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Honeycomb rectangular disks

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Abstract

In this paper, we propose a variation of honeycomb meshes. A honeycomb rectangular disk HReD(m,n) is obtained from the honeycomb rectangular mesh HReM(m,n) by adding a boundary cycle. A honeycomb rectangular disk HReD(m,n) is a 3-regular planar graph. It is obvious that the honeycomb rectangular mesh HReM(m,n) is a subgraph of HReD(m,n). We also prove that HReD(m,n) is hamiltonian. Moreover, HReD$(m,n) - f$ remains hamiltonian for any $f \in V(\text{HReD}(m,n)) \cup E(\text{HReD}(m,n))$ if $n \geqslant 6$.
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1. Introduction

Network topology is a crucial factor for an interconnection network since it determines the performance of the network. Many interconnection network topologies have been proposed in the literature for the purpose of connecting a large number of processing elements. Network topology is always represented by a graph where

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the nodes represent processors and the edges represent the links between processors. One of the most popular architectures is mesh connected computers [4]. Each processor is placed into a square or rectangular grid and connected by a communication link to its neighbors in up to four directions.

It is well known that there are three possible tessellations of a plane with regular polygons of the same kind: square, triangular, and hexagonal, corresponding to dividing a plane into regular squares, triangles, and hexagons, respectively. Some computer and communication networks have been built based on this observation. The square tessellation is the basis for mesh-connected computers. The triangle tessellation is the basis for defining hexagonal meshed multiprocessors [3,7]. The hexagonal tessellation is the basis for defining honeycomb meshes [2,6].

Stojmenovic [6] introduced three different honeycomb meshes-the honeycomb rectangular mesh, honeycomb rhombic mesh, and honeycomb hexagonal mesh. Most of these meshes are not regular. Moreover, such meshes are not hamiltonian unless it is small in size [5]. To remedy these drawbacks, the honeycomb rectangular torus, honeycomb rhombic torus and honeycomb hexagonal torus are proposed [6]. Any such torus is 3-regular. Moreover, all honeycomb tori are not planar. In this paper, we propose a variation of honeycomb meshes, called honeycomb rectangular disk. A honeycomb rectangular disk HReD(m,n) is obtained from the honeycomb rectangular mesh HReM(m,n) by adding a boundary cycle. Any HReD(m,n) is a planar 3-regular hamiltonian graph. Moreover, HReD$(m,n) - f$ remains hamiltonian for any $f \in V(\text{HReD}(m,n)) \cup E(\text{HReD}(m,n))$ if $n \geqslant 6$. These hamiltonian properties are optimal. Thus, the honeycomb retangular disk network has superior basic characteristics compared with commercial mesh connected computers, which belong to the same family of planar bounded degree networks.

In the following section, we give some graph terms that are used in this paper and a formal definition of honeycomb rectangular disk. Obviously, such HReD(m,n) is a super graph of the honeycomb rectangular mesh HReM(m,n). Assume that m and n are positive even integers with $m \geqslant 4$ and $n \geqslant 6$. In Section 3, we present four basic recursive algorithms to obtain hamiltonian cycle for such HReD$(m,n) - f$. In Section 4, we prove that such HReD$(m,n) - e$ remains hamiltonian for any $e \in E$. In Section 5, we prove that such HReD$(m,n) - v$ remains hamiltonian for any $v \in V$. In the final section, we cover the general HReD(m,n) and present our conclusion.

2. Honeycomb rectangular disks

Usually, computer networks are represented by graphs where nodes represent processors and edges represent links between processors. In this paper, a network is represented as an undirected graph. For the graph definition and notation, we follow [1]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(a,b) | (a,b)$ is an unordered pair of $V\}$. We say that V is the *node set* and E is the *edge set* of G. Two nodes a and b are *adjacent* if $(a,b) \in E$. A *path* is a sequence of nodes such that two consecutive nodes are adjacent. A path is delimited by $\langle x_0, x_1, x_2, \ldots, x_n \rangle$. We use P^{-1}

to denote the path $\langle x_n, x_{n-1}, \ldots, x_1, x_0\rangle$ if P is the path $\langle x_0, x_1, x_2, \ldots, x_n\rangle$. A *cycle* is a path of at least three nodes such that the first node is the same as the last node.

A *hamiltonian path* is a path such that its nodes are distinct and span V. A *hamiltonian cycle* is a cycle such that its nodes are distinct except for the first node and the last node and span V. A *hamiltonian graph* is a graph with a hamiltonian cycle. A graph $G = (V,E)$ is *1-edge hamiltonian* if $G - e$ is hamiltonian for any $e \in E$, and a graph $G = (V,E)$ is *1-node hamiltonian* if $G - v$ is hamiltonian for any $v \in V$. Obviously, any 1-edge hamiltonian graph is hamiltonian. A graph $G = (V,E)$ is *1-hamiltonian* if $G - f$ is hamiltonian for any $f \in E \cup V$.

The *honeycomb rectangular mesh* HReM(m,n) is the graph with

$$\begin{aligned} V(\mathrm{HReM}(m,n)) &= \{(i,j) \mid 0 \leqslant i < m, 0 \leqslant j < n\}, \text{ and} \\ E(\mathrm{HReM}(m,n)) &= \{((i,j),(k,l)) \mid i = k \text{ and } j = l \pm 1\} \\ &\quad \cup \{((i,j),(k,l)) \mid j = l \text{ and } k = i + 1 \text{ with } i + j \text{ is odd}\}. \end{aligned}$$

For example, the honeycomb rectangular mesh HReM(8,6) is shown in Fig. 1.

For easy presentation, we first assume that m and n are positive even integers with $m \geqslant 4$ and $n \geqslant 6$. A *honeycomb rectangular disk* HReD(m,n) is the graph obtained from HReM(m,n) by adding a boundary cycle. More precisely,

$$\begin{aligned} V(\mathrm{HReD}(m,n)) &= (\{(i,j) \mid 0 \leqslant i < m, -1 \leqslant j \leqslant n\} \\ &\quad - \{(0,-1),(m-1,-1)\}) \\ &\quad \cup \{(i,j) \mid i \in \{-1,m\}, 0 < j < n, j \text{ is even}\}, \text{ and} \\ E(\mathrm{HReD}(m,n)) &= \{((i,j),(k,l)) \mid i = k \text{ and } j = l \pm 1\} \\ &\quad \cup \{((i,j),(k,l)) \mid j = l \text{ and } k = i + 1 \text{ with } i + j \text{ is odd}\} \\ &\quad \cup \{((i,j),(k,l)) \mid i = k \in \{-1,m\} \text{ and } j = l \pm 2\} \\ &\quad \cup \{((0,0),(-1,2)),((-1,n-2),(0,n)), \\ &\qquad ((m-1,n),(m,n-2))\} \\ &\quad \cup \{((m,2),(m-1,0)),((m-1,0),(m-2,-1)), \\ &\qquad ((1,-1),(0,0))\}. \end{aligned}$$

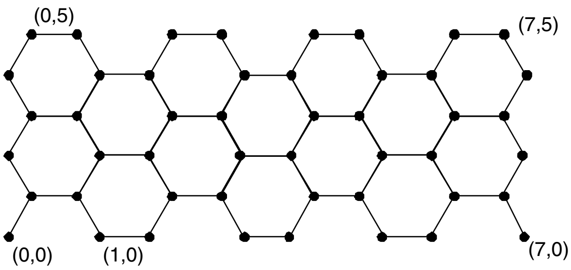

Fig. 1. The Honeycomb rectangular mesh HReM(8,6).

