Least-squares finite element methods for the elasticity problem[1]

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Abstract

A new first-order system formulation for the linear elasticity problem in displacement-stress form is proposed. The formulation is derived by introducing additional variables of derivatives of the displacements, whose combinations represent the usual stresses. Standard and weighted least-squares finite element methods are then applied to this extended system. These methods offer certain advantages such as that they need not satisfy the inf–sup condition which is required in the mixed finite element formulation, that a single continuous piecewise polynomial space can be used for the approximation of all the unknowns, that the resulting algebraic systems are symmetric and positive definite, and that accurate approximations of the displacements and the stresses can be obtained simultaneously. With displacement boundary conditions, it is shown that both methods achieve optimal rates of convergence in the H^1-norm and in the L^2-norm for all the unknowns. Numerical experiments with various Poisson ratios are given to demonstrate the theoretical error estimates.

Keywords: Elasticity; Poisson ratios; Elliptic systems; Least squares; Finite elements; Convergence; Error estimates

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1. Introduction

Over the past decade, increasing attention has been drawn to the use of least-squares principles in connection with finite element applications in the field of computational fluid dynamics (see, e.g., [4, 9, 10, 19, 22–26, 36], etc.). In this paper, we attempt to apply the methodology to develop two least-squares finite element methods for approximating the solution to the following two-dimensional linear

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elasticity problem [13, 27]

$$-\mu\Delta\boldsymbol{u} - (\lambda + \mu)\nabla(\nabla\cdot\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_0, \tag{1.2}$$

$$\sum_{j=1}^{2} \sigma_{ij}(\boldsymbol{u})n_j = g_i \quad \text{on } \Gamma_1, \ i = 1, 2 \tag{1.3}$$

with the following notation:

- $\Omega \subset \mathbb{R}^2$ is a bounded domain representing the region occupied by an elastic body.
- $\partial\Omega = \Gamma_0 \cup \Gamma_1$ is the smooth boundary of Ω partitioned into two disjoint parts Γ_0 and Γ_1 with the measure of Γ_0 being strictly positive.
- μ, λ are the Lamé coefficients where

$$\mu = \frac{E}{2(1 + v)} > 0$$

with v the Poisson ratio, $0 < v < 0.5$, and E the Young modulus and

$$\lambda = \frac{Ev}{(1 + v)(1 - 2v)} > 0.$$

The upper limit of the Poisson ratio, $v \to 0.5^-$, corresponds to an incompressible material.

- $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$ is the displacement vector field.
- $\mathbf{f} = (f_1, f_2)^{\mathrm{T}}$ is the density of a body force acting on the body.
- $\mathbf{g} = (g_1, g_2)^{\mathrm{T}}$ is the density of a surface force acting on Γ_1.
- $\mathbf{n} = (n_1, n_2)^{\mathrm{T}}$ is the outward unit normal vector to $\partial\Omega$.
- $\sigma_{ij}(\mathbf{u})$ are the stresses defined by

$$\sigma_{ij}(\mathbf{u}) = \sigma_{ji}(\mathbf{u}) = \lambda\left(\sum_{k=1}^{2} \varepsilon_{kk}(\mathbf{u})\right)\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{u}), \quad 1 \leqslant i, \ j \leqslant 2.$$

- $\varepsilon_{ij}(\mathbf{u})$ are the strains with

$$\varepsilon_{ij}(\mathbf{u}) = \varepsilon_{ji}(\mathbf{u}) = \tfrac{1}{2}(\partial_j u_i + \partial_i u_j), \quad 1 \leqslant i, \ j \leqslant 2.$$

- δ_{ij} is the Kronecker symbol so that $\delta_{ij} = 0$ if $i \neq j$, and $\delta_{ij} = 1$ when $i = j$.

In the analysis of structural mechanics, the knowledge of the stresses σ_{ij} (strains ε_{ij}) is often of greater interest than the knowledge of the displacements u_i. It is well known that the approximation of the stresses can be recovered from the displacements by postprocessing in the standard finite element formulation for solving problem (1.1)–(1.3). From a numerical point of view, however, their computation requires the derivatives of the displacement field $\mathbf{u}$ which implies a loss of precision. Thus, the most widely used approach for obtaining a better approximation of the stresses is based on the mixed finite element formulation which allows the stresses as new variables along with the primary variables (see [13] and many references therein). Consequently, the accurate stresses can be obtained directly from the discretized problem. Unfortunately, the approximation spaces in the mixed method must be required to satisfy the inf–sup condition which precludes the application of many seemingly natural finite elements.

We provide herein an alternate way to avoid these difficulties by exploiting the least-squares principles on a new first-order system formulation of the elasticity problem. Introducing additional variables of derivatives of the displacements, whose combinations represent the usual stresses (strains), the original system of second-order Eqs. (1.1) can be recast as an equivalent first-order square system in 6 equations with 6 unknowns, which is called the displacement–stress formulation here. The new formulation is very different from the standard one which is extensively studied in the mixed finite element method (see, e.g., [3, 13, 33, 34, 40, 41], etc.). We show that the first-order formulation is an elliptic system in the sense of Petrovski, and that, with the displacement boundary conditions, it satisfies the Lopatinski condition [43]. As a result, the problem can then be solved by using least-squares finite element methods (LSFEMs).

The least-squares approach represents a fairly general methodology that can produce a variety of algorithms. In this paper, we shall consider two LSFEMs. According to the boundary treatment, the first method is based on the minimization of a least-squares functional that involves only the sum of the squared L^2-norms of the residuals in the differential equations. In this case, the trial and test functions are required to fulfill the boundary conditions. We refer it as the *standard* least-squares finite element method (SLSFEM) (cf. [9, 14, 19, 20, 22, 24–26, 31, 32, 36, 39], etc.). The other is based on the minimization of a least-squares functional which consists of the sum of the squared L^2-norms of the residuals both in the differential equations and the boundary conditions with the same weight h^{-1}, where h is the mesh parameter. This method will be referred as the *weighted* least-squares finite element method (WLSFEM) (cf. [4, 5, 11, 21, 43], etc.).

