A Stimulus–Response Model of Day-to-Day Network Dynamics

Hsun-Jung Cho and Ming-Chorng Hwang

***Abstract*—A general structure of stimulus–response formula is presented to specify the interacted network dynamics under the assumption of a daily learning and adaptive travel behavior. By taking the time derivative of system variable as a response term, the evolution is formulated as a dynamic system. Issues of existence, uniqueness, and stability for the proposed differential equations are briefly discussed. Approximation of a time-varying route-choice model is derived from the addressed path-flow dynamics. Threshold effects on path-flow dynamics are encapsulated into the proposed general structure by incorporating a discontinuous stimulus term. Then, the quasi user equilibrium is achieved when all users feel indifferent between the experienced and predicted travel time provided by intelligent transportation systems, i.e., the whole system dynamics stay within a bounded range. The derived quasi user equilibrium is reduced to Wardrop's user equilibrium as the threshold effects of path-flow dynamics vanish.**

***Index Terms*—Day-to-day network dynamics, dynamic traffic assignment, Lyapunov stability, stimulus–response formula, threshold effect.**

I. Introduction

THE ABILITY to forecast how the information predicted and provided by advanced traveler information systems (ATIS) influences time trajectories of network flows is essential in the era of intelligent transportation systems (ITS). The concentration of this paper is to develop an analytical approach that captures the effects of travel information on network flow evolutions, especially the day-to-day interactions among traffic variables, network performances, and travel information. Network flow evolution has been studied analytically in [1]–[4] or simulation-oriented in [5]–[10]. An analytical approach is able to derive theoretical properties and to foresee asymptotic behaviors of simplified system based on a well-established mathematical foundation. On the contrary, complex manners of travel decisions are possibly handled by a simulation method. However, it is not easy to achieve a fully satisfied mathematical base pertaining to issues of existence, uniqueness, and stability for simulation-based approaches.

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Some other researchers were concerned about formulations and solution algorithms of the dynamic traffic assignment (DTA) problem to compute the flow pattern of system optimum [11], dynamic user equilibrium [12], or dynamic user optimum [13], [14]. These studies did not simulate the evolutions of network flows, but only provide a unique flow solution that coincided with some optimal criteria.

Flow evolution models are able to capture the transition states of system and, consequently, reach equilibrium if the simulation time period is long enough. They are classified into two broad categories according to the time scale of system adjustments, the transitions of system variables within each single day (intraday or time-of-day dynamics) and between subsequent "days" or, more generally, observation periods of similar characteristics (interday or day-to-day dynamics). The experiments on route-choice behaviors of commuters directed by Chang and Mahmassani [5]–[7] had indicated that the learning and adaptive processes for this choice may take weeks, partly because of the dynamic feedbacks from the traffic system and can indeed be lengthened by complex switching that resulted from the provision of better information. Friesz *et al.* in [4] addressed Chang and Mahmassani's design theoretically by introducing a tatonnement process for modeling the fluctuations of disequilibria from one state to another without touching the existence and uniqueness of the proposed models. Cho and Hwang [18] developed a system of differential equations to characterize the time-varying interdependence between vehicular flows and predicted travel information with a partially similar concept presented in Carey [15], Smith [2], and Friesz *et al.* [4]. This work provided a congestion-sensitive evolution behavior of network dynamics with multiple user classes and an easy way to derive the Wardrop's user equilibrium [16] directly from the steady state of a dynamical system with entirely analytical materials about the existence, uniqueness, and stability of solutions. We refer readers to Peeta and Ziliaskopoulos [22], Mahmassani [23], Ben-Akiva *et al.* [24], and Boyce *et al.* [25] for a more-detailed and well-organized review.

Most of the studies discussed have sought to solve the presumed unique user equilibrium or evolution toward such a user equilibrium state without a positive indication of the validity of this presumption. Alternative behavioral rules founded on the bounded-rationality notion and the associated satisfying decision rules had been explored by Mahmassani and Chang [26] for the departure time choices of urban commuters in the context of computer-simulation experiments. Boundedly rational user equilibrium (BRUE) is attained when all users given a system satisfied with their current travel choices; thus, they feel no need to improve their outcome by changing decisions. How-

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ever, the existence of a BRUE is not guaranteed, depending on the specific decision rule followed by users and the characteristics of corresponding system. For example, in the departure-time choices pursued in Mahmassani and Chang [26], the indifference band of tolerable schedule delay had to exceed a certain threshold before a BRUE commuting pattern could exist. Further analytical exploration on system dynamics, particularly the convergence to and stability of equilibrium, remains vacuum in this study.

A general structure of stimulus–response formula (SRF) is proposed in this paper to specify the interacted day-to-day network dynamics under the assumption of a daily learning and adaptive behavioral process. In particular, by defining a discontinuous stimulus, path-flow dynamics encapsulating threshold effect are formulated successfully. The remainder of this paper is organized as follows. In Section II, we mention the assumptions, variable definitions, and basic framework of stimulus–response function. Developments of network dynamics in terms of SRF are described in Section III and analysis of the steady state is presented in Section IV. Issues of existence, uniqueness, and stability are briefly discussed in Section V, based on the results of Cho and Hwang [18]. Numerical examples are demonstrated in Section VI and the conclusion of this research is outlined in Section VII.

II. Assumptions, Notation, and Stimulus–Response Formula

A. Assumptions

Interday dynamics of peak-period commuter trips are examined in this paper. The learning and adjustable behavior process is assumed to guide daily travel decisions. This process is specified that travel information about the upcoming day's travel time for an origin–destination (O–D) pair is provided with users and compared with the actual travel time experienced by all corresponding path users. The deviation between provided and experienced travel time results in path-flow adjustments of the next day. Experienced travel time is estimated by a link travel-time function in terms of an average in peak period. Predicted travel time of an O–D pair is assumed similarly to adjust as the difference between travel demand and the sum of corresponding path flows is detected. Link cost function is assumed to be a smooth and strict monotone function of link flow. In addition, travel demand is presumably fixed in this study under the supposition of no structural changes from competing transportation facilities over the entire period of interest. The authors are concerned that the variations of path flow and path travel time are much more sensitive than that of O–D demand if travel information provided by ATIS is the only perturbation of transportation system. In order to prevent any links from oversaturation, it is also assumed that travel demands of all O–D pairs do not jointly violate any capacity constraints of links.