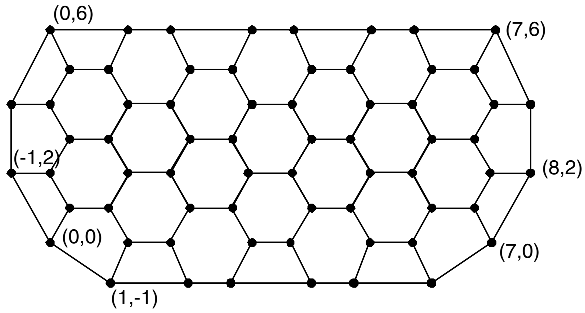

Fig. 2. The Honeycomb rectangular disk HReD(8,6).

For example, the honeycomb rectangular disk HReD(8,6) is shown in Fig. 2. Obviously, HReM(m,n) is a subgraph of HReD(m,n). Moreover, any honeycomb rectangular disk is a planar 3-regular graph. With Fig. 2, we can easily observe that that any HReD(m,n) is *left-right symmetric*; i.e., symmetric with respect to $x = \frac{m-2}{2}$.

3. Four basic algorithms

The honeycomb rectangular disk has a good symmetric property which we shall take advantage of it to construct a hamiltonian cycle. In the following, we shall first establish four basic algorithms. Let F be a subset of $V(\text{HReD}(m,n)) \cup E(\text{HReD}(m,n))$. The purpose of these basic algorithms are to extend a hamiltonian cycle of $\text{HReD}(m,n) - F$ to a hamiltonian cycle of $\text{HReD}(m+2,n) - F$. For $1 \leqslant i \leqslant m-2$, we say a hamiltonian cycle HC of $\text{HReD}(m,n) - F$ is *i-regular* if either $((i,n),(i+1,n))$ or $((i,-1),(i+1,-1))$ is incident with HC. We call a hamiltonian cycle HC of $\text{HReD}(m,n) - F$ is *0-regular* if either $((0,n),(1,n))$ or $((0,0),(1,-1))$ is incident with HC. Assume that $0 \leqslant i < m-1$. We define a function f_i from $V(\text{HReD}(m,n))$ into $V(\text{HReD}(m+2,n))$ by assigning $f_i(k,l) = (k,l)$ if $k \leqslant i$ and $f_i(k,l) = (k+2,l)$ if otherwise. Then we define

$$\begin{aligned} f_i(F) = {} & \{f_i(k,l) \mid (k,l) \in V(\text{HReD}(m,n)) \cap F\} \\ & \cup \{(f_i(k,l), f_i(k',l')) \mid ((k,l),(k',l')) \in E(\text{HReD}(m,n)) \cap F; \\ & \quad \{k,k'\} \neq \{i,i+1\}\} \\ & \cup \{((i,l),(i+1,l')) \mid ((i,l),(i+1,l')) \in E(\text{HReD}(m,n)) \cap F\}. \end{aligned}$$

We will present four basic algorithms to obtain a hamiltonian cycle of $\text{HReD}(m+2,n) - f_i(F)$ from a hamiltonian cycle of $\text{HReD}(m,n) - F$ for some F.

For $-1 \leqslant i \leqslant m$, $-1 \leqslant j$, and $k \leqslant n$, let $H_i(j,k)$ denote the path $\langle (i,j), (i,j+1),(i,j+2), \ldots, (i,k-2),(i,k-1),(i,k)\rangle$.

Algorithm 1. Suppose that HC is a hamiltonian cycle of $\text{HReD}(m,n) - F$ containing the edge $((i,-1),(i+1,-1))$ with $1 \leqslant i < m-2$. We construct $g_i^1(\text{HC})$ as follows:

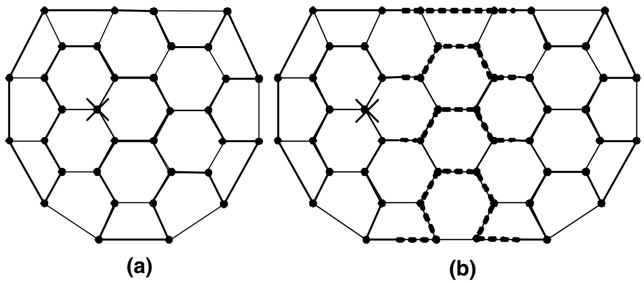

Fig. 3. Illustration for Algorithm 1.

Let $-1 \leqslant k_0 < k_1 < \ldots < k_{(t-1)} \leqslant n$ be the indices such that $((i,k_j),(i+1,k_j)) \in E(\text{HC})$. We set $k_t = n$. Let $\overline{\text{HC}_i}$ be the image of $\text{HC}-\{((i,k_j),(i+1,k_j))|-1 \leqslant k_j \leqslant n\}$ under g_i^1. We define P_j as

$$\langle (i,k_j),(i+1,k_j) \overset{H_{i+1}(k_j,k_{j+1}-1)}{\rightarrow} (i+1,k_j-1),$$
$$(i+2,k_j-1) \overset{H^{-1}_{i+2}(k_j,k_{j+1}-1)}{\rightarrow} (i+2,k_j),(i+3,k_j)\rangle.$$

It is easy to see that edges of $\overline{\text{HC}_i}$ together with edges of P_j, with $0 \leqslant j < t$ form a hamiltonian cycle of $\text{HReD}(m+2,n) - f_i(F)$. We denote this cycle as $g_i^1(\text{HC})$. For example, a hamiltonian cycle HC of HReD(4,6)−(1,3) is shown in Fig. 3(a). The corresponding $g_1^1(\text{HC})$ is shown in Fig. 3(b).

Algorithm 2. Suppose that HC is a hamiltonian cycle of $\text{HReD}(m,n) - F$ containing the edge $((i,n),(i+1,n))$ with $1 \leqslant i < m-2$. We construct $g_i^2(\text{HC})$ as follows:

Let $-1 \leqslant k_0 < k_1 < \ldots < k_{(t-1)} \leqslant n$ be the indices such that $((i,k_j), (i+1,k_j)) \in E(\text{HC})$. We set $k_{-1} = -2$. Let $\overline{\text{HC}_i}$ be the image of $\text{HC}-\{((i,k_j),(i+1,k_j))|-1 \leqslant k_j \leqslant n\}$ under g_i^2. We define Q_j as

$$\langle (i,k_j),(i+1,k_j) \overset{H^{-1}_{i+1}(k_{j-1}+1,k_j)}{\rightarrow} (i+1,k_{j-1}+1),$$
$$(i+2,k_{j-1}+1) \overset{H_{i+2}(k_j,k_{j-1}+1)}{\rightarrow} (i+2,k_j),(i+3,k_j)\rangle.$$

It is easy to see that edges of $\overline{\text{HC}_i}$ together with edges of Q_j, with $0 \leqslant j < t$ form a hamiltonian cycle of $\text{HReD}(m+2,n) - f_i(F)$. We denote this cycle as $g_i^2(\text{HC})$. For

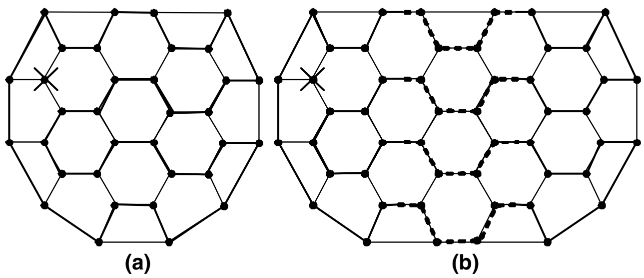

Fig. 4. Illustration for Algorithm 2.

example, a hamiltonian cycle HC of HReD(4,6)−(0,4) is shown in Fig. 4(a). The corresponding g_1^2(HC) is shown in Fig. 4(b).