Recently, Cai, Manteuffel, and McCormick and their coworkers have developed a series of first-order systems least-squares (FOSLS) for the general second-order elliptic scalar equations [14, 16], the Stokes equations [15, 17], and the linear elasticity equations [15, 17, 18]. They have pointed out that one of the benefits of least-squares approach is the freedom to incorporate additional equations and impose additional boundary conditions as long as the system is consistent. Instead of applying Agmon–Douglis–Nirenberg (ADN) [1] theory, which is restricted to square systems, they use more direct tools of analysis for their overdetermined FOSLS (see also [31]). For example, the FOSLS of [17] for the elasticity equations with the pure displacement (homogeneous) boundary conditions, namely, (1.1) and (1.2) with $\Gamma_1 = \emptyset$, consists of 11 equations and 7 unknowns. They prove that the FOSLS is uniformly coercive in the Poisson ratio in an H^1-norm appropriately scaled by the Lamé constants. These FOSLS can be classified into the *standard* least-squares category mentioned above since the least-squares functionals involve only the sum of the squared L^2-norms (or with the squared H^{-1}-norms) of the residuals in the differential equations, and thus the trial and test functions are required to fulfill the boundary conditions. On the other hand, our formulation for (1.1)–(1.3) results in a 6×6 FOSLS in order to stay in the regime of the ADN theory. The advantages of the ADN-type FOSLS are that the system is smaller and that both SLSFEM and WLSFEM can be applied to the system. More specifically, the trial and test functions in the WLSFEM need not satisfy the boundary requirements, and thus, it is more convenient for treating nonhomogeneous boundary conditions. Convergence results of both approximations can be established in the natural norms associated with the least-squares bilinear forms. Furthermore, it is shown that, with displacement boundary conditions, both LSFEMs achieve optimal rates of convergence in the H^1-norm and in the L^2-norm for all the unknowns. However, we do not obtain the uniform coercivity in the Poisson ratio under the standard H^1-norm without any scaling, although numerical results given in Section 5 show the uniformity.

When compared with the classical mixed FEM formulation, the least-squares approach appears to require increased regularity and results in a larger system, i.e., with more equations and unknowns. Nevertheless, with a closer inspection, these shortcomings may be dispelled by the following important features in practice:

- Since the approach is not subject to the Babuška–Brezzi condition, more flexible finite element spaces can be used. In fact, a single continuous piecewise polynomial space can be used for the approximation of all the unknowns (cf. Section 5).
- The resulting linear algebraic systems are symmetric and positive definite and are highly vectorizable and parallelizable. The approach thus admits efficient solvers such as multigrid methods [16] or conjugate gradient methods [35].
- The solution of FOSLS can be accelerated by using two-stage algorithms [6, 19] that first solve for the gradients of displacement (which immediately yield deformation and stress), then for the displacement itself (if desired), see [18].

The layout of the remainder of the paper is as follows. In Section 2, we propose the displacement–stress formulation for (1.1)–(1.3). The LSFEMs are given in Section 3, as well as their fundamental properties. A priori error estimates with the displacement boundary conditions are derived in Section 4. In Section 5, some numerical results are presented to demonstrate the approach. Finally, some concluding remarks are addressed in Section 6.

2. A new displacement–stress formulation

We first rewrite the system of Eqs. (1.1) as follows:

$$-\frac{\partial}{\partial x}\left((\lambda+2\mu)\frac{\partial u_1}{\partial x}+\lambda\frac{\partial u_2}{\partial y}\right)-\frac{\partial}{\partial y}\left(\mu\frac{\partial u_1}{\partial y}+\mu\frac{\partial u_2}{\partial x}\right)=f_1 \quad \text{in } \Omega, \tag{2.1}$$

$$-\frac{\partial}{\partial x}\left(\mu\frac{\partial u_1}{\partial y}+\mu\frac{\partial u_2}{\partial x}\right)-\frac{\partial}{\partial y}\left(\lambda\frac{\partial u_1}{\partial x}+(\lambda+2\mu)\frac{\partial u_2}{\partial y}\right)=f_2 \quad \text{in } \Omega. \tag{2.2}$$

Introducing the auxiliary variables

$$\varphi_1=\frac{\partial u_1}{\partial x}, \tag{2.3}$$

$$\varphi_2=\frac{\partial u_2}{\partial y}, \tag{2.4}$$

$$\varphi_3=\frac{\partial u_1}{\partial y}, \tag{2.5}$$

$$\varphi_4=\frac{\partial u_2}{\partial x}, \tag{2.6}$$

defined on $\bar{\Omega}$ and letting $\alpha = \lambda + 2\mu$, we can rewrite Eqs. (2.1)–(2.2) as

$$-\frac{\partial}{\partial x}(\alpha\varphi_1 + \lambda\varphi_2) - \frac{\partial}{\partial y}(\mu\varphi_3 + \mu\varphi_4) = f_1 \quad \text{in } \Omega, \tag{2.7}$$

$$-\frac{\partial}{\partial x}(\mu\varphi_3 + \mu\varphi_4) - \frac{\partial}{\partial y}(\lambda\varphi_1 + \alpha\varphi_2) = f_2 \quad \text{in } \Omega. \tag{2.8}$$

Note that a combination of φ_i, $i = 1, 2, 3, 4$, can represent the usual stresses σ_{ij}, $i, j = 1, 2$. Also, by (2.3) with (2.5) and (2.4) with (2.6), we obtain the following two compatibility equations:

$$\frac{\partial\varphi_1}{\partial y} - \frac{\partial\varphi_3}{\partial x} = 0 \quad \text{in } \Omega, \tag{2.9}$$

$$\frac{\partial\varphi_2}{\partial x} - \frac{\partial\varphi_4}{\partial y} = 0 \quad \text{in } \Omega. \tag{2.10}$$

To recover the displacements, we have the equations

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} - \varphi_1 - \varphi_2 = 0 \quad \text{in } \Omega, \tag{2.11}$$

$$\frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial x} - \varphi_3 + \varphi_4 = 0 \quad \text{in } \Omega. \tag{2.12}$$

Eqs. (2.7)–(2.12) are the so-called displacement–stress formulation of (1.1) and may be written in the matrix form

$$\mathscr{L}U = AU_x + BU_y + DU = F \quad \text{in } \Omega, \tag{2.13a}$$

where

$$A = \begin{pmatrix} -\alpha & -\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu & -\mu & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & -\mu & -\mu & 0 & 0 \\ -\lambda & -\alpha & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} f_1 \\ f_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The system of differential Eqs. (2.13a) will be also supplemented with the boundary conditions (1.2)–(1.3) which may be written as

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} U = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{on } \Gamma_0, \tag{2.14}$$

$$\begin{pmatrix} \alpha n_1 & \lambda n_1 & \mu n_2 & \mu n_2 & 0 & 0 \\ \lambda n_2 & \alpha n_2 & \mu n_1 & \mu n_1 & 0 & 0 \end{pmatrix} U = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \quad \text{on } \Gamma_1. \tag{2.15}$$

The boundary condition (2.14) implies that the tangential derivatives of u_i, $i = 1, 2$, vanish

$$n_2 \varphi_1 - n_1 \varphi_3 = 0 \quad \text{on } \Gamma_0,$$

$$-n_1 \varphi_2 + n_2 \varphi_4 = 0 \quad \text{on } \Gamma_0,$$

and also that

$$n_1 u_1 + n_2 u_2 = 0 \quad \text{on } \Gamma_0.$$

So, we have

$$\begin{pmatrix} n_2 & 0 & -n_1 & 0 & 0 & 0 \\ 0 & -n_1 & 0 & n_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_1 & n_2 \end{pmatrix} U = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{on } \Gamma_0. \tag{2.16}$$

Conversely, we can show that (2.16) together with (2.3)–(2.6) implies (2.14) as well. The boundary condition (2.16) will play an important role in the later theoretical error analysis.