B. Notation

Some notation based on typical equilibrium models of commuter route choice are employed and augmented to meet our concerns. In particular, we take all vectors to be column vectors. Vectors and matrices are expressed in boldface. Notation common throughout this paper is summarized in Table I.

TABLE I
NOTATION

Symbol	Descriptions	Units
t	time index	
W	the full set of O-D pairs with $\overline{W}$ O-D pairs	
P	the full set of paths with $\overline{P}$ paths	
A	the full set of links with $\overline{A}$ links	
P_w	the set of paths connecting O-D pair w with $\overline{P}_w$ paths	
h_p^t	the non-negative peak-flow of path p at day t	vehicle/hour
f_a^t	the non-negative peak volume of link a at day t, $f_a^t = \sum_p \delta_{ap} h_p^t$ where $\delta_{ap} = 1$ if link a belongs to path p, otherwise $\delta_{ap} = 0$	vehicle/hour
$c_a(f_a^t)$	the unit average travel time on link a at day t, a smooth and strict monotone function of f_a^t	hour(min)
k_a	the capacity of link a;	vehicle/hour
c_p^t	the unit average travel time on path p at day t, without considering node travel time, $c_p^t = \sum_a \delta_{ap} c_a(f_a^t)$	hour(min)
c_w^t	the average travel time on O-D pair w at day t predicted and provided by ATIS	hour(min)
D_w	the travel demand of O-D pair w over whole time period of interest	vehicle/hour
α_p	the positive and path-specific parameter to denote the propensity of path flow dynamics	vehicle/hour^2
β_w	the positive and O-D-specific parameter to denote the sensitivity of predicted travel time dynamics	hour^2/vehicle
$\dot{x}$	the ordinary derivative of x with respect to t	
$\boldsymbol{x}'$	the transpose of vector (matrix) $\boldsymbol{x}$	

C. Stimulus–Response Formula

The general structure of SRF is composed of stimulus, response, and sensitivity. The basic equation of these models is of the form

$$\text{Response}(t+T) = \text{Sensitivity} \times \text{Stimulus}(t) \qquad (1)$$

which can be interpreted as that a response is in proportional to the magnitude of stimulus at time t and begins after a time lag T. A well-known application of SRF in traffic analysis is the car-following model [20]. The response term in (1) usually represents the acceleration (or deceleration) of the following vehicle, while the stimuli is often expressed by the velocity difference between the leading vehicle and the following one. The variants of car-following models concentrate on the representation of sensitivity.

In the learning and adaptive behavior process of daily commuter trips, the stimuli is specified by the travel discrepancy between experienced and predicted (expected) travel time and results in path-flow diversion the next day, which is denoted as the response. Sensitivity is the propensity of path-flow diversion due to travel discrepancy. For the adjustment of predicted travel time of an O–D pair, the stimuli is described by the difference between predicted travel demand and the actual sum of corresponding path flows and causes predicted travel time changed in the next time point interpreted as the response. Sensitivity is supposed to be the tendency of predicted travel time to adapt due to corresponding stimuli. Based on this analysis, a general model encapsulating stimulus–response nature to specify the interacted network dynamics in a day-to-day time scale can be shown as

$$\text{Path flow diversion at day } (t+T) \equiv F\ (\text{Sensitivity}_p, \text{the difference between experienced travel time and predicted (expected) O-D travel time at day}(t)) \tag{2}$$

and

$$\text{Predicted travel time adjustment at day } (t+T) \equiv G\ (\text{Sensitivity}_w, \text{the difference between predicted travel demand and the actual sum of corresponding path flows at day } (t)) \tag{3}$$

where $F(\cdot)$ and $G(\cdot)$ are functional and Sensitivity_p and Sensitivity_w denote sensitivities of path-flow dynamics and predicted travel time evolutions, respectively.

III. Modelling Network Dynamics

A. Path-Flow Dynamics

Path-flow dynamics are formulated by further developments of (2). Time lag T is obviously equal to 1 d. Let us take the time derivative of path flow as a reaction in the stimulus–response equation; then, we transform (2) into a differential equation. The stimulus in (2) can be expressed as

$$\text{Stimulus} \equiv c_p^t - c_w^t. \tag{4}$$

The congestion effect is considered in sensitivity to reflect that the propensity of path-flow evolution is dependent on the path-flow state. If we further suppose that sensitivity obeys a linear function of path flow, it can be illustrated as

$$\text{Sensitivity}_p \equiv \text{function}\left(h_p^t\right) \equiv \alpha_p h_p^t. \tag{5}$$

From (1), (4), and (5), path-flow dynamics are derived as

$$\dot{h}_p^t = -\left(\alpha_p h_p^t\right)\left(c_p^t - c_w^t\right) \qquad \forall p \in P_w \tag{6}$$

where

$$0 < \alpha_p < \inf_{c_p^t - c_w^t > 0}\left(\frac{1}{c_p^t - c_w^t}\right) \tag{7a}$$

$$\sum_{w\in W}\sum_{\forall p\in P_w}\left(-\delta_{ap}\alpha_p \text{PTTL}_{p,w}^t\right) \le k_a - \sum_{\forall p \in P}\delta_{ap}h_p^t \tag{7b}$$

and

$$\text{PTTL}_{p,w}^t \equiv h_p^t(c_p^t - c_w^t) \tag{7c}$$

The minus sign in (6) meets the general rule of least travel-time seeking. The right-hand side (RHS) of (6) can be interpreted as a pseudo travel-time saving (or loss) perceived by all path p users at day t [18]. Parameter α_p, consequently, means time-change rate of path flow when one unit amount of this pseudo value is generated. (7a) and (7b) ensure that (6) avoids the infeasibility of nonnegative path-flow and link-capacity constraints. It is obvious that the inequalities (7a) and (7b) are naturally satisfied if α_p is carefully calibrated from the empirical data. $\text{PTTL}_{p,w}^t$ in (7c) can be interpreted as the estimation of total perceived travel time loss (or savings) for all users selecting path p of the O–D pair w at day t.