Suppose that HC is i-regular. We can apply Algorithm 1 to obtain a hamiltonian cycle g_i^1(HC) of HReD$(m+2,n) - f_i(F)$ if $((i,-1),(i+1,-1))$ is incident with HC, and apply Algorithm 2 to obtain a hamiltonian cycle g_i^2(HC) of HReD-$(m+2,n) - f_i(F)$ if otherwise. It is easy to see that the resultant hamiltonian cycle is i-regular, $(i+1)$-regular, and $(i+2)$-regular. So we can further extend a hamiltonian cycle in HReD$(m+4) - F'$ for some $F' \subseteq E$(HReD $(m+4)) \cup E$(HReD $(m+4)$). However, the above discussion only works for column i with $1 \leqslant i \leqslant m-2$. We use the following two algorithms to obtain similar results for column 0.

Algorithm 3. Suppose that HC is a hamiltonian cycle of HReD$(m,n) - F$ containing the edge ((0,0),(1,−1)). Now, we construct g^3(HC) as follows:

Let $1 \leqslant k_1 < k_2 < \ldots < k_t \leqslant n$ be the indices such that $((0,k_j),(1,k_j)) \in E$(HC). We set $k_{t+1} = n$. Let $\overline{\text{HC}}$ be the image of HC$-\{((0,k_j),(1,k_j)) | 1 \leqslant k_j \leqslant n\} \cup \{((0,0),(1,-1))\}$ under g^3. We define R_0 as

$$\langle(0,0),(1,-1)\overset{H_1(-1,k_1-1)}{\rightarrow}(1,k_1-1),(2,k_1-1)\overset{H_2^{-1}(-1,k_1-1)}{\rightarrow}(2,-1),(3,-1)\rangle.$$

For $1 \leqslant j \leqslant t$, we define R_j as

$$\langle(0,k_j),(1,k_j)\overset{H_1(k_j,k_{j+1}-1)}{\rightarrow}(1,k_j-1),(2,k_j-1)\overset{H_2^{-1}(k_j,k_{j+1}-1)}{\rightarrow}(2,k_j),(3,k_j)\rangle.$$

It is easy to see that edges of $\overline{\text{HC}}$ together with edges of R_j, with $0 \leqslant j \leqslant t$ form a hamiltonian cycle of HReD$(m+2,n) - f_i(F)$. We denote this cycle as g^3(HC).

Algorithm 4. Suppose that HC is a hamiltonian cycle of HReD$(m,n) - F$ containing the edge $((0,n),(1,n))$. We construct g^4(HC) as follows:

Let $1 \leqslant k_0 < k_1 < \ldots < k_{(t-1)} \leqslant n$ be the indices such that $((0,k_j),(1,k_j))$ is an edge of HC. We set $k_{-1} = -2$. Let $\overline{\text{HC}}$ be the image of HC$-\{((0,k_j),(1,k_j)) | 1 \leqslant k_j \leqslant n\} \cup \{((0,0),(1,-1))\}$ under g^4. We define S_0 as

$$\langle(0,k_0),(1,k_0)\overset{H_1^{-1}(-1,k_0)}{\rightarrow}(1,-1),(2,-1)\overset{H_2(-1,k_0)}{\rightarrow}(2,k_0),(3,k_0)\rangle.$$

For $1 \leqslant j < t$, we define S_j as

$$\langle(0,k_j),(1,k_j)\overset{H_1^{-1}(k_{j-1}+1,k_j)}{\rightarrow}(1,k_{j-1}+1),(2,k_{j-1}+1)\overset{H_2(k_j,k_{j-1}+1)}{\rightarrow}(2,k_j),(3,k_j)\rangle.$$

It is easy to see that edges of $\overline{\text{HC}}$ together with edges of S_j, with $0 \leqslant j < t$ form a hamiltonian cycle of HReD$(m+2,n) - f_i(F)$. We denote this cycle as g^4(HC).

Suppose that HC is 0-regular. We can apply Algorithm 3 to obtain a hamiltonian cycle g^3(HC) of HReD$(m+2,n) - f_0(F)$ if ((0,0),(1,−1)) is incident with HC, and apply Algorithm 4 to obtain a hamiltonian cycle g^4(HC) of HReD$(m+2,n) - f_0(F)$ if otherwise. It is easy to see that the resultant hamiltonian cycle is 0-regular, 1-regular, and 2-regular. So we can further extend a hamiltonian cycle in HReD$(m+4) - F'$ for some $F' \subseteq E$(HReD$(m+4)) \cup E$(HReD$(m+4)$).

4. HReD(m,n) is 1-edge hamiltonian

In this section, we shall show that if m,n are even integers with $m \geqslant 4$ and $n \geqslant 6$, then HReD(m,n) is 1-edge hamiltonian. We say an edge e of HReD(m,n) is *regular* if there exists a hamiltonian cycle C of HReD(m,n) $- e$ such that C is $(\frac{m}{2}-1)$-regular and 0-regular.

Lemma 1. *Any edge e of HReD(4,n) that is incident with at least one vertex in* $\{(i,j)|-1 \leqslant i \leqslant 2\}$ *is regular.*

Proof. Assume that e is any edge of HReD(4,n) that is incident with at least one vertex in $\{(i,j)|-1 \leqslant i \leqslant 2\}$. Obviously, e is in one of the following 6 sets: namely,

$$\begin{aligned}
A =&\{((i,j),(i+1,j)) \mid -1 \leqslant i \leqslant 1, -1 \leqslant j \leqslant n\}\\
&- \{((1,0),(2,0)),((1,n),(2,n)),((0,1),(1,1))\},\\
B =&\{((-1,n-2),(0,n)),((0,0),(1,-1))\}\\
&\cup \{((i,j),(i+1,j)) \mid -1 \leqslant i \leqslant 1, -1 \leqslant j \leqslant n\}\\
&- \{((1,-1),(2,-1)),((0,n-3),(1,n-3)),((0,n),(1,n))\}\\
&- \{((-1,n-2),(0,n-2)),((1,n-2),(2,n-2))\},\\
C =&\{((i,j),(i,j+1)) \mid 0 \leqslant i \leqslant 1, j \text{ is odd}, 1 \leqslant j \leqslant n-1\}\\
&\cup \{((-1,j+1),(-1,j+3)) \mid j \text{ is odd}, 1 \leqslant j \leqslant n-1\},\\
D =&\{((i,j),(i,j+1)) \mid 0 \leqslant i \leqslant 1, j \text{ is even}, j \geqslant 4\},\\
E =&\{((i,j),(i,j+1)) \mid 0 \leqslant i \leqslant 1, j = 0,2\},\\
F =&\{((-1,2),(0,0)),((1,-1),(1,0))\}.
\end{aligned}$$

Suppose that $e \in A$. Then

$$\begin{aligned}
&\langle(1,0),(1,-1),(0,0),(-1,2)\overset{H_{-1}(2,n-2)}{\rightarrow}(-1,n-2),(0,n)\\
&\overset{H_0^{-1}(1,n)}{\rightarrow}(0,1),(1,1)\overset{H_1(1,n)}{\rightarrow}(1,n),(2,n)\overset{H_2^{-1}(1,n)}{\rightarrow}(2,1),(3,1)\\
&\overset{H_3(1,n)}{\rightarrow}(3,n),(4,n-2)\overset{H_4^{-1}(2,n-2)}{\rightarrow}(4,2),(3,0),(2,-1),(2,0),(1,0)\rangle
\end{aligned}$$

is the desired hamiltonian cycle. See Fig. 5(a) for illustration.