Rewrite (2.15) and (2.16) as the following operator form:

$$\mathscr{R}U = G \quad \text{on } \partial\Omega. \tag{2.13b}$$

It is easily seen that Eqs. (1.1) and (2.13a) are equivalent for smooth solutions.

Theorem 2.1. $u = (u_1, u_2)^T \in [C^2(\bar{\Omega})]^2$ satisfies (1.1) if and only if $U = (\varphi_1, \varphi_2, \varphi_3, \varphi_4, u_1, u_2)^T \in [C^1(\bar{\Omega})]^4 \times [C^2(\bar{\Omega})]^2$ satisfies (2.13a).

The existence, uniqueness, and smoothness of the solution of problem (1.1)–(1.3) are well known (cf. [30, 29]). Therefore, in the sequel, we shall always assume that problem (2.13a/b) has a unique solution $U \in [H^1(\Omega)]^6$ with the given functions $F \in [L^2(\Omega)]^6$ and $g \in [L^2(\Gamma_1)]^2$. Our LSFEMs will be performed over the first-order system (2.13a/b) to obtain approximations of the displacements and the stresses simultaneously.

3. Least-squares finite element methods

We shall require some function spaces defined on Ω, Γ_0, and Γ_1 throughout this paper [27, 38]. The classical Sobolev spaces $H^s(\Omega)$, $s \geq 0$ integer, $L^2(\Gamma_0)$, and $L^2(\Gamma_1)$ with their associated inner products $(\cdot, \cdot)_{s,\Omega}$, $(\cdot, \cdot)_{0,\Gamma_0}$, $(\cdot, \cdot)_{0,\Gamma_1}$ and norms $\|\cdot\|_{s,\Omega}$, $\|\cdot\|_{0,\Gamma_0}$, $\|\cdot\|_{0,\Gamma_1}$ are employed. As usual, $L^2(\Omega) = H^0(\Omega)$. For

the Cartesian product spaces $[H^s(\Omega)]^6$, $[L^2(\Gamma_0)]^3$, and $[L^2(\Gamma_1)]^2$, the corresponding inner products and norms are also denoted by $(\cdot, \cdot)_{s,\Omega}$, $(\cdot, \cdot)_{0,\Gamma_0}$, $(\cdot, \cdot)_{0,\Gamma_1}$, and $\|\cdot\|_{s,\Omega}$, $\|\cdot\|_{0,\Gamma_0}$, $\|\cdot\|_{0,\Gamma_1}$ when there is no chance for confusion.

Let $H_0^s(\Omega)$ be the closure of $\mathscr{D}(\Omega)$ for the norm $\|\cdot\|_{s,\Omega}$, where $\mathscr{D}(\Omega)$ denotes the linear space of infinitely differentiable functions on Ω with compact support. We denote by $H^{-s}(\Omega)$ the dual space of $H_0^s(\Omega)$ normed by

$$\|u\|_{-s,\Omega} = \sup_{0 \neq v \in H_0^s(\Omega)} \frac{|u(v)|}{\|v\|_{s,\Omega}}.$$

Since the boundary $\partial\Omega$ of the bounded domain Ω is smooth, there exists an operator $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, linear and continuous, such that

$$\gamma_0 v = \text{restriction of } v \text{ on } \partial\Omega \text{ for every } v \in C^1(\bar{\Omega}).$$

The space $\gamma_0(H^1(\Omega))$ is not the whole space $L^2(\partial\Omega)$, it is denoted by $H^{1/2}(\partial\Omega)$ and define its norm by

$$\|\varphi\|_{1/2,\partial\Omega} = \inf\{\|v\|_{1,\Omega}; \ v \in H^1(\Omega), \ \gamma_0 v = \varphi\},$$

which makes it a Hilbert space. Its dual is denoted by $H^{-1/2}(\partial\Omega)$ with the norm $\|\cdot\|_{-1/2,\partial\Omega}$. Also, the associated norms of the product spaces $[H^{1/2}(\partial\Omega)]^3$ and $[H^{-1/2}(\partial\Omega)]^3$ are still denoted by $\|\cdot\|_{1/2,\partial\Omega}$ and $\|\cdot\|_{-1/2,\partial\Omega}$, respectively.

We now introduce the standard and the weighted LSFEMs for solving problem (2.13a/b) in the following two subsections. For simplicity, we assume that $G = 0$ on $\partial\Omega$, i.e., $\boldsymbol{g} = \boldsymbol{0}$ on Γ_1.

3.1. The standard least-squares finite element method

Let

$$\mathscr{V}^s = \{V \in [H^1(\Omega)]^6; \ \mathscr{R}V = \boldsymbol{0}\}, \tag{3.1}$$

then define a standard least-squares energy functional $\mathscr{E}^s : \mathscr{V}^s \to \mathbb{R}$ as

$$\mathscr{E}^s(V) = \int_\Omega (\mathscr{L}V - F) \cdot (\mathscr{L}V - F). \tag{3.2}$$

Obviously, the exact solution $U \in \mathscr{V}^s$ of problem (2.13a/b) is the unique zero minimizer of the functional $\mathscr{E}^s$ on $\mathscr{V}^s$, i.e.,

$$\mathscr{E}^s(U) = 0 = \min\{\mathscr{E}^s(V); V \in \mathscr{V}^s\}. \tag{3.3}$$

Applying the techniques of variations, we can find that (3.3) is equivalent to

$$\int_\Omega \mathscr{L}U \cdot \mathscr{L}V = \int_\Omega F \cdot \mathscr{L}V, \quad \forall V \in \mathscr{V}^s. \tag{3.4}$$

The SLSFEM for problem (2.13a/b) is therefore to determine $U_h^s \in \mathscr{V}_h^s$ such that

$$\int_\Omega \mathscr{L}U_h^s \cdot \mathscr{L}V_h = \int_\Omega F \cdot \mathscr{L}V_h, \quad \forall V_h \in \mathscr{V}_h^s, \tag{3.5}$$

where the finite element space $\mathscr{V}_h^s \subset \mathscr{V}^s$ is assumed to satisfy the following approximation property. For any $V \in \mathscr{V}^s \cap [H^{p+1}(\Omega)]^6$, $p \geqslant 0$ integer, there exists $V_h \in \mathscr{V}_h^s$ such that

$$\|V - V_h\|_{0,\Omega} + h\|V - V_h\|_{1,\Omega} \leqslant Ch^{p+1}\|V\|_{p+1,\Omega}, \tag{3.6}$$

with the positive constant C independent of V and h. Throughout this paper, in any estimate or inequality the quantity C will denote a generic positive constant and need not necessarily be the same constant in different places.