B. Predicted Travel-Time Dynamics

Predicted travel-time dynamics are accomplished by similar treatments. We take the time derivative of predicted travel time as a response in (3). Stimulus in (3) is replaced in our notation as

$$\text{Stimulus} \equiv D_w - \sum_{p\in P_w} h_p^t. \tag{8}$$

The RHS of (8) is interpreted as excess demand in [4], [15], and [18] to note the relative scarcity (or surplus) of resources. Sensitivity of predicted travel-time dynamics is assumed to be an O–D pair specific constant and is expressed as

$$\text{Sensitivity}_w \equiv \beta_w. \tag{9}$$

From (1), (8), and (9), predicted travel-time dynamics are denoted as

$$\dot{c}_w^t = \beta_w\left(D_w - \sum_{p\in P_w} h_p^t\right) \qquad \forall w \in W \tag{10}$$

where

$$0 < \beta_w < \min\left\{\inf_{\forall(D_w - h_w^t < 0)}\left(\frac{\breve{c}_w - c_w^t}{D_w - h_w^t}\right) \inf_{\forall(D_w - h_w^t > 0)}\left(\frac{\widehat{c}_w - c_w^t}{D_w - h_w^t}\right)\right\} \tag{11}$$

if we further suppose that the travel time in the next time point for an O–D pair is transformed from the current status at a rate scaled to Stimulus_w^t. Inequality (11) guarantees that all predicted travel times stay in the feasible region without being greater than $\frown{c}_w$, the travel time at maximal flow, or less than $\smile{c}_w$, the travel time at free flow, for O-D pair w. We define

$$\breve{c}_w \equiv \min_{j \in P_w} \left(\sum_a \delta_{aj} c_a (f_a = 0) \right) \tag{11a}$$

and

$$\widehat{c}_w \equiv \max_{j \in P_w} \left(\sum_a \delta_{aj} c_a (f_a = k_a) \right). \tag{11b}$$

As mentioned in (8), predicted travel-time dynamics attempt to forward network traffic to a steady status by tuning itself in accordance with the relative scarcity (or surplus) of transportation facilities [4], [18]. Accordingly, parameter β_w is positive and means that the time-change rate of predicted travel time when one unit amount of excess demand is deviated.

Therefore, the final version of network dynamics is proposed as (6) and (10) with constraints (7a), (7b), and (11), which are collected as shown in the equations at the bottom of the page for all O–D pairs $w \in W, p \in P_w, a \in A$ at day t.

C. Approximation of the Time-Dependent Route-Choice Model

The path-flow dynamics discussed previously do not touch individual route-choice behavior, but directly formulate the collective outcomes path-flow adjustment based on network performance. The following context is the derivation of an approximated time-varying route-choice model based on the mentioned path-flow dynamics.

Route-choice behavior had been formulated as a stochastic model in [8]–[10] or as a deterministic model in [5]–[7] and [21]. The essential results of these models are to predict the peak-period (or some other time intervals of interest) path flows for the upcoming day. This measure can be expressed in our terms as

$$\begin{aligned} \text{Pr}_{p,w}^t &\approx \frac{\sum_{n=1}^{D_w} \text{Pr}_{n,p}^t}{D_w} \\ &= \frac{\sum_{n=1}^{D_w} \text{Pr}^t \left(U_n^p > U_n^k, \forall k \in P_w, k \neq p \right)}{D_w} \end{aligned} \tag{12}$$

for stochastic models, where

n	user index of O-D pair w;
Pr	probability;
$\text{Pr}_{p,w}^t$	probability of choosing path $p \in P_w$ in D_w at day t;
$\text{Pr}_{n,p}^t$	probability of user n choosing path $p \in P_w$ at day t;
U_n^p	utility of user n choosing path $p \in P_w$;

or as

$$R_{p,w}^t \equiv \frac{\sum_{n=1}^{D_w} \rho_{n,p}^t}{D_w} = \frac{\sum_{n=1}^{D_w} \Phi(Z_n^t)}{D_w} \tag{13}$$

for deterministic models, where

$R_{p,w}^t$	proportion of selecting path $p \in P_w$ in D_w at day t;
$\rho_{n,p}^t, \rho_{n,p}^t = 1$	if user n selecting path $p \in P_w$ at day t, otherwise zero;
Z_n^t	attributes of user n at day t;
$\Phi(\cdot)$	route decision function, $\Phi(Z_n^t)$ is equivalent to $\rho_{n,p}^t$ [21].