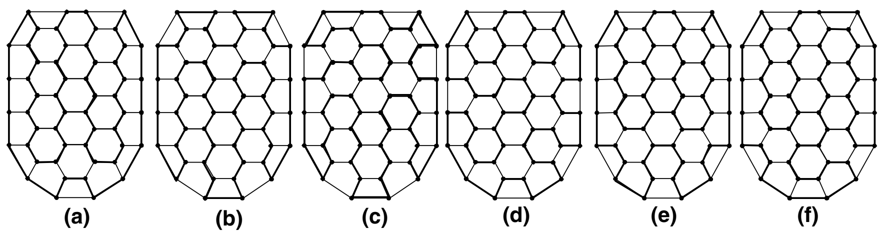

Fig. 5. Illustration for Lemma 1.

Suppose that $e \in B$. Then

$$\begin{aligned}\langle(0,0),(-1,2) &\stackrel{H_{-1}(2,n-2)}{\rightarrow}(-1,n-2),(0,n-2),(0,n-1),(0,n),(1,n),\\ &\quad(1,n-1),(1,n-2),(2,n-2),(2,n-1),(2,n),(3,n),\\ &\quad(3,n-1),(3,n-2),(4,n-2)\\ &\stackrel{H_4^{-1}(2,n-2)}{\rightarrow}(4,2),(3,0)\stackrel{H_3(0,n-3)}{\rightarrow}(3,n-3),(2,n-3)\\ &\stackrel{H_2^{-1}(-1,n-3)}{\rightarrow}(2,-1),(1,-1)\\ &\stackrel{H_1(-1,n-3)}{\rightarrow}(1,n-3),(0,n-3)\stackrel{H_0^{-1}(0,n-3)}{\rightarrow}(0,0)\rangle\end{aligned}$$

is the desired hamiltonian cycle. See Fig. 5(b) for illustration.

Suppose that $e \in C$. Assume that $e = ((i,j),(i,j+1))$ for some $0 \leqslant i \leqslant 1$. We set $x = j$ if $1 \leqslant j \leqslant n-5$ and $x = n-5$ if otherwise. Assume that $e = ((-1,j+1),(-1,j+3))$. We set $x = j$. Then

$$\begin{aligned}\langle(0,0),(-1,2) &\stackrel{H_{-1}(2,x+1)}{\rightarrow}(-1,x+1),(0,x+1),(0,x+2),(1,x+2),(1,x+1),\\ &\quad(2,x+1),(2,x+2),(2,x+3),(1,x+3)\\ &\stackrel{H_1(x+3,n-1)}{\rightarrow}(1,n-1),(0,n-1)\stackrel{H_0^{-1}(x+3,n-1)}{\rightarrow}(0,x+3),(-1,x+3)\\ &\stackrel{H_{-1}(x+3,n-2)}{\rightarrow}(-1,n-2),(0,n),(1,n),(2,n)\\ &\stackrel{H_2^{-1}(x+4,n)}{\rightarrow}(2,x+4),(3,x+4)\stackrel{H_3(x+4,n)}{\rightarrow}(3,n),(4,n-2)\\ &\stackrel{H_4^{-1}(x+3,n-2)}{\rightarrow}(3,x+3),(3,x+2),(3,x+1),(4,x+1)\\ &\stackrel{H_4^{-1}(2,x+1)}{\rightarrow}(4,2),(3,0)\stackrel{H_3(0,x)}{\rightarrow}(3,x),(2,x)\stackrel{H_2^{-1}(-1,x)}{\rightarrow}(2,-1),(1,-1)\\ &\stackrel{H_1(-1,x)}{\rightarrow}(1,x),(0,x)\stackrel{H_0^{-1}(0,x)}{\rightarrow}(0,0)\rangle.\end{aligned}$$

is the desired hamiltonian cycle. See Fig. 5(c) for illustration.

Suppose that $e \in D$. Assume that $e = ((i,j),(i,j+1))$. We set $x = j$ if $j \geqslant 6$ and $x = 6$ if otherwise. Then

$$\begin{aligned}\langle(0,0),(-1,2) &\stackrel{H_{-1}(2,x-2)}{\rightarrow}(-1,x-2),(0,x-2)\stackrel{H_0^{-1}(1,x-2)}{\rightarrow}(0,1),(1,1)\\ &\stackrel{H_1(1,x-2)}{\rightarrow}(1,x-2),(2,x-2),(2,x-1),(2,x),(1,x),(1,x-1),\\ &\quad(0,x-1),(0,x),(-1,x)\\ &\stackrel{H_{-1}(x,n-2)}{\rightarrow}(-1,n-2),(0,n)\stackrel{H_0^{-1}(x+1,n)}{\rightarrow}(0,x+1),(1,x+1)\\ &\stackrel{H_1(x+1,n)}{\rightarrow}(1,n),(2,n)\stackrel{H_2^{-1}(x+1,n)}{\rightarrow}(2,x+1),(3,x+1)\\ &\stackrel{H_3(x+1,n)}{\rightarrow}(3,n),(4,n-2)\stackrel{H_4^{-1}(x,n-2)}{\rightarrow}(4,x),(3,x),(3,x-1),\\ &\quad(3,x-2),(4,x-2)\\ &\stackrel{H_4^{-1}(2,x-2)}{\rightarrow}(4,2),(3,2)\stackrel{H_3(2,x-3)}{\rightarrow}(3,x-3),(2,x-3)\\ &\stackrel{H_2^{-1}(1,x-3)}{\rightarrow}(2,1),(3,1),(3,0),(2,-1),\\ &\quad(2,0),(1,0),(1,-1),(0,0)\rangle.\end{aligned}$$

is the desired hamiltonian cycle. See Fig. 5(d) for illustration.

Suppose that $e \in E$. Assume that $e = ((i,j),(i+1,j))$. Then

$$\langle (0,0) \overset{H_0(0,j)}{\rightarrow} (0,j), (-1,2) \overset{H_{-1}(2,n-2)}{\rightarrow} (-1,n-2), (0,n)$$
$$\overset{H_0^{-1}(j+1,n)}{\rightarrow} (0,j+1), (1,j+1) \overset{H_1(j+1,n)}{\rightarrow} (1,n), (2,n)$$
$$\overset{H_2^{-1}(j+1,n)}{\rightarrow} (2,j+1), (3,j+1) \overset{H_3(j+1,n)}{\rightarrow} (3,n), (4,n-2)$$
$$\overset{H_4^{-1}(2,n-2)}{\rightarrow} (4,2), (3,j) \overset{H_3^{-1}(0,j)}{\rightarrow} (3,0), (2,-1)$$
$$\overset{H_2(-1,j)}{\rightarrow} (2,j), (1,j) \overset{H_1^{-1}(-1,j)}{\rightarrow} (1,-1), (0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 5(e) for illustration.

Suppose that $e \in F$. Then

$$\langle (0,0), (1,-1), (2,-1), (3,0), (3,1), (2,1), (2,0), (1,0), (1,1), (1,2), (2,2)$$
$$\overset{H_2(2,n-1)}{\rightarrow} (2,n-1), (3,n-1) \overset{H_3^{-1}(2,n-1)}{\rightarrow} (3,2), (4,2)$$
$$\overset{H_4(2,n-2)}{\rightarrow} (4,n-2), (3,n), (2,n), (1,n) \overset{H_1^{-1}(3,n)}{\rightarrow} (1,3), (0,3) \overset{H_0(3,n)}{\rightarrow} (0,n), (-1,n-2)$$
$$\overset{H_{-1}^{-1}(2,n-2)}{\rightarrow} (-1,2), (0,2), (0,1), (0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 5(f) for illustration.

The lemma is proved. □

With the left-right symmetric property of HReD$(4,n)$, we have the following corollary.