3.2. The weighted least-squares finite element method

Similar to the standard least-squares case, we define

$$\mathscr{V}^w = [H^1(\Omega)]^6, \tag{3.7}$$

and define a weighted least-squares energy functional $\mathscr{E}^w : \mathscr{V}^w \to \mathbb{R}$ as

$$\mathscr{E}^w(V) = \int_\Omega (\mathscr{L}V - F) \cdot (\mathscr{L}V - F) + h^{-1} \int_{\partial\Omega} \mathscr{R}V \cdot \mathscr{R}V, \tag{3.8}$$

where h is the mesh parameter. The exact solution $U \in \mathscr{V}^w$ of problem (2.13a/b) is the unique zero minimizer of the weighted least-squares functional $\mathscr{E}^w$ on $\mathscr{V}^w$, i.e.,

$$\mathscr{E}^w(U) = 0 = \min\{\mathscr{E}^w(V);\ V \in \mathscr{V}^w\}. \tag{3.9}$$

Taking the first variation, we can find that (3.9) is equivalent to

$$\int_\Omega \mathscr{L}U \cdot \mathscr{L}V + h^{-1} \int_{\partial\Omega} \mathscr{R}U \cdot \mathscr{R}V = \int_\Omega F \cdot \mathscr{L}V, \quad \forall V \in \mathscr{V}^w. \tag{3.10}$$

The WLSFEM for problem (2.13a/b) is then to determine $U_h^w \in \mathscr{V}_h^w$ such that

$$\int_\Omega \mathscr{L}U_h^w \cdot \mathscr{L}V_h + h^{-1} \int_{\partial\Omega} \mathscr{R}U_h^w \cdot \mathscr{R}V_h = \int_\Omega F \cdot \mathscr{L}V_h, \quad \forall V_h \in \mathscr{V}_h^w, \tag{3.11}$$

where the finite element space $\mathscr{V}_h^w \subset \mathscr{V}^w$ is also required to satisfy the following approximation property. For any $V \in \mathscr{V}^w \cap [H^{p+1}(\Omega)]^6$, $p \geqslant 0$ integer, there exists $V_h \in \mathscr{V}_h^w$ such that

$$\|V - V_h\|_{0,\Omega} + h\|V - V_h\|_{1,\Omega} \leqslant Ch^{p+1}\|V\|_{p+1,\Omega}, \tag{3.12}$$

where C is a positive constant independent of V and h.

3.3. Some fundamental properties

In this subsection, we shall discuss the unique solvability of the numerical schemes (3.5), (3.11), and some of their fundamental properties. Before presenting these properties, it is of interest to note that the trial and test functions in the WLSFEM (3.11) need not satisfy the boundary conditions. In contrast, in the SLSFEM (3.5), both the trial and test functions are required to fulfill the boundary requirements. Moreover, since the original system of second-order Eqs. (1.1) is transformed into the system of first-order Eqs. (2.13a), the same C^0 piecewise polynomials can be used to approximate all the unknown functions.

Denote the bilinear form and the linear form in (3.4) as

$$a^{s}(V, W) = \int_{\Omega} \mathscr{L} V \cdot \mathscr{L} W, \tag{3.13}$$

$$\ell^{s}(V) = \int_{\Omega} F \cdot \mathscr{L} V, \tag{3.14}$$

for all $V, W \in \mathscr{V}^{s}$. Then (3.4) and (3.5) can be rewritten as

$$a^{s}(U, V) = \ell^{s}(V), \quad \forall V \in \mathscr{V}^{s}, \tag{3.15}$$

and

$$a^{s}(U_{h}^{s}, V_{h}) = \ell^{s}(V_{h}), \quad \forall V_{h} \in \mathscr{V}_{h}^{s}, \tag{3.16}$$

respectively. Similarly, denote the bilinear form and the linear form in (3.10) as

$$a^{w}(V, W) = \int_{\Omega} \mathscr{L} V \cdot \mathscr{L} W + h^{-1} \int_{\partial\Omega} \mathscr{R} V \cdot \mathscr{R} V, \tag{3.17}$$

$$\ell^{w}(V) = \int_{\Omega} F \cdot \mathscr{L} V, \tag{3.18}$$

for all $V, W \in \mathscr{V}^{w}$. Then (3.10) and (3.11) can be rewritten as

$$a^{w}(U, V) = \ell^{w}(V), \quad \forall V \in \mathscr{V}^{w}, \tag{3.19}$$

and

$$a^{w}(U_{h}^{w}, V_{h}) = \ell^{w}(V_{h}), \quad \forall V_{h} \in \mathscr{V}_{h}^{w}, \tag{3.20}$$

respectively.

It is clear that $a^{s}(\cdot, \cdot)$ and $a^{w}(\cdot, \cdot)$ define inner products on $\mathscr{V}^{s} \times \mathscr{V}^{s}$ and $\mathscr{V}^{w} \times \mathscr{V}^{w}$, respectively, since the positive-definiteness is implied by the fact that the problem (2.13a/b) possesses the unique solution $U = 0$ for $F = 0$ and $G = 0$. Denote the associated norms as

$$\|V\|_{a^{s}} = \{a^{s}(V, V)\}^{1/2}, \quad \forall V \in \mathscr{V}^{s}, \tag{3.21}$$

$$\|V\|_{a^{w}} = \{a^{w}(V, V)\}^{1/2}, \quad \forall V \in \mathscr{V}^{w}. \tag{3.22}$$

We first state the fundamental properties of the SLSFEM (3.16).

Theorem 3.1. *Let* $U \in [H^{1}(\Omega)]^{6}$ *be the exact solution of* (2.13a/b) *with the given functions* $F \in [L^{2}(\Omega)]^{6}$ *and* $G = 0$.

(i) *Problem* (3.16) *has a unique solution* $U_{h}^{s} \in \mathscr{V}_{h}^{s}$ *which satisfies the following stability estimate*

$$\|U_{h}^{s}\|_{a^{s}} \leqslant \|F\|_{0, \Omega}. \tag{3.23}$$

(ii) *The matrix of the linear system associated with problem* (3.16) *is symmetric and positive definite.*

(iii) *The following orthogonality relation holds*

$$a^{s}(U - U_{h}^{s}, V_{h}) = 0, \quad \forall V_{h} \in \mathscr{V}_{h}^{s}. \tag{3.24}$$

(iv) *The approximate solution U_h^s is a best approximation of U in the $\|\cdot\|_{a^s}$-norm,*

$$\|U - U_h^s\|_{a^s} = \inf_{V_h \in \mathscr{V}_h^s} \|U - V_h\|_{a^s}. \tag{3.25}$$

(v) *If $U \in [H^{p+1}(\Omega)]^6$, $p \geqslant 0$ integer, then there exists a positive constant C independent of h such that*

$$\|U - U_h^s\|_{a^s} \leqslant Ch^p \|U\|_{p+1,\Omega}. \tag{3.26}$$

Proof. To prove the unique solvability, it suffices to prove the uniqueness of solution since the finite dimensionality of $\mathscr{V}_h^s$. Let U_h^s be a solution of (3.16) then, by the Cauchy–Schwarz inequality,

$$
\begin{aligned}
\|U_h^s\|_{a^s}^2 &= a^s(U_h^s, U_h^s) = (F, \mathscr{L} U_h^s)_{0,\Omega} \\
&\leqslant \|F\|_{0,\Omega} \|\mathscr{L} U_h^s\|_{0,\Omega} \\
&\leqslant \|F\|_{0,\Omega} \|U_h^s\|_{a^s}.
\end{aligned}
$$

Thus, we obtain (3.23). Consequently, the solution U_h^s of (3.16) is unique.