Rewriting (6) in a difference equation and replacing $(c_p^t - c_w^t)$ with $Q_{p,w}^t$, we have

$$h_p^{t+1} = h_p^t \times \left(1 - \alpha_p Q_{p,w}^t\right). \tag{14}$$

Let (14) divided by D_w; then we have

$$\frac{h_p^{t+1}}{D_w} = \frac{h_p^t}{D_w} \times \left(1 - \alpha_p Q_{p,w}^t\right). \tag{15}$$

Furthermore, the definitions of (12) and (13) imply that

$$\begin{aligned} \text{Pr}_{p,w}^{t+1} &\approx \frac{\sum_{n=1}^{D_w} \text{Pr}_{n,p}^{t+1}}{D_w} \approx \frac{h_p^{t+1}}{D_w} = \frac{h_p^t}{D_w} \times \left(1 - \alpha_p Q_{p,w}^t\right) \\ &\approx \frac{\sum_{n=1}^{D_w} \text{Pr}_{n,p}^t}{D_w} \times \left(1 - \alpha_p Q_{p,w}^t\right) \\ &= \text{Pr}_{p,w}^t \left(1 - \alpha_p Q_{p,w}^t\right) \end{aligned} \tag{16}$$

and

$$R_{p,w}^{t+1} \approx R_{p,w}^t \left(1 - \alpha_p Q_{p,w}^t\right) \tag{17}$$

respectively. Equations (16) and (17) are aggregate dynamic route-choice models approximated by the proposed path-flow dynamics.

D. Threshold Effect on Path-Flow Dynamics

Based on the results in (6), the response term here is reformulated as a path-flow adjustments prompt only if $(c_p^t - c_w^t) > B$ or $(c_p^t - c_w^t) < -B$. B is a positive real number to denote the threshold value of stimulus. This supposition tells us that path-flow evolution is not fully sensitive to stimulus. If experienced travel time is not good enough to attract users or severely ill to be abandoned by users, the system will remain tranquil. This idea is fulfilled by defining a discontinuous stimulus in (6), depicted as

$$\begin{cases} \dot{h}_p^t = -\alpha_p h_p^t \left(c_p^t - c_w^t\right), & \text{if } \left|c_p^t - c_w^t\right| > B \\ \dot{h}_p^t = 0, & \text{if } \left|c_p^t - c_w^t\right| \leq B. \end{cases} \tag{18}$$

$$\begin{cases} \dot{h}_p^t = -\alpha_p h_p^t \left(c_p^t - c_w^t\right) \\ \dot{c}_w^t = \beta_w \left(D_w - h_w^t\right) \\ 0 < \alpha_p < \inf_{c_p^t - c_w^t > 0} \left(\frac{1}{c_p^t - c_w^t}\right) \quad \text{and} \quad \sum_{w \in W} \sum_{p \in P_w} \left(-\delta_{ap} \alpha_p \text{PTTL}_{p,w}^t\right) \leq k_a - \sum_{p \in P} \delta_{ap} h_p^t \\ 0 < \beta_w < \min\left\{ \inf_{\forall (D_w - h_w^t < 0)} \left(\frac{\breve{c}_w - c_w^t}{D_w - h_w^t}\right), \inf_{\forall (D_w - h_w^t > 0)} \left(\frac{\widehat{c}_w - c_w^t}{D_w - h_w^t}\right) \right\} \end{cases}$$

IV. Analysis of the Steady State

A. Steady State of Network Dynamics in (6) and (10)

It is useful to recall the definition of Wardrop's static-user equilibrium in our terms before elaborating on the steady state of the proposed network dynamics. Let the symbol "$^-$" denote the steady-state or equilibrium point. The Wardrop's user equilibrium can be described as

$$\begin{cases} \bar{h}_p > 0 \rightarrow c_p^{\bar{t}} = \bar{c}_w \\ c_p^{\bar{t}} > \bar{c}_w \rightarrow \bar{h}_p = 0 \qquad \forall\, w \in W, \quad p \in P_w \\ \sum_{p \in P_w} \bar{h}_p = D_w \end{cases} \tag{19}$$

which states a condition that is stable only when no traveler can improve his travel time by unilaterally changing path. All path travel times with positive flow of the same O–D pair are equal and minimal at this status.

The steady state of (6) and (10) implies that $\dot{h}_p = 0$ and $\dot{c}_w = 0$ for all O–D pair $w \in W, p \in P_w$. After some algebraic reasoning, we have the following conditions jointly equivalent to the steady state of (6) and (10):

$$\begin{cases} h_p^t > 0 \rightarrow c_p^t = c_w^t \quad \text{or} \quad h_p^t < 0 \rightarrow c_p^t = c_w^t \\ c_p^t > c_w^t \rightarrow h_p^t = 0 \quad \text{or} \quad c_p^t < c_w^t \rightarrow h_p^t = 0 \\ c_w^t > 0 \rightarrow D_w = \sum_{p \in P_w} h_p^t \end{cases} \tag{20}$$

for all O–D pairs $w \in W, p \in P_w$. $h_p^t < 0 \rightarrow c_p^t = c_w^t$ in (20) never happens due to violating nonnegative flow constraint. If initial conditions with positive path flows are assumed, $c_p^t < c_w^t \rightarrow h_p^t = 0$ in (20) will be infeasible. The third equilibrium state in (20) is evidently held on. Then, we refine the critical components in (20) as

$$\begin{cases} h_p^t > 0 \rightarrow c_p^t = c_w^t \\ c_p^t > c_w^t \rightarrow h_p^t = 0 \\ D_w = \sum_{p \in P_w} h_p^t \end{cases} \tag{21}$$

for all O–D pairs $w \in W, p \in P_w$. It is easy to infer that the experienced travel time is equal to the predicted travel time and is minimal among all positive-flow paths of an O–D pair simultaneously in (21). The travel demand of an O–D pair is equal to the sum of corresponding path flows. Based on these results, we claim that the steady state of (6) and (10) is identical to Wardrop's user equilibrium.