Corollary 2. *Every HReD$(4,n)$ is 1-edge hamiltonian for any even integer n with $n \geqslant 6$.*

Theorem 3. *Assume that m,n are even integers with $m \geqslant 4$ and $n \geqslant 6$. Any edge e of HReD(m,n) that is incident with any vertex of $\{(i,j) \mid 0 \leqslant i \leqslant \frac{m}{2}\}$ is regular. Hence, HReD(m,n) is 1-edge hamiltonian.*

Proof. We prove this theorem by induction. The inductive basis $m = 4$ is proved in Lemma 1. Let $e = ((i,j),(i',j'))$ be any edge of HReD$(m+2,n)$ that is incident with any vertex of $\{(i,j) \mid 0 \leqslant i \leqslant \frac{m}{2}\}$.

Suppose that $e = ((\frac{m}{2},j),(\frac{m}{2}+1,j))$ for some j. By induction, there exists a regular hamiltonian cycle C of HReD$(m,n) - ((\frac{m}{2}-2,j),(\frac{m}{2}-1,j))$. Then $g_0(C)$ is a regular hamiltonian cycle of HReD$(m+2,n) - e$. Moreover, $g_0(C)$ is both 0-regular and $\frac{m}{2}$-regular. Hence, e is regular.

Suppose that $e \notin \{((\frac{m}{2},j),(\frac{m}{2}+1,j)) \mid 0 \leqslant j \leqslant n\}$. By induction, there exists a regular hamiltonian cycle C of HReD$(m,n)-((i,j),(i',j'))$. Then $g_{\frac{m}{2}-1}(C)$ is a regular hamiltonian cycle of HReD($m+2, n$) $-$ e. Moreover, $g_{\frac{m}{2}-1}(C)$ is both 0-regular and $\frac{m}{2}$-regular. Hence, e is regular.

By the left-right symmetric property, HReD($m+2, n$) is 1-edge hamiltonian. The theorem is proved. □

5. HReD(m, n) is 1-node hamiltonian

In this section, we shall show that if m,n are even integers with $m \geqslant 4$ and $n \geqslant 6$, then HReD(m,n) is 1-node hamiltonian. We say a vertex $v = (i,j)$ of HReD(m,n) is *regular* if there exists a hamiltonian cycle C of HReD(m,n) $-$ v such that C is $(\frac{m}{2}-1)$-regular and 0-regular.

Lemma 4. *Any vertex* $v = (i,j)$ *of HReD*(4,n) *with* $i \in \{-1,0,1\}$ *is regular.*

Proof. Suppose that (i,j) is not in $\{(0,0),(0,1),(0,n-1),(0,n),(1,-1),(1,0),(1,1),(1,n-1),(1,n),(-1,2),(-1,n-2)\}$. Then v is in one of the following 7 sets: namely,

$$A = \left\{(0,j) \mid j = 0 \pmod 2,\ j < \frac{n}{2}\right\},$$
$$B = \left\{(0,j) \mid j = 0 \pmod 2,\ j \geqslant \frac{n}{2}\right\},$$
$$C = \{(0,j) \mid j = 1 \pmod 2,\ j \neq 1, n-1\},$$
$$D = \left\{(1,j) \mid j = 0 \pmod 2,\ 0 < j < \frac{n}{2}\right\},$$
$$E = \left\{(1,j) \mid j = 0 \pmod 2,\ j \geqslant \frac{n}{2}\right\},$$
$$F = \{(1,j) \mid j = 1 \pmod 2,\ j \notin \{-1,1,n-1\}\},\ \text{and}$$
$$G = \{(-1,j) \mid j = 0 \pmod 2,\ j \neq 2, n-2\}.$$

Suppose that $v \in A$. Let $v = (0,j)$. Then

$$\langle (0,0),(-1,2) \overset{H_{-1}(2,n-2)}{\rightarrow} (-1,n-2),(0,n),(1,n),(2,n),(3,n),(4,n-2)$$
$$\overset{H_4^{-1}(j+2,n-2)}{\rightarrow} (4,j+2),(3,j+2) \overset{H_3(j+2,n-1)}{\rightarrow} (3,n-1),(2,n-1)$$
$$\overset{H_2^{-1}(j+2,n-1)}{\rightarrow} (2,j+2),(1,j+2) \overset{H_1(j+2,n-1)}{\rightarrow} (1,n-1),(0,n-1)$$
$$\overset{H_0^{-1}(j+1,n-1)}{\rightarrow} (0,j+1),(1,j+1),(1,j),(2,j),(2,j+1),(3,j+1),(3,j),(4,j)$$
$$\overset{H_4^{-1}(2,j)}{\rightarrow} (4,2),(3,0) \overset{H_3(0,j-1)}{\rightarrow} (3,j-1),(2,j-1) \overset{H_2^{-1}(-1,j-1)}{\rightarrow} (2,-1),(1,-1)$$
$$\overset{H_1(-1,j-1)}{\rightarrow} (1,j-1),(0,j-1) \overset{H_0^{-1}(0,j-1)}{\rightarrow} (0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(a) for illustration.

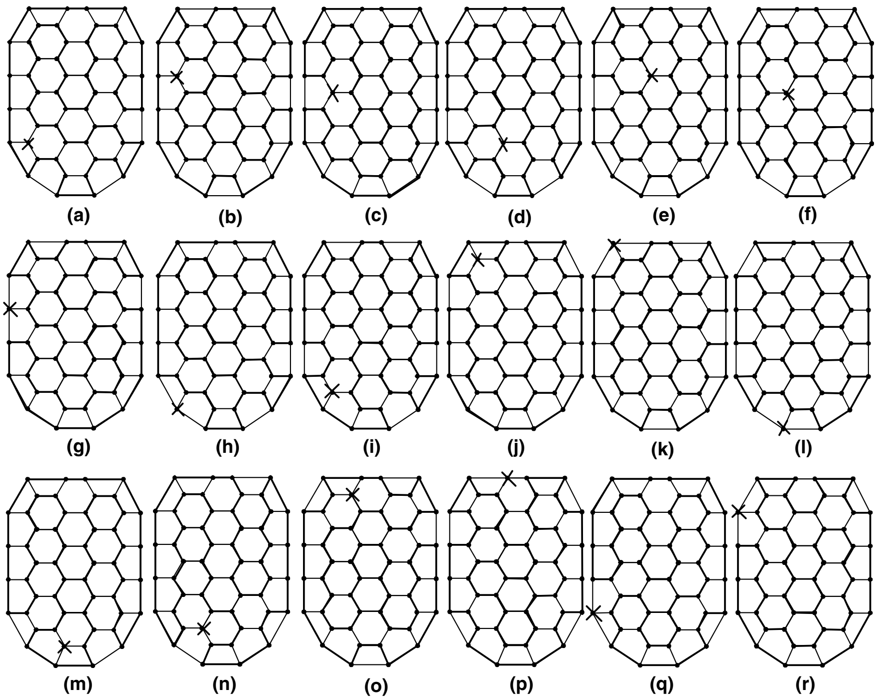

Fig. 6. Illustration for Lemma 4.