Assertion (ii) follows from the fact that the bilinear form $a^s(\cdot, \cdot)$ is symmetric and positive definite. To prove (iii), subtracting Eq. (3.16) from Eq. (3.15), we get (3.24).

To prove (iv), by (3.24) and the Cauchy–Schwarz inequality,

$$
\begin{aligned}
\|U - U_h^s\|_{a^s}^2 &= a^s(U - U_h^s, U - U_h^s) \\
&= a^s(U - U_h^s, U - V_h), \quad \forall V_h \in \mathscr{V}_h^s \\
&\leqslant \|U - U_h^s\|_{a^s} \|U - V_h\|_{a^s}.
\end{aligned}
$$

Thus, we have (3.25).

Finally, assume that $U \in [H^{p+1}(\Omega)]^6$. Let $V_h \in \mathscr{V}_h^s$ such that (3.6) holds with V replaced by U. Then, by (3.25), we have

$$\|U - U_h^s\|_{a^s} \leqslant \|U - V_h\|_{a^s} \leqslant C \|U - V_h\|_{1,\Omega} \leqslant Ch^p \|U\|_{p+1,\Omega}.$$

In the second inequality above, we use the fact that $\mathscr{L}$ is a first-order differential operator with constant coefficients. $\square$

Similarly, we have the following results for the WLSFEM (3.20).

Theorem 3.2. *Let $U \in [H^1(\Omega)]^6$ be the exact solution of (2.13a/b) with the given functions $F \in [L^2(\Omega)]^6$ and $G = 0$.*

(i) *Problem (3.20) has a unique solution $U_h^w \in \mathscr{V}_h^w$ which satisfies the following stability estimate:*

$$\|U_h^w\|_{a^w} \leqslant \|F\|_{0,\Omega}. \tag{3.27}$$

(ii) *The matrix of the linear system associated with problem (3.20) is symmetric and positive definite.*

(iii) *The following orthogonality relation holds*

$$a^w(U - U_h^w, V_h) = 0, \quad \forall V_h \in \mathscr{V}_h^w. \tag{3.28}$$

(iv) *The approximate solution U_h^w is a best approximation of U in the $\|\cdot\|_{a^w}$-norm, that is,*

$$\|U - U_h^w\|_{a^w} = \inf_{V_h \in \mathcal{V}_h^w} \|U - V_h\|_{a^w}. \tag{3.29}$$

(v) *If $U \in [H^{p+1}(\Omega)]^6$, $p \geqslant 0$ integer, then there exists a positive constant C independent of h such that*

$$\|U - U_h^w\|_{a^w} \leqslant Ch^p \|U\|_{p+1,\Omega}. \tag{3.30}$$

Proof. The proofs for (i)–(iv) are similar to the standard least-squares case. For proving part (v), we need the following result whose proof can be found in [12]: there exists a positive constant C such that, for any $V \in [H^1(\Omega)]^6$ and any $\varepsilon > 0$,

$$\|V\|_{0,\partial\Omega} \leqslant C \left(\varepsilon \|V\|_{1,\Omega} + \frac{1}{\varepsilon} \|V\|_{0,\Omega} \right).$$

Taking $\varepsilon = h^{1/2}$ and V replaced by $U - V_h$, where $V_h \in \mathcal{V}_h^w$ is chosen such that (3.12) holds with V replaced by U, then we have

$$\begin{aligned} \|U - V_h\|_{0,\partial\Omega} &\leqslant C(h^{1/2}\|U - V_h\|_{1,\Omega} + h^{-1/2}\|U - V_h\|_{0,\Omega}) \\ &\leqslant Ch^{p+1/2}\|U\|_{p+1,\Omega}. \end{aligned}$$

Thus,

$$\begin{aligned} \|U - U_h^w\|_{a^w}^2 &\leqslant \|U - V_h\|_{a^w}^2 \\ &= \|\mathscr{L}(U - V_h)\|_{0,\Omega}^2 + h^{-1}\|\mathscr{R}(U - V_h)\|_{0,\partial\Omega}^2 \\ &\leqslant C(\|U - V_h\|_{1,\Omega}^2 + h^{-1}\|U - V_h\|_{0,\partial\Omega}^2) \\ &\leqslant Ch^{2p}\|U\|_{p+1,\Omega}^2. \end{aligned}$$

This completes the proof. $\quad\square$

As a consequence of part (v) in the above theorems, the consistency of the approximations follows.

Corollary 3.3. *Let U be the exact solution of problem (2.13a/b) with the given functions $F \in [L^2(\Omega)]^6$ and $G = \mathbf{0}$. If $U \in [H^{p+1}(\Omega)]^6$, $p \geqslant 0$ integer, then there exists a positive constant C independent of h such that*

$$\|\mathscr{L}U_h^s - F\|_{0,\Omega} \leqslant Ch^p \|U\|_{p+1,\Omega}, \tag{3.31}$$

$$\|\mathscr{L}U_h^w - F\|_{0,\Omega} \leqslant Ch^p \|U\|_{p+1,\Omega}, \tag{3.32}$$

$$\|\mathscr{R}U_h^w - G\|_{0,\partial\Omega} \leqslant Ch^{p+1/2} \|U\|_{p+1,\Omega}. \tag{3.33}$$

4. Error analysis

The error estimates of the previous approximations in the H^1- and L^2-norm are primarily based on the theories of ADN and of Dikanskij [28]. Our approach in exploiting these theories follows principally that of Wendland [43, Section 3.1, ch. 8] for two-dimensional first-order elliptic systems in the sense of Petrovski. The application of the theories to our problem involves some unavoidable difficulties concerning the Lopatinski condition if the boundary condition (2.13b) is taken to be as that general. For simplicity, we only consider the displacement boundary conditions

$$\mathscr{R}U = \begin{pmatrix} n_2 & 0 & -n_1 & 0 & 0 & 0 \\ 0 & -n_1 & 0 & n_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_1 & n_2 \end{pmatrix} U = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{on } \Gamma_0 = \partial\Omega. \tag{2.13b'}$$

4.1. A priori estimates

We first show that $\mathscr{L}$ is an elliptic operator in the sense of Petrovski, and that the boundary operator $\mathscr{R}$ in (2.13b') satisfies the Lopatinski condition. So (2.13a/b') is a regular elliptic boundary value problem and then $(\mathscr{L}, \mathscr{R})$ is a Fredholm operator with zero nullity. This enables us to get the coercive type a priori estimates (see Theorem 4.1).