B. Steady State of Network Dynamics in (18)

The steady state of (18) implies

$$\begin{cases} c_p^t > c_w^t + B \rightarrow h_p^t = 0 \quad \text{or} \quad c_p^t < c_w^t - B \rightarrow h_p^t = 0 \\ h_p^t > 0 \rightarrow c_w^t - B \le c_p^t \le c_w^t + B \end{cases} \tag{22}$$

for all O–D pairs $w \in W, p \in P_w$. $c_p^t < c_w^t \bar{B} \rightarrow h_p^t = 0$ in (22) never prompts if the initial conditions with positive path flows are provided. Then, we have an equilibrium state of (10) and, together with (22), as

$$\begin{cases} c_p^t > c_w^t + B \rightarrow h_p^t = 0 \\ h_p^t > 0 \rightarrow c_w^t - B \le c_p^t \le c_w^t + B \\ D_w = \sum_{p \in P_w} h_p^t \end{cases} \tag{23}$$

for all O–D pairs $w \in W, p \in P_w$. Two facts in (23) are observed. The first is that path flow is zero if the experienced travel time is larger than the predicted travel time plus threshold value. The second one is that positive path flow implies that the experienced travel time lies within the close set $[c_w^t - B, c_w^t + B]$. Equation (23) is not identical to Wardrop's user equilibrium. But the less the threshold value is, the closer (23) reaches to Wardrop's user equilibrium. Or we can say that Wardrop's user equilibrium is a special case of (23) with zero threshold value, i.e., $B = 0$. Now we conclude that as (18) and (10) settle down, the experienced travel times of paths with positive flow are not necessarily the same, but with deviations less than twice of threshold value B. Now we give a new definition called quasi user equilibrium and the corresponding theorem to state the relationship between (23) and Wardrop's user equilibrium.

Definition. (Quasi User Equilibrium): No travelers *desire to* improve their travel time by unilaterally changing paths.

Theorem. (Quasi User Equilibrium): Wardrop's user equilibrium (19) is a special case of *quasi user equilibrium* (23) and they are equivalent as B defined in (18) is zero.

Proof: The proof is immediate from the analysis of the steady state of (18) and (10), mentioned before.

V. Existence, Uniqueness, and Stability

In this section, existence and uniqueness of solutions in dynamical systems (6) and (10) are briefly discussed in Section VA. Lyapunov function is given in Section VB to deliberate the asymptotic behavior of (6) and (10). A detailed proof of existence, uniqueness, and stability of solutions is omitted here; we refer readers to [18] for a complete proof. Because (18) is not a continuously differentiable function, detailed statements of existence, uniqueness, and stability of (18) are not discussed in this paper. Time indices, day t, and inequalities (7a), (7b) and (11) are neglected for conciseness in subsequent sections. Then, (6) and (10) are rewritten as

$$\begin{cases} \dot{h}_p = -\alpha_p h_p (c_p - c_w) \\ \dot{c}_w = \beta_w (D_w - h_w). \end{cases} \tag{24}$$

A. Existence and Uniqueness of (24)

A dynamic system is a way of describing the time passage of all the points for a given space E. Mathematically, the space E might be an Euclidean space R or a subset of R. For the network dynamics mentioned in Section III, the set of possible nonnegative path flows and predicted travel times clearly is a convex subset of $R_+^{\bar{P}+\bar{W}}$, denoted as S, from the fundamental theorem in [17].

Fundamental Theorem: Suppose that $V \in C^1(E)$ and that $V(\boldsymbol{x})$ satisfies the global Lipschitz condition denoted as $|V(\boldsymbol{x}) - V(\boldsymbol{y})| \le L|\boldsymbol{x} - \boldsymbol{y}|$ for all $\boldsymbol{x}, \boldsymbol{y} \in E$. Then, for $\boldsymbol{x_0} \in R^n$, the initial value problem $\dot{\boldsymbol{x}} = V(\boldsymbol{x})$ with $\boldsymbol{x}(0) = \boldsymbol{x}_0$ has a unique solution $\boldsymbol{x}(t)$ defined for all $t \in R$.

The existence and uniqueness of (24) are achieved if the vector field V of (24) is continuously differentiable and satisfies the global Lipschitz condition. The assumptions of a smooth and strict monotone function of link travel time ensure that V

is $C^1(R_+^{\bar{P}+\bar{W}})$. For convenience, $V : S \to R_+^{\bar{P}+\bar{W}}$ in (24) is reindexed as

$$V(\boldsymbol{h}', \boldsymbol{c}') \equiv \begin{cases} v_j(\boldsymbol{h}', \boldsymbol{c}') = -\alpha_j h_j (c_j - c_w) \\ \quad \forall w \in W, j \in \{1, 2, \ldots, \bar{P}\}, \quad \text{if path } j \in P_w \\ v_j(\boldsymbol{h}', \boldsymbol{c}') = \beta_w [D_w - h_w] \\ \quad \forall w \in W, \quad j \in \{1 + \bar{P}, 2 + \bar{P}, \ldots, w + \bar{P}\}. \end{cases} \tag{25}$$

Now, we give the lemma of the Lipschitz condition for (25).

Lemma (Lipschitz Condition, [18]): $V(\boldsymbol{h}', \boldsymbol{c}')$, defined in (25), satisfies the global Lipschitz condition with a Lipschitz constant $L = \max\{\max_{\forall w}\{\beta_w\}, L_p\}$, where

$$L_p = \max_{1 \le j \le \bar{P}} \left\{ \sup_{j \in P_w} \left(\alpha_j \left((\overset{\frown}{c}_w - \overset{\smile}{c}_w) + h_j \left. \frac{\partial c_j}{\partial h_j} \right|_{h_j = \overset{\frown}{h}_j} \right) \right) \right. \left. \sup_{j, l \ne j} \left(\alpha_j \overset{\frown}{h}_j \left. \frac{\partial c_j}{\partial h_l} \right|_{h_l = \overset{\frown}{h}_l} \right), \sup_j (\alpha_j \overset{\frown}{h}_j) \right\}.$$

$\overset{\smile}{c}_w$, $\overset{\frown}{c}_w$, and $\overset{\frown}{h}_p$ are defined in (11a), (11b), and $\overset{\frown}{h}_p \equiv \min_{\forall a \in p}\{k_a\}$. Now, we have the sufficient conditions listed in fundamental theorem. The proof of global existence and uniqueness of (25) is standard; we refer readers to [17].