Suppose that $v \in B$. Let $v = (0,j)$. Then

$$\langle(0,0),(-1,2)\overset{H_{-1}(2,n-2)}{\rightarrow}(-1,n-2),(0,n)$$
$$\overset{H_0^{-1}(j+1,n)}{\rightarrow}(0,j+1),(1,j+1)\overset{H_1(j+1,n)}{\rightarrow}(1,n),(2,n)$$
$$\overset{H_2^{-1}(j+1,n)}{\rightarrow}(2,j+1),(3,j+1)\overset{H_3(j+1,n)}{\rightarrow}(3,n),(4,n-2)$$
$$\overset{H_4^{-1}(j,n-2)}{\rightarrow}(4,j),(3,j),(3,j-1),(2,j-1),(2,j),(1,j),(1,j-1),(0,j-1)$$
$$\overset{H_0^{-1}(1,j-1)}{\rightarrow}(0,1),(1,1)\overset{H_1(1,j-2)}{\rightarrow}(1,j-2),(2,j-2)\overset{H_2^{-1}(1,j-2)}{\rightarrow}(2,1),(3,1)$$
$$\overset{H_3(1,j-2)}{\rightarrow}(3,j-2),(4,j-2)$$
$$\overset{H_4^{-1}(2,j-2)}{\rightarrow}(4,2),(3,0),(2,-1),(2,0),(1,0),(1,-1),(0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(b) for illustration.

Suppose that $v \in C$. Let $v = (0,j)$. Then

$$\langle (0,0),(-1,2) \overset{H_{-1}(2,j-1)}{\rightarrow} (-1,j-1),(0,j-1)$$
$$\overset{H_0^{-1}(1,j-1)}{\rightarrow} (0,1),(1,1),(1,0),(2,0),(2,1),(3,1)$$
$$\overset{H_3(1,n-1)}{\rightarrow} (3,n-1),(2,n-1) \overset{H_2^{-1}(2,n-1)}{\rightarrow} (2,2),(1,2)$$
$$\overset{H_1(2,n-1)}{\rightarrow} (1,n-1),(0,n-1) \overset{H_0^{-1}(j+1,n-1)}{\rightarrow} (0,j+1),(-1,j+1)$$
$$\overset{H_{-1}(j+1,n-2)}{\rightarrow} (-1,n-2),(0,n),(1,n),(2,n),(3,n),(4,n-2),$$
$$\overset{H_4^{-1}(2,n-2)}{\rightarrow} (4,2),(3,0),(2-1),(1,-1),(0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(c) for illustration.

Suppose that $v \in D$. Let $v = (1,j)$. Then

$$\langle (0,0),(-1,2) \overset{H_{-1}(2,j)}{\rightarrow} (-1,j),(0,j),(0,j+1),(1,j+1)$$
$$\overset{H_1(j+1,n-1)}{\rightarrow} (1,n-1),(0,n-1) \overset{H_0^{-1}(j+2,n-1)}{\rightarrow} (0,j+2),(-1,j+2)$$
$$\overset{H_{-1}(j+2,n-2)}{\rightarrow} (-1,n-2),(0,n),(1,n),(2,n),(3,n),(4,n-2)$$
$$\overset{H_4^{-1}(2,n-2)}{\rightarrow} (4,2),(3,0) \overset{H_3(0,n-1)}{\rightarrow} (3,n-1),(2,n-1)$$
$$\overset{H_2^{-1}(-1,n-1)}{\rightarrow} (2,-1),(1,-1) \overset{H_1(-1,j-1)}{\rightarrow} (1,j-1),(0,j-1) \overset{H_0^{-1}(0,j-1)}{\rightarrow} (0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(d) for illustration.

Suppose that $v \in E$. Let $v = (1,j)$. Then

$$\langle (0,0),(-1,2), \overset{H_{-1}(2,j-2)}{\rightarrow} (-1,j-2),(0,j-2), \overset{H_0^{-1}(1,j-2)}{\rightarrow} (0,1),(1,1)$$
$$\overset{H_1(1,j-1)}{\rightarrow} (1,j-1),(0,j-1),(0,j),(-1,j) \overset{H_{-1}(j,n-2)}{\rightarrow} (-1,n-2),(0,n)$$
$$\overset{H_0^{-1}(j+1,n)}{\rightarrow} (0,j+1),(1,j+1) \overset{H_1(j+1,n)}{\rightarrow} (1,n),(2,n)$$
$$\overset{H_2^{-1}(1,n)}{\rightarrow} (2,1),(3,1) \overset{H_3(1,n)}{\rightarrow} (3,n),(4,n-2)$$
$$\overset{H_4^{-1}(2,n-2)}{\rightarrow} (4,2),(3,0),(2,-1),(2,0),(1,0),(1,-1),(0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(e) for illustration.

Suppose that $v \in F$. Let $v = (1,j)$. Then

$$\langle (0,0),(-1,2) \overset{H_{-1}(2,n-2)}{\rightarrow} (-1,n-2),(0,n),(1,n),(2,n) \overset{H_2^{-1}(j+2,n)}{\rightarrow} (2,j+2),(3,j+2)$$
$$\overset{H_3(j+2,n)}{\rightarrow} (3,n),(4,n-2) \overset{H_4^{-1}(j+1,n-2)}{\rightarrow} (4,j+1),(3,j+1),(3,j),(3,j-1),(4,j-1)$$
$$\overset{H_4^{-1}(2,j-1)}{\rightarrow} (4,2),(3,0) \overset{H_3(0,j-2)}{\rightarrow} (3,j-2),(2,j-2)$$
$$\overset{H_2^{-1}(-1,j-2)}{\rightarrow} (2,-1),(1,-1) \overset{H_1(-1,j-1)}{\rightarrow} (1,j-1),(2,j-1),(2,j),(2,j+1),(1,j+1)$$
$$\overset{H_1(j+1,n-1)}{\rightarrow} (1,n-1),(0,n-1) \overset{H_0^{-1}(0,n-1)}{\rightarrow} (0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(f) for illustration.

Suppose that $v \in G$. Let $v = (-1,j)$. Then

$$\langle (0,0),(-1,2) \overset{H_{-1}(2,j-2)}{\rightarrow} (-1,j-2),(0,j-2)$$
$$\overset{H_0^{-1}(1,j-2)}{\rightarrow} (0,1),(1,1),(1,0),(2,0),(2,1),(3,1) \overset{H_3(1,j-1)}{\rightarrow} (3,j-1),(2,j-1)$$
$$\overset{H_2^{-1}(2,j-1)}{\rightarrow} (2,2),(1,2) \overset{H_1(2,j-1)}{\rightarrow} (1,j-1),(0,j-1)$$
$$\overset{H_0(j-1,j+2)}{\rightarrow} (0,j+2),(-1,j+2) \overset{H_{-1}(j+2,n-2)}{\rightarrow} (-1,n-2),(0,n)$$
$$\overset{H_0^{-1}(j+3,n)}{\rightarrow} (0,j+3),(1,j+3) \overset{H_1(j+3,n)}{\rightarrow} (1,n),(2,n),(3,n),(4,n-2)$$
$$\overset{H_4^{-1}(j+2,n-2)}{\rightarrow} (4,j+2),(3,j+2) \overset{H_3(j+2,n-1)}{\rightarrow} (3,n-1),(2,n-1)$$
$$\overset{H_2^{-1}(j+2,n-1)}{\rightarrow} (2,j+2),(1,j+2),(1,j+1),(1,j),(2,j),(2,j+1),(3,j+1),(3,j),(4,j)$$
$$\overset{H_4^{-1}(2,j)}{\rightarrow} (4,2),(3,0),(2,-1),(1,-1),(0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(g) for illustration.

Suppose that $v = (0,0)$. Then

$$\langle (-1,2) \overset{H_{-1}(2,n-2)}{\rightarrow} (-1,n-2),(0,n),(1,n),(2,n),(3,n),(4,n-2)$$
$$\overset{H_4^{-1}(4,n-2)}{\rightarrow} (4,4),(3,4) \overset{H_3(4,n-1)}{\rightarrow} (3,n-1),(2,n-1)$$
$$\overset{H_2^{-1}(4,n-1)}{\rightarrow} (2,4),(1,4) \overset{H_1(4,n-1)}{\rightarrow} (1,n-1),(0,n-1)$$
$$\overset{H_0^{-1}(3,n-1)}{\rightarrow} (0,3),(1,3),(1,2),(2,2),(2,3),$$
$$(3,3),(3,2),(4,2),(3,0),(3,1),(2,1),(2,0),(2,-1),(1,-1),(1,0),$$
$$(1,1),(0,1),(0,2),(-1,2)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(h) for illustration.