For all $(\xi, \eta) \in \mathbb{R}^2$ and $(\xi, \eta) \neq (0, 0)$,

$$\det(\xi A + \eta B) = -(\lambda\mu + 2\mu^2)(\xi^2 + \eta^2)^3$$
$$\neq 0.$$

Thus, (2.13a) is an elliptic system in the sense of Petrovski. Obviously, by taking $(\xi, \eta) = (1, 0)$, the matrix A is nonsingular and its inverse is

$$A^{-1} = \begin{pmatrix} -1/\alpha & 0 & 0 & -\lambda/\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1/\mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then the original elliptic system (2.13a) can be transformed into the following form:

$$U_x + \tilde{B}U_y + \tilde{D}U = \tilde{F} \quad \text{in } \Omega,$$

where

$$\tilde{B} = A^{-1}B = \begin{pmatrix} 0 & 0 & \mu/\alpha & (\mu + \lambda)/\alpha & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ (\lambda + \mu)/\mu & \alpha/\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\tilde{D} = A^{-1}D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{pmatrix},$$

$$\tilde{F} = A^{-1}F = \begin{pmatrix} -f_1/\alpha \\ 0 \\ 0 \\ -f_2/\mu \\ 0 \\ 0 \end{pmatrix}.$$

We now check the Lopatinski condition as follows. After elementary operations, we find that the eigenvalues of the matrix $\tilde{B}^{\mathrm{T}}$ are the imaginary numbers i and $-$i both with multiplicities three. Consider the eigenvalue $\tau_+ = $ i in the upper half-plane, to which there exists a chain of linearly independent generalized eigenvectors p_1 and p_2 of $\tilde{B}^{\mathrm{T}}$ defined by

$$\tilde{B}^{\mathrm{T}}p_1 - \tau_+ p_1 = 0,$$
$$\tilde{B}^{\mathrm{T}}p_2 - \tau_+ p_2 = p_1,$$

and a third eigenvector p_3 is given by

$$\tilde{B}^{\mathrm{T}}p_3 - \tau_+ p_3 = 0,$$

where

$$p_1 = \left(1, 1, -\frac{\mu}{\alpha}\mathrm{i}, \frac{\mu}{\alpha}\mathrm{i}, 0, 0\right)^{\mathrm{T}},$$
$$p_2 = \left(\frac{2\mu}{\lambda + \mu}\mathrm{i}, 0, \frac{\mu(\lambda + 3\mu)}{\alpha(\lambda + \mu)}, \frac{\mu}{\alpha}, 0, 0\right)^{\mathrm{T}},$$
$$p_3 = (0, 0, 0, 0, 1, -\mathrm{i})^{\mathrm{T}}.$$

Then

$$\mathscr{P} = (p_1, \bar{p}_1, p_2, \bar{p}_2, p_3, \bar{p}_3)^{\mathrm{T}}$$

is nonsingular. Let

$$\mathscr{Q} = (q_1, \bar{q}_1, q_2, \bar{q}_2, q_3, \bar{q}_3)$$

be the inverse matrix of $\mathscr{P}$, then

$$
\mathscr{Q} = \begin{pmatrix}
0 & 0 & -((\lambda+\mu)/4\mu)\,\mathrm{i} & ((\lambda+\mu)/4\mu)\,\mathrm{i} & 0 & 0 \\
1/2 & 1/2 & ((\lambda+\mu)/4\mu)\,\mathrm{i} & -((\lambda+\mu)/4\mu)\,\mathrm{i} & 0 & 0 \\
((\lambda+\mu)/4\mu)\,\mathrm{i} & -((\lambda+\mu)/4\mu)\,\mathrm{i} & (\lambda+\mu)/4\mu & (\lambda+\mu)/4\mu & 0 & 0 \\
-((\lambda+3\mu)/4\mu)\,\mathrm{i} & ((\lambda+3\mu)/4\mu)\,\mathrm{i} & (\lambda+\mu)/4\mu & (\lambda+\mu)/4\mu & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 1/2 \\
0 & 0 & 0 & 0 & (1/2)\,\mathrm{i} & -(1/2)\,\mathrm{i}
\end{pmatrix}.
$$

Now, check the following determinant:

$$
\det\left\{ 2 \begin{pmatrix}
n_2 & 0 & -n_1 & 0 & 0 & 0 \\
0 & -n_1 & 0 & n_2 & 0 & 0 \\
0 & 0 & 0 & 0 & n_1 & n_2
\end{pmatrix} (q_1, q_2, q_3) \right\}
$$

$$
= -(1/4\mu^2)(\lambda+3\mu)(\lambda+\mu)(n_1+n_2\mathrm{i})^3 \neq 0,
$$

since $(n_1, n_2) \neq (0,0)$. That is, the Lopatinski condition is satisfied for the boundary conditions (2.13b′).

The following estimates then follow the standard results of [43].

Theorem 4.1. *For the boundary value problem* (2.13a/b′), (2.13a) *is an elliptic system in the sense of Petrovski, and the boundary condition* (2.13b′) *satisfies the Lopatinski condition. Thus, we have the a priori estimates: for each* $l \geqslant 0$ *there is a constant* $C > 0$ *such that if* $V \in [H^{l+1}(\Omega)]^6$, *then*

$$
\|V\|_{l+1,\Omega} \leqslant C(\|\mathscr{L}V\|_{l,\Omega} + \|\mathscr{R}V\|_{l+1/2,\partial\Omega}). \tag{4.1}
$$

By an interpolation argument [28] (cf. [43, Lemma 8.2.1]), the estimate (4.1) can be extended to the case $l \geqslant -1$. Taking $l = 1$, $l = 0$, and $l = -1$ in (4.1), we have

$$
\|V\|_{2,\Omega} \leqslant C\|\mathscr{L}V\|_{1,\Omega}, \quad \forall V \in \mathscr{V}^s \cap [H^2(\Omega)]^6, \tag{4.2}
$$

$$
\|V\|_{1,\Omega} \leqslant C\|\mathscr{L}V\|_{0,\Omega}, \quad \forall V \in \mathscr{V}^s, \tag{4.3}
$$

$$
\|V\|_{0,\Omega} \leqslant C\|\mathscr{L}V\|_{-1,\Omega}, \quad \forall V \in \mathscr{V}^s, \tag{4.4}
$$

$$
\|V\|_{2,\Omega} \leqslant C(\|\mathscr{L}V\|_{1,\Omega} + \|\mathscr{R}V\|_{3/2,\partial\Omega}), \quad \forall V \in \mathscr{V}^w \cap [H^2(\Omega)]^6, \tag{4.5}
$$

$$
\|V\|_{1,\Omega} \leqslant C(\|\mathscr{L}V\|_{0,\Omega} + \|\mathscr{R}V\|_{1/2,\partial\Omega}), \quad \forall V \in \mathscr{V}^w, \tag{4.6}
$$

$$
\|V\|_{0,\Omega} \leqslant C(\|\mathscr{L}V\|_{-1,\Omega} + \|\mathscr{R}V\|_{-1/2,\partial\Omega}), \quad \forall V \in \mathscr{V}^w. \tag{4.7}
$$

Two sets of estimates (4.2)–(4.4) and (4.5)–(4.7) imply, respectively, the error estimates for SLS-FEM and WLSFEM in the following two subsections.