B. Asymptotic Behavior of (24)

The following context mainly relies on the well-known definition of the Lyapunov function and the corresponding stability theorem [19].

Definition (Lyapunov Function): Let $\bar{\boldsymbol{v}}$ be a steady state of a dynamical system $V \in C^1(E)$. A function $L : E \to R$ is called a strict Lyapunov function for $\bar{\boldsymbol{v}}$ if the following conditions are satisfied:

$$(i) \times L(\bar{\boldsymbol{v}}) = 0 \,\text{and}\, L(\boldsymbol{v}) > 0 \quad \forall\, \boldsymbol{v} \ne \bar{\boldsymbol{v}}, \boldsymbol{v} \in E$$
$$(ii) \times \dot{L}(\boldsymbol{v}) < 0 \qquad \forall\, \boldsymbol{v} \ne \bar{\boldsymbol{v}}, \in E.$$

Theorem (Lyapunov's Stability): Let $\bar{\boldsymbol{v}}$ be a steady state of $\dot{\boldsymbol{v}} = V(\boldsymbol{v})$. If there exists a strict Lyapunov function $\forall\, \boldsymbol{v} \ne \bar{\boldsymbol{v}}, \boldsymbol{v} \in E$, then $\bar{\boldsymbol{v}}$ is asymptotically stable.

We rewrite (24) in a vector and matrix form as

$$\begin{aligned} \dot{\boldsymbol{s}} &= -\begin{pmatrix} \hat{\boldsymbol{h}}_{\boldsymbol{r}} & 0 \\ 0 & \boldsymbol{I} \end{pmatrix} \boldsymbol{\varphi} \begin{pmatrix} \boldsymbol{\Delta}' \boldsymbol{c_a}(\boldsymbol{\Delta h}) - \boldsymbol{\Gamma c} \\ \boldsymbol{\Gamma}' \boldsymbol{h} - \boldsymbol{O} \end{pmatrix} \\ &= -\begin{pmatrix} \hat{\boldsymbol{h}}_{\boldsymbol{r}} & 0 \\ 0 & \boldsymbol{I} \end{pmatrix} \boldsymbol{\varphi}(\boldsymbol{M}(\boldsymbol{s}) + \boldsymbol{\Omega s}). \end{aligned} \tag{26}$$

$\hat{\boldsymbol{h}}_{\boldsymbol{r}} \equiv (\begin{smallmatrix} \boldsymbol{h}_+ & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{h}_- \end{smallmatrix})$ is with its components of $\boldsymbol{h}_+$ and $\boldsymbol{h}_-$ being both diagonal matrices with elements of path flows such that $(h_p - \bar{h}_p)(c_p - c_w) \ge 0$ and $(h_p - \bar{h}_p)(c_p - c_w) < 0$, respectively.

$$\boldsymbol{s} \equiv (\boldsymbol{h} \quad \boldsymbol{c})'$$
$$\boldsymbol{M}(\boldsymbol{s}) \equiv (\Delta' \boldsymbol{c_a}(\Delta \boldsymbol{h}) \quad -\boldsymbol{O})'$$
$$\boldsymbol{\Omega} \equiv \begin{pmatrix} \boldsymbol{0} & -\boldsymbol{\Gamma} \\ \boldsymbol{\Gamma}' & \boldsymbol{0} \end{pmatrix}$$
$$\varphi \equiv \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\beta} \end{pmatrix}$$

and $\boldsymbol{I}, \boldsymbol{c_a}(\Delta \boldsymbol{h}), \Delta, \boldsymbol{O}, \boldsymbol{0}$, and $\boldsymbol{\Gamma}$ denote identity matrix, full link-cost vector, link-path incident matrix, full O–D pair demand vector, zero matrix, and path-O–D pair incident matrix with suitable dimension, respectively, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are diagonal matrices with elements for all α_p and β_w, respectively. If we further let

$$\bar{\boldsymbol{h}}_{\boldsymbol{\varepsilon}} \equiv \begin{pmatrix} \bar{\boldsymbol{h}}_{\boldsymbol{\varepsilon}+} & \boldsymbol{0} \\ \boldsymbol{0} & \bar{\boldsymbol{h}}_{\boldsymbol{\varepsilon}-} \end{pmatrix}$$

where $\bar{\boldsymbol{h}}_{\boldsymbol{\varepsilon}+}$ and $\bar{\boldsymbol{h}}_{\boldsymbol{\varepsilon}-}$ are two diagonal matrices and there exists $\varepsilon_p^+ \in R^+$ and $\varepsilon_p^- \in R^+$ such that the elements of $\bar{\boldsymbol{h}}_{\boldsymbol{\varepsilon}+}$ and $\bar{h}_{\boldsymbol{\varepsilon}-}$ are $\bar{h}_p^* \varepsilon_p^+$ and $\bar{h}_p^* \varepsilon_p^-$, respectively, with

$$\frac{h_p}{\bar{h}_p^* \varepsilon_p^+} > 1 \qquad \forall h_p \in \boldsymbol{h}_+ \tag{27}$$

and

$$\frac{h_p}{\bar{h}_p^* \varepsilon_p^-} < 1 \qquad \forall h_p \in \boldsymbol{h}_- \tag{28}$$

where $\bar{h}_p^* = \sup_{\forall h_p \in h}(\bar{h}_p)$ and $\bar{h}_p$ is the steady state of h_p. Then, a strict Lyapunov function is presented as

$$L(\boldsymbol{s}) \equiv \frac{1}{2}(\boldsymbol{s} - \bar{\boldsymbol{s}})' \begin{pmatrix} \bar{\boldsymbol{h}}_{\boldsymbol{\varepsilon}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}^{-1} \varphi^{-1}(\boldsymbol{s} - \bar{\boldsymbol{s}}). \tag{29}$$

We refer readers to [18] for the detail proof of the asymptotical stability for dynamical system (26).