Suppose that $v = (0,1)$. Then

$$\langle(0,0),(-1,2),(0,2),(0,3),(0,4),(-1,4)\overset{H_{-1}(4,n-2)}{\rightarrow}(-1,n-2),(0,n)$$
$$\overset{H_0^{-1}(5,n)}{\rightarrow}(0,5),(1,5)\overset{H_1(5,n)}{\rightarrow}(1,n),(2,n),\overset{H_2^{-1}(5,n)}{\rightarrow}(2,5),(3,5)\overset{H_3(5,n)}{\rightarrow}(3,n),(4,n-2)$$
$$\overset{H_4^{-1}(4,n-2)}{\rightarrow}(4,4),(3,4),(3,3),(3,2),(4,2),(3,0),(3,1),(2,1)\overset{H_2(1,4)}{\rightarrow}(2,4),(1,4)$$
$$\overset{H_1^{-1}(0,4)}{\rightarrow}(1,0),(2,0),(2,-1),(1,-1),(0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(i) for illustration.

Suppose that $v = (0,n-1)$. Then

$$\langle(0,0),(-1,2)\overset{H_{-1}(2,n-4)}{\rightarrow}(-1,n-4),(0,n-4)$$
$$\overset{H_0^{-1}(1,n-4)}{\rightarrow}(0,1),(1,1),(1,0),(2,0),(2,1),(3,1)$$
$$\overset{H_3(1,n-3)}{\rightarrow}(3,n-3),(2,n-3)\overset{H_2^{-1}(2,n-3)}{\rightarrow}(2,2),(1,2)$$
$$\overset{H_1(2,n-3)}{\rightarrow}(1,n-3),(0,n-3),(0,n-2),(-1,n-2),(0,n),$$
$$(1,n),(1,n-1),(1,n-2),(2,n-2),(2,n-1),(2,n),$$
$$(3,n),(3,n-1),(3,n-2),(4,n-2)$$
$$\overset{H_4^{-1}(2,n-2)}{\rightarrow}(4,2),(3,0),(2,-1),(1,-1),(0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(j) for illustration.

Suppose that $v = (0,n)$. Then

$$\langle(0,0),(-1,2)\overset{H_{-1}(2,n-2)}{\rightarrow}(-1,n-2),(0,n-2),(0,n-1),(1,n-1),(1,n),(2,n),$$
$$(2,n-1),(2,n-2),(1,n-2),(1,n-3),(0,n-3)$$
$$\overset{H_0^{-1}(1,n-3)}{\rightarrow}(0,1),(1,1)\overset{H_1(1,n-4)}{\rightarrow}(1,n-4),(2,n-4),(2,n-3),(3,n-3),(3,n-2),$$
$$(3,n-1),(3,n),(4,n-2),(4,n-4),(3,n-4),(3,n-5),(2,n-5)$$
$$\overset{H_2^{-1}(1,n-5)}{\rightarrow}(2,1),(3,1)\overset{H_3(1,n-6)}{\rightarrow}(3,n-6),(4,n-6)$$
$$\overset{H_4^{-1}(2,n-6)}{\rightarrow}(4,2),(3,0),(2,-1),(2,0),(1,0),(1,-1),(0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(k) for illustration.

Suppose that $v = (1,-1)$. Then

$$\langle(0,0),(-1,2),(0,2),(0,3),(0,4),(-1,4)\overset{H_{-1}(4,n-2)}{\rightarrow}(-1,n-2),(0,n-2)$$
$$\overset{H_0^{-1}(5,n-2)}{\rightarrow}(0,5),(1,5)\overset{H_1(5,n-1)}{\rightarrow}(1,n-1),(0,n-1),(0,n),(1,n),(2,n)$$
$$\overset{H_2^{-1}(5,n)}{\rightarrow}(2,5),(3,5)\overset{H_3(5,n)}{\rightarrow}(3,n),(4,n-2)$$
$$\overset{H_4^{-1}(4,n-2)}{\rightarrow}(4,4),(3,4),(3,3),(2,3),(2,4),(1,4),(1,3),(1,2),(2,2)(2,1),$$
$$(3,1),(3,2),(4,2),(3,0),(2,-1),(2,0),(1,0),(1,1),(0,1),(0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(l) for illustration.

Suppose that $v = (1,0)$. Then

$$\langle (0,0),(0,1),(1,1) \overset{H_1(1,n-1)}{\rightarrow} (1,n-1),(0,n-1) \overset{H_0^{-1}(2,n-1)}{\rightarrow} (0,2),(-1,2) \overset{H_{-1}(2,n-2)}{\rightarrow} (-1,n-2),(0,n),(1,n),(2,n),(3,n),(4,n-2) \overset{H_4^{-1}(2,n-2)}{\rightarrow} (4,2),(3,0) \overset{H_3(0,n-1)}{\rightarrow} (3,n-1),(2,n-1) \overset{H_2^{-1}(-1,n-1)}{\rightarrow} (2,-1),(1,-1),(0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(m) for illustration.

Suppose that $v = (1,1)$. Then

$$\langle (0,0),(-1,2) \overset{H_{-1}(2,n-2)}{\rightarrow} (-1,n-2),(0,n),(1,n),(2,n) \overset{H_2^{-1}(3,n)}{\rightarrow} (2,3),(3,3) \overset{H_3(3,n)}{\rightarrow} (3,n),(4,n-2) \overset{H_4^{-1}(2,n-2)}{\rightarrow} (4,2),(3,2),(3,1),(3,0),(2,-1),(1,-1),(1,0),(2,0),(2,1),(2,2),(1,2) \overset{H_1(2,n-1)}{\rightarrow} (1,n-1),(0,n-1) \overset{H_0^{-1}(0,n-1)}{\rightarrow} (0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(n) for illustration.

Suppose that $v = (1,n-1)$. Then

$$\langle (0,0),(-1,2) \overset{H_{-1}(2,n-2)}{\rightarrow} (-1,n-2),(0,n-2),(0,n-1),(0,n),(1,n),(2,n),(2,n-1),(3,n-1),(3,n),(4,n-2),(3,n-2),(3,n-3),(2,n-3),(2,n-2),(1,n-2),(1,n-3),(0,n-3) \overset{H_0^{-1}(1,n-3)}{\rightarrow} (0,1),(1,1) \overset{H_1(1,n-4)}{\rightarrow} (1,n-4),(2,n-4) \overset{H_2^{-1}(1,n-4)}{\rightarrow} (2,1),(3,1) \overset{H_3(1,n-4)}{\rightarrow} (3,n-4),(4,n-4) \overset{H_4^{-1}(2,n-4)}{\rightarrow} (4,2),(3,0),(2,-1),(2,0),(1,0),(1,-1),(0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(o) for illustration.