4.2. Error estimates for the SLSFEM

For the standard least-squares case, by (4.3), we have

$$a^s(V,V) = \|\mathscr{L}V\|_{0,\Omega}^2 \geq C\|V\|_{1,\Omega}^2, \quad \forall V \in \mathscr{V}^s, \tag{4.8}$$

i.e., the bilinear form $a^s(\cdot,\cdot)$ is coercive on $\mathscr{V}^s$. Thus, by using the standard argument, we have

Theorem 4.2. *Let* $U \in \mathscr{V}^s \cap [H^{p+1}(\Omega)]^6$, $U_h^s \in \mathscr{V}_h^s$ *be the solutions of* (2.13a/b') *and* (3.16), *respectively. Then*

$$\|U - U_h^s\|_{1,\Omega} \leq Ch^p \|U\|_{p+1,\Omega}. \tag{4.9}$$

Proof. Utilizing (4.8) and (3.24), we have

$$\begin{aligned}
\|U - U_h^s\|_{1,\Omega}^2 &\leq Ca^s(U - U_h^s, U - U_h^s) \\
&= Ca^s(U - U_h^s, U - V_h), \quad \forall V_h \in \mathscr{V}_h^s \\
&\leq C\|U - U_h^s\|_{1,\Omega}\|U - V_h\|_{1,\Omega}.
\end{aligned}$$

Thus,

$$\|U - U_h^s\|_{1,\Omega} \leq C\|U - V_h\|_{1,\Omega}, \quad \forall V_h \in \mathscr{V}_h^s.$$

Taking $V_h \in \mathscr{V}_h^s$ such that (3.6) holds with V replaced by U, we obtain (4.9). $\quad\square$

Theorem 4.2 shows that the SLSFEM (3.16) achieves optimal convergence in the H^1-norm. For deriving the optimal L^2-estimates, we need the following regularity assumption: assume that, for any $V \in [H_0^1(\Omega)]^6$ and $Q \in [H^{1/2}(\partial\Omega)]^3$, the unique solution U^* of the following problem

$$\begin{aligned}
\mathscr{L}U^* &= V \quad \text{in } \Omega, \\
\mathscr{R}U^* &= Q \quad \text{on } \partial\Omega,
\end{aligned} \tag{4.10}$$

belongs to $[H^2(\Omega)]^6$, where $\mathscr{R}$ is the displacement boundary operator (cf. (2.13b')). This assumption is reasonable since $\mathscr{L}$ is a first-order differential operator.

Theorem 4.3. *Let* $U \in \mathscr{V}^s \cap [H^{p+1}(\Omega)]^6$, $U_h^s \in \mathscr{V}_h^s$ *be the solutions of* (2.13a/b') *and* (3.16), *respectively. If the regularity assumption of* (4.10) *holds with* $Q = 0$, *then*

$$\|U - U_h^s\|_{0,\Omega} \leq Ch^{p+1}\|U\|_{p+1,\Omega}. \tag{4.11}$$

Proof. For $V \in [H_0^1(\Omega)]^6$, let $U^* \in [H^2(\Omega)]^6$ be the solution of (4.10) with $Q = 0$. Then,

$$\begin{aligned}
&|(\mathscr{L}(U - U_h^s), V)_{0,\Omega}| \\
&= |(\mathscr{L}(U - U_h^s), \mathscr{L}U^*)_{0,\Omega}| \\
&= |(\mathscr{L}(U - U_h^s), \mathscr{L}(U^* - V_h))_{0,\Omega}|, \quad \forall V_h \in \mathscr{V}_h^s \quad \text{(by (3.24))}
\end{aligned}$$

$$\leqslant C \|\mathscr{L}(U - U_h^{\mathrm{s}})\|_{0,\Omega} \|\mathscr{L}(U^* - V_h)\|_{0,\Omega}, \quad \forall V_h \in \mathscr{V}_h^{\mathrm{s}}$$

$$\leqslant C \|U - U_h^{\mathrm{s}}\|_{1,\Omega} \|U^* - V_h\|_{1,\Omega}, \quad \forall V_h \in \mathscr{V}_h^{\mathrm{s}}$$

$$\leqslant Ch \|U - U_h^{\mathrm{s}}\|_{1,\Omega} \|U^*\|_{2,\Omega}, \quad (\text{by } (3.6))$$

$$\leqslant Ch \|U - U_h^{\mathrm{s}}\|_{1,\Omega} \|\mathscr{L}U^*\|_{1,\Omega}, \quad (\text{by } (4.2))$$

$$= Ch \|U - U_h^{\mathrm{s}}\|_{1,\Omega} \|V\|_{1,\Omega}.$$

In addition, the L^2-inner product $(\mathscr{L}(U - U_h^{\mathrm{s}}), V)_{0,\Omega}$ defines a bounded linear functional on $[H_0^1(\Omega)]^6$, since

$$|(\mathscr{L}(U - U_h^{\mathrm{s}}), V)_{0,\Omega}| \leqslant \|\mathscr{L}(U - U_h^{\mathrm{s}})\|_{0,\Omega} \|V\|_{1,\Omega}, \qquad \forall V \in [H_0^1(\Omega)]^6.$$

Therefore, by the definition of the $\|\cdot\|_{-1,\Omega}$-norm,

$$\|\mathscr{L}(U - U_h^{\mathrm{s}})\|_{-1,\Omega} \leqslant Ch \|U - U_h^{\mathrm{s}}\|_{1,\Omega}. \tag{4.12}$$

The proof is completed by combining (4.12), (4.4), and (4.9). □

4.3. Error estimates for the WLSFEM

Following the techniques developed in [43, pp. 352–356], we shall first present the optimal L^2-estimates and then the optimal H^1-estimates for the WLSFEM.

Similar to the proof of part (v) in Theorem 3.2, we note that, for any $W \in [H^{p+1}(\Omega)]^6$, $p \geqslant 0$ integer, there exists $W_h \in \mathscr{V}_h^{\mathrm{w}}$ such that

$$\|W - W_h\|_{a^{\mathrm{w}}} \leqslant Ch^p \|W\|_{p+1,\Omega}, \tag{4.13}$$

where C is a positive constant independent of W and h.

Theorem 4.4. *Let* $U \in \mathscr{V}^{\mathrm{w}} \cap [H^{p+1}(\Omega)]^6$, $U_h^{\mathrm{w}} \in \mathscr{V}_h^{\mathrm{w}}$ *be the solutions of* (2.13a/b′) *and* (3.20), *respectively. Assume that the regularity assumption of* (4.10) *holds, then*

$$\|U - U_h^{\mathrm{w}}\|_{0,\Omega} \leqslant Ch^{p+1} \|U\|_{p+1,\Omega}. \tag{4.14}$$