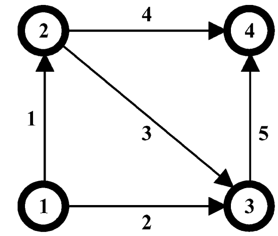

Fig. 1. Graph of the numerical example.

TABLE II
FUNCTION FORM[1] AND PARAMETERS OF THE LINK COST

Link No.	A_a	B_a	k_a
1	40	20	80
2	60	30	80
3	20	10	120
4	50	25	80
5	30	15	80

[1] $c_a(f_a^t) = A_a + B_a\left(\frac{f_a^t}{k_a}\right)^4$

VI. NUMERICAL EXAMPLES

A simple network with four nodes and five links, illustrated as Fig. 1, is used to show the numerical results of the proposed models solved by high-order Runge–Kutta method. There is only one O–D pair $w = \{\text{node } 1, \text{node } 4\}$, which is connected by three paths denoted as: path $1 = \{\text{link } 1, \text{link } 4\}$; path $2 = \{\text{link } 2, \text{link } 5\}$; and path $3 = \{\text{link } 1, \text{link } 3, \text{link } 5\}$, respectively. The parameters of the link cost functions are set in Table II.

The three examples were solved with the O–D demand fixed as 120. Initial conditions with assumed identical parameters α_p of path flow dynamics of model are given as $(h_1^0, h_2^0, h_3^0, c_1^0, \alpha, \beta) = (40, 50, 30, 125, 0.0006, 0.1)$. Table III

TABLE III
NUMERICAL RESULTS OF (6) AND (10)

	Flow			Experienced travel time		
	Initial	State at t=200	Steady state	Initial	State at t=200	Steady state
Path 1	40	51.06	56.16	103.29	103.84	103.79
Path 2	50	53.13	56.95	109.58	104.05	103.79
Path 3	30	15.69	6.89	116.76	107.91	103.80
Link 1	70	66.75	63.05	51.72	49.69	47.72
Link 2	50	53.13	56.95	64.58	65.84	67.70
Link 3	30	15.69	6.89	20.04	20.01	20.00
Link 4	40	51.06	56.16	51.56	55.15	56.07
Link 5	80	68.82	63.84	45.00	38.22	36.08
Predicted travel time				125.00	104.25	103.79

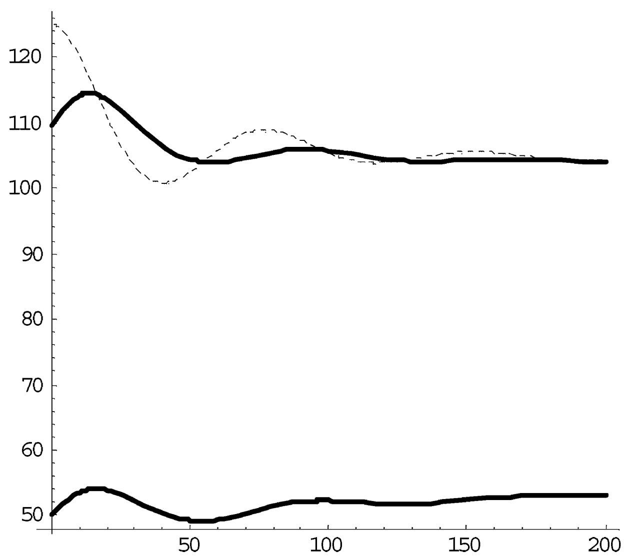

Fig. 2. Evolutions of path 1 and stimulus$_{\mathrm{p1}}$. Dashed line, predicted travel time; upper black line, experienced travel time; lower black line, flow.

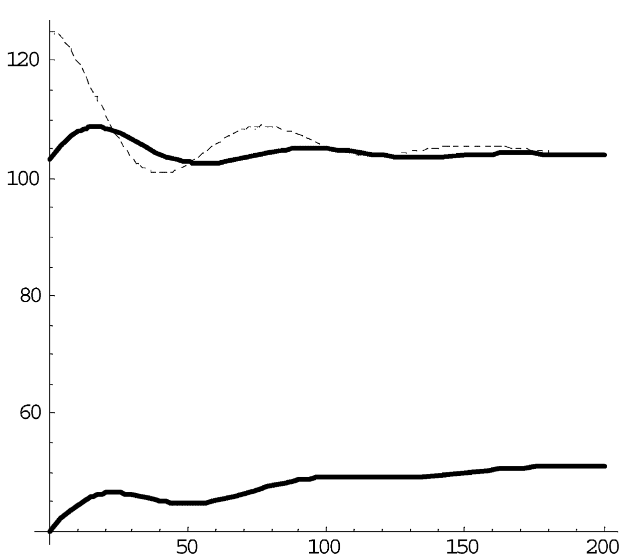

Fig. 3. Evolutions of path 2 and stimulus$_{\mathrm{p2}}$. Dashed line, predicted travel time; upper line, experienced travel time; lower black line, flow.

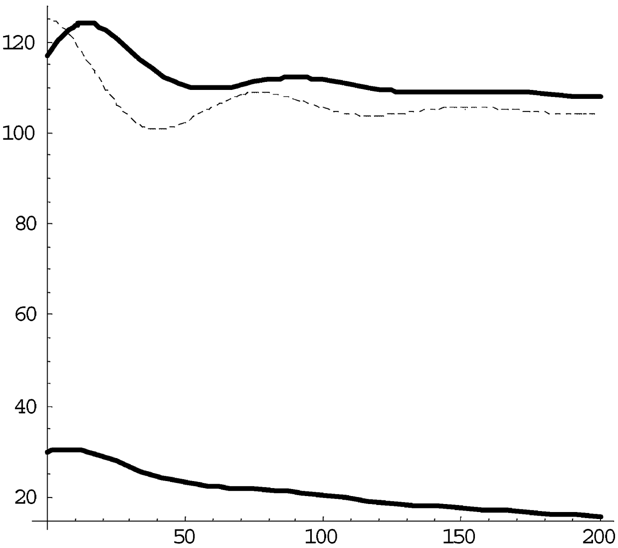

Fig. 4. Evolutions of path 3 and stimulus$_{\mathrm{p3}}$. Dashed line, predicted travel time; upper line, experienced travel time; lower black line, flow.