Suppose that $v = (1,n)$. Then

$$\langle (0,0),(0,1),(0,2),(-1,2) \overset{H_{-1}(2,n-4)}{\rightarrow} (-1,n-4),(0,n-4),(0,n-3),(0,n-2),(-1,n-2),(0,n),(0,n-1),(1,n-1) \overset{H_1^{-1}(n-4,n-1)}{\rightarrow} (1,n-4),(2,n-4) \overset{H_2^{-1}(4,n-4)}{\rightarrow} (2,4),(1,4) \overset{H_1(4,n-5)}{\rightarrow} (1,n-5),(0,n-5) \overset{H_0^{-1}(3,n-5)}{\rightarrow} (0,3),(1,3),(1,2),(1,1),(1,0),(2,0),(2,1),(2,2),(2,3),(3,3) \overset{H_3(3,n-3)}{\rightarrow} (3,n-3),(2,n-3),(2,n-2),(2,n-1),(2,n),(3,n),(3,n-1),(3,n-2),(4,n-2) \overset{H_4^{-1}(2,n-2)}{\rightarrow} (4,2),(3,2),(3,1),(3,0),(2,-1),(1,-1),(0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(p) for illustration.

Suppose that $v = (-1,2)$. Then

$$\langle(0,0) \overset{H_0(0,3)}{\rightarrow} (0,3),(1,3) \overset{H_1^{-1}(0,3)}{\rightarrow} (1,0),(2,0) \overset{H_2(0,3)}{\rightarrow} (2,3),(3,3) \overset{H_3(3,n-1)}{\rightarrow} (3,n-1),(2,n-1) \overset{H_2^{-1}(4,n-1)}{\rightarrow} (2,4),(1,4) \overset{H_1(4,n-1)}{\rightarrow} (1,n-1),(0,n-1) \overset{H_0^{-1}(4,n-1)}{\rightarrow} (0,4),(-1,4) \overset{H_{-1}(4,n-2)}{\rightarrow} (-1,n-2),(0,n),(1,n),(2,n),(3,n),(4,n-2) \overset{H_4^{-1}(2,n-2)}{\rightarrow} (4,2),(3,2),(3,1),(3,0),(2,-1),(1,-1),(0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(q) for illustration.

Suppose that $v = (-1,n-2)$. Then

$$\langle(0,0),(-1,2) \overset{H_{-1}(2,n-4)}{\rightarrow} (-1,n-4),(0,n-4) \overset{H_0^{-1}(1,n-4)}{\rightarrow} (0,1),(1,1),(1,0),(2,0),(2,1),(3,1) \overset{H_3(1,n-3)}{\rightarrow} (3,n-3),(2,n-3) \overset{H_2^{-1}(2,n-3)}{\rightarrow} (2,2),(1,2) \overset{H_1(2,n-3)}{\rightarrow} (1,n-3),(0,n-3),(0,n-2),(0,n-1),(0,n),(1,n),(1,n-1),(1,n-2),(2,n-2),(2,n-1),(2,n),(3,n),(3,n-1),(3,n-2),(4,n-2) \overset{H_4^{-1}(2,n-2)}{\rightarrow} (4,2),(3,0),(2,-1),(1,-1),(0,0)\rangle.$$

is the desired hamiltonian cycle. See Fig. 6(r) for illustration.

The lemma is proved. □

With the left-right symmetric property of HReD(4,n), we have the following corollary.

Corollary 5. *HReD(4,n) is 1-node hamiltonian if $n \geqslant 6$ and n is even.*

Theorem 6. *Assume that m,n are even integers with $m \geqslant 4$ and $n \geqslant 6$. Any vertex $v = (i,j)$ of HReD(m,n) with $i \leqslant \frac{m}{2}$ is regular. Hence, HReD(m,n) is 1-node hamiltonian.*

Proof. We prove this theorem by induction. The inductive basis $m = 4$ is proved in Lemma 4. Let $v = (i,j)$ be any node of HReD$(m+2,n)$.

Assume that $i \in \{\frac{m}{2}, \frac{m}{2}+1\}$. By induction, there exists a hamiltonian cycle C of HReD$(m,n) - (i-2,j)$ which is 0-regular. Then $g_0(C)$ is a hamiltonian cycle of HReD$(m+2,n) - v$. Moreover, $g_0(C)$ is both 0-regular and $\frac{m}{2}$-regular. Hence, (i,j) is regular.

Assume that $i \notin \{\frac{m}{2}, \frac{m}{2}+1\}$. By induction, there exists a hamiltonian cycle C of HReD$(m,n) - (i-2,j)$ which is $\frac{m-2}{2}$-regular. Then $g_{\frac{m}{2}-1}(C)$ is a hamiltonian cycle of HReD$(m+2,n) - v$. Moreover, $g_{\frac{m}{2}-1}(C)$ is both 0-regular and $\frac{m}{2}$-regular. Hence, (i,j) is regular.

By the left-right symmetric property, HReD$(m+2,n)$ is 1-node hamiltonian. The theorem is proved. □

Combining Theorems 3 and 6, we have the following theorem.

Theorem 7. *HReD(m,n) is 1-hamiltonian for any even integer m,n with* $m \geqslant 4$ *and* $n \geqslant 6$.

6. Conclusion

We have seen the hamiltonian properties of honeycomb rectangular disk HReD(m,n) for any positive even m and n integers with $m \geqslant 4$ and $n \geqslant 6$. The honeycomb rectangular disk HReD(m,n) is obtained by adding a boundary cycle to the honeycomb rectangular mesh HReM(m,n). Any such HReD(m,n) is a 3-regular hamiltonian planar graph. Moreover, HReD(m,n) $- F$ remains hamiltonian for any fault $F \in V(\text{HReD}(m,n)) \cup E(\text{HReD}(m,n))$ with $|F| = 1$. Suppose that two faults occur to the neighbor of some vertex x. Then $\deg_{\text{HReD}(m,n) - F}(x) = 1$. Obviously, HReD($m,n$) $- F$ is not hamiltonian. Hence, such hamiltonian property is optimal.

We may also define HReD(m,n) for $m \geqslant 4$ and $n = 4$ by adding a boundary cycle to HReM(m,n). For example, the HReD(6,4) is shown in Fig. 7. By brute force, we can check that such honeycomb rectangular disk is 1-edge hamiltonian but not 1-node hamiltonian.

We may use similar concept to define other cases of HReD(m,n). For example, the HReD(5,6), HReD(5,7), and HReD(6,7) are shown in Fig. 8. With similar discussion as above, we can prove that any HReD(m,n) for odd integer m and even

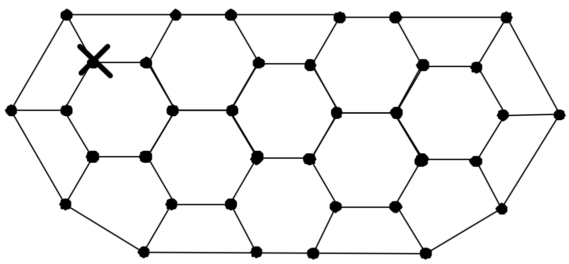

Fig. 7. The Honeycomb rectangular disk HReD(6,4).

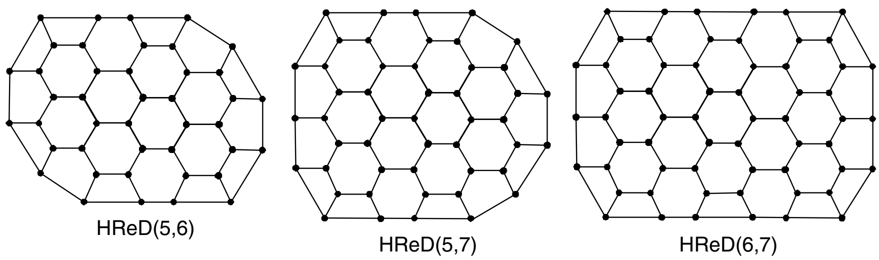

Fig. 8. The Honeycomb rectangular disk HReD(5,6), HReD(5,7), and HReD(6,7).

integer n with $n \geqslant 4$ is 1-edge hamiltonian. Moreover, it is 1-node hamiltonian if and only if $n \geqslant 6$.

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