=0}^{n_2} \frac{b_i(11+i)}{C_{12}}, \quad m_{58} = 0,$$

$$m_{66} = \sum_{i=0}^{n_1} \frac{a_i(78 + 17i + i^2)R}{C_{12}}, \quad m_{67} = 0, \quad m_{68} = -\sum_{i=0}^{n_1} \frac{a_i(11 + i)R}{C_{12}},$$

$$m_{77} = 2 \sum_{i=0}^{n_2} \frac{b_i}{C_{12}}, \quad m_{78} = 0, \quad m_{88} = 2 \sum_{i=0}^{n_1} \frac{a_i R}{C_{12}},$$

Here

$$C_1 = 60 + 47i + 12i^2 + i^3, \quad C_2 = 720 + 1044i + 580i^2 + 155i^3 + 20i^4 + i^5,$$

$$C_3 = 120 + 154i + 71i^2 + 14i^3 + i^4, \quad C_4 = 360 + 342i + 119i^2 + 18i^3 + i^4,$$

$$C_5 = 120 + 74i + 15i^2 + i^3,$$

$$C_6 = 2520 + 5274i + 3929i^2 + 1420i^3 + 270i^4 + 26i^5 + i^6,$$

$$C_7 = 240 + 508i + 372i^2 + 121i^3 + 18i^4 + i^5,$$

$$C_8 = 5040 + 8028i + 5104i^2 + 1665i^3 + 295i^4 + 27i^5 + i^6,$$

$$C_9 = 2520 + 2754i + 1175i^2 + 245i^3 + 25i^4 + i^5,$$

$$C_{10} = 120 + 274i + 225i^2 + 85i^3 + 15i^4 + i^5,$$

$$C_{11} = 840 + 638i + 179i^2 + 22i^3 + i^4,$$

$$C_{12} = 210 + 107i + 18i^2 + i^3, \quad S = \kappa G l^2 / E, \quad R = 1/l^2.$$