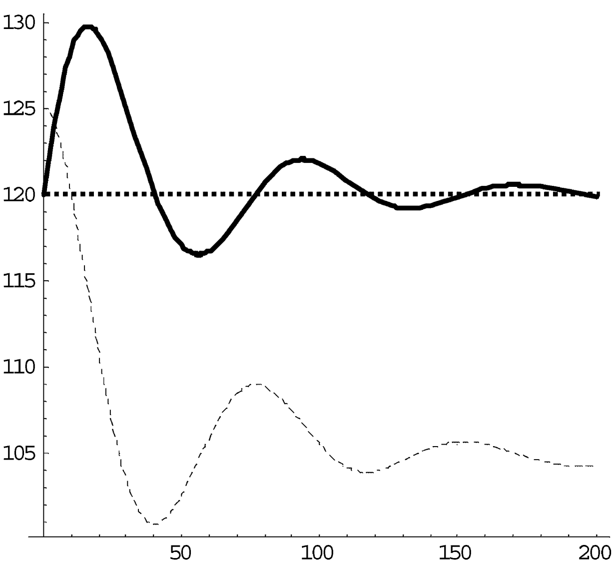

Fig. 5. Evolutions of predicted travel time versus stimulus$_{\mathrm{cw}}$. Dashed line, predicted travel time; dotted line, O–D demand; black line, path flows sum.

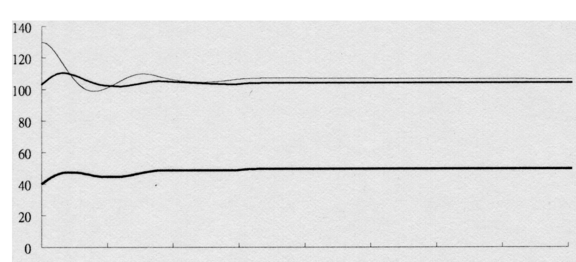

Fig. 6. Evolutions of path 1 and stimulus$_{p1}$. Thinner line, predicted travel time; middle line, experienced travel time; black line, path 2; dotted black line, path 3.

shows the dynamics of flows, experienced travel times, and predicted travel times by ATIS at three different states for (6) and (10). It is clear that the steady state satisfies the Wardrop's user equilibrium and that the predicted travel time is equal to the path travel times of which path flows are positive simultaneously. Numerical results by the evolutions of network dynamics illustrated from Figs. 2–5 also show that the path flow increases (decreases) as experienced travel time is less

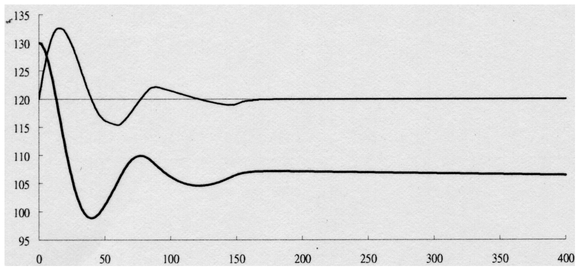

Fig. 7. Evolutions of predicted travel time versus stimulus$_{cw}$. Thinner line, O–D demand; middle line, path flows sum; thicker line, predicted travel time

TABLE IV
NUMERICAL RESULTS OF (18) AND (10)

	Flow		Experienced travel time	
	Initial	Steady state (T=405)	Initial	Steady state (T=405)
Path 1	40	49.72	103.29	104.20
Path 2	50	51.96	109.58	104.27
Path 3	30	18.33	116.76	109.41
Link 1	70	68.04	51.72	50.47
Link 2	50	51.96	64.58	65.34
Link 3	30	18.33	20.04	20.01
Link 4	40	49.72	51.56	53.37
Link 5	80	70.28	45.00	38.94
Predicted O-D travel time by ATIS			130.00	106.44

Quasi user equilibrium is reached at the 405th time step.

(more) than predicted travel time and predicted travel time increases (decreases) as O–D demand is more (less) than the sum of corresponding path flows.

Selected numerical results of models (10) and (18) with the same inputs as the above example, but with additional $B = 3$ and $c_1^0 = 130$ are presented in Figs. 6 and 7. Quasi user equilibrium is reached at the 405th time step, shown in Table IV. The experienced travel time in quasi user equilibrium lies within the close set $[c_w^t - B, c_w^t + B]$, which is centered at the predicted travel time and with the length of being $2B$.

VII. CONCLUSION

Day-to-day network dynamics are successfully formulated by using the stimulus–response formula under the assumption of a daily learning and adaptive behavioral process. The time-change rate of flow and the difference between the experienced and predicted travel times for a path are depicted as the response and stimulus, respectively, in path flow dynamics. The time derivative of the predicted travel time and the excess travel demand are introduced to be the response and the stimulus, respectively, in predicted travel-time dynamics. The approximated dynamic route-choice model is derived from the proposed path-flow dynamics. By defining a discontinuous stimulus in path-flow dynamics, the path-flow adjustments encapsulated threshold effects are developed. Then, the quasi user equilibrium is researched to be a steady state when all users feel indifferent between the experienced travel time and the predicted travel time. The proposed quasi user equilibrium is reduced to Wardrop's user equilibrium if the threshold effect is vanished. Based on these results, the proposed models build an analytical linkage between the Wardrop's user equilibrium and the empirical adaptability of route preference under the operations of ITS.

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