Proof. For $V \in [H_0^1(\Omega)]^6$, let $U^* \in [H^2(\Omega)]^6$ be the solution of (4.10) with $Q = 0$. Then,

$$|(\mathscr{L}(U - U_h^{\mathrm{w}}), V)_{0,\Omega}|$$

$$= |(\mathscr{L}(U - U_h^{\mathrm{w}}), \mathscr{L}U^*)_{0,\Omega}|$$

$$= |a^{\mathrm{w}}(U - U_h^{\mathrm{w}}, U^*)|$$

$$= |a^{\mathrm{w}}(U - U_h^{\mathrm{w}}, U^* - V_h)|, \quad \forall V_h \in \mathscr{V}_h^{\mathrm{w}}, \quad (\text{by } (3.28))$$

$$\leqslant \{a^{\mathrm{w}}(U - U_h^{\mathrm{w}}, U - U_h^{\mathrm{w}})\}^{1/2} \{a^{\mathrm{w}}(U^* - V_h, U^* - V_h)\}^{1/2}, \quad \forall V_h \in \mathscr{V}_h^{\mathrm{w}}$$

$$\leqslant Ch \{a^{\mathrm{w}}(U - U_h^{\mathrm{w}}, U - U_h^{\mathrm{w}})\}^{1/2} \|U^*\|_{2,\Omega} \quad (\text{by } (4.13))$$

$$\leqslant Chh^{p}\|U\|_{p+1,\Omega}\|\mathscr{L}U^{*}\|_{1,\Omega} \quad \text{(by (3.30), (4.5))}$$
$$= Ch^{p+1}\|U\|_{p+1,\Omega}\|V\|_{1,\Omega}.$$

Therefore, we have

$$\|\mathscr{L}(U - U_{h}^{\mathrm{w}})\|_{-1,\Omega} \leqslant Ch^{p+1}\|U\|_{p+1,\Omega}. \tag{4.15}$$

On the other hand, take $V = \mathbf{0} \in [H_{0}^{1}(\Omega)]^{6}$ in (4.10), then for any $Q \in [H^{1/2}(\partial\Omega)]^{3}$,

$$|h^{-1}(\mathscr{R}(U - U_{h}^{\mathrm{w}}), Q)_{0,\partial\Omega}|$$
$$= |h^{-1}(\mathscr{R}(U - U_{h}^{\mathrm{w}}), \mathscr{R}U^{*})_{0,\partial\Omega}|$$
$$= |a^{\mathrm{w}}(U - U_{h}^{\mathrm{w}}, U^{*})|$$
$$= |a^{\mathrm{w}}(U - U_{h}^{\mathrm{w}}, U^{*} - V_{h})|, \quad \forall V_{h} \in \mathscr{V}_{h}^{\mathrm{w}}, \quad \text{(by (3.28))}$$
$$\leqslant \{a^{\mathrm{w}}(U - U_{h}^{\mathrm{w}}, U - U_{h}^{\mathrm{w}})\}^{1/2}\{a^{\mathrm{w}}(U^{*} - V_{h}, U^{*} - V_{h})\}^{1/2}, \quad \forall V_{h} \in \mathscr{V}_{h}^{\mathrm{w}}$$
$$\leqslant C\{a^{\mathrm{w}}(U - U_{h}^{\mathrm{w}}, U - U_{h}^{\mathrm{w}})\}^{1/2}\|U^{*}\|_{1,\Omega} \quad \text{(by (4.13))}$$
$$\leqslant Ch^{p}\|U\|_{p+1,\Omega}\|\mathscr{R}U^{*}\|_{1/2,\partial\Omega} \quad \text{(by (3.30),(4.6))}$$
$$= Ch^{p}\|U\|_{p+1,\Omega}\|Q\|_{1/2,\partial\Omega}.$$

Thus, for any $Q \in [H^{1/2}(\partial\Omega)]^{3}$ we have

$$|(\mathscr{R}(U - U_{h}^{\mathrm{w}}), Q)_{0,\partial\Omega}| \leqslant Ch^{p+1}\|U\|_{p+1,\Omega}\|Q\|_{1/2,\partial\Omega}.$$

Hence,

$$\|\mathscr{R}(U - U_{h}^{\mathrm{w}})\|_{-1/2,\partial\Omega} \leqslant Ch^{p+1}\|U\|_{p+1,\Omega}. \tag{4.16}$$

The proof is completed by combining (4.7), (4.15), and (4.16). $\quad\square$

Note that in the proof of Theorem 4.4, we utilize the estimate (3.30) to circumvent the use of the optimal H^{1}-estimates which is not yet established. In order to give the optimal H^{1}-estimates, we need to define the following Gauss projection [43]:

$$\mathscr{G}: \mathscr{V}^{\mathrm{w}} \rightarrow \mathscr{V}_{h}^{\mathrm{w}}, \quad \mathscr{G}W \equiv W_{h}^{\mathrm{w}}, \tag{4.17}$$

where W_{h}^{w} is the solution of the discretized problem (3.20) corresponding to problem (2.13a/b$'$) with suitable data function F such that its unique exact solution is W. Since problem (3.20) is uniquely solvable, the Gauss mapping is well defined, and we have

$$\mathscr{G}V_{h} = V_{h}, \quad \forall V_{h} \in \mathscr{V}_{h}^{\mathrm{w}}. \tag{4.18}$$

Taking $p = 0$ in (4.14), then we get

$$\|\mathscr{G}U\|_{0,\Omega} = \|U_{h}^{\mathrm{w}}\|_{0,\Omega}$$
$$\leqslant \|U\|_{0,\Omega} + \|U - U_{h}^{\mathrm{w}}\|_{0,\Omega}$$
$$\leqslant \|U\|_{0,\Omega} + Ch\|U\|_{1,\Omega}.$$

Thus, we can conclude that, for any $V \in \mathscr{V}^{\mathrm{w}}$,

$$\|\mathscr{G}V\|_{0,\Omega} \leqslant \|V\|_{0,\Omega} + Ch\|V\|_{1,\Omega}. \tag{4.19}$$

We also need the following inverse assumption on the finite element space $\mathscr{V}_h^{\mathrm{w}}$: there exists a constant $C > 0$ independent of h such that

$$\|V_h\|_{1,\Omega} \leqslant Ch^{-1}\|V_h\|_{0,\Omega}, \quad \forall V_h \in \mathscr{V}_h^{\mathrm{w}}. \tag{4.20}$$

The inverse assumption is commonly used in many least-squares finite element analyses [4, 43]. More precisely, if the regular family $\{\mathscr{T}_h\}$ of triangulations of $\bar{\Omega}$ associated with the finite element space $\mathscr{V}_h^{\mathrm{w}}$ is quasi-uniform [27, 37], i.e., there exists a positive constant C independent of h such that

$$h \leqslant C \ \mathrm{diam}(\Omega_i^h), \quad \forall \Omega_i^h \in \mathscr{T}_h, \ \mathscr{T}_h \in \{\mathscr{T}_h\},$$

then (4.20) is satisfied.

The optimal order of convergence for the WLSFEM in the H^1-norm is thus concluded.

Theorem 4.5. *Let* $U \in \mathscr{V}^{\mathrm{w}} \cap [H^{p+1}(\Omega)]^6$, $U_h^{\mathrm{w}} \in \mathscr{V}_h^{\mathrm{w}}$ *be the solutions of* (2.13a/b′) *and* (3.20), *respectively. Suppose that the regularity assumption of* (4.10) *and the inverse assumption* (4.20) *hold, then*

$$\|U - U_h^{\mathrm{w}}\|_{1,\Omega} \leqslant Ch^p \|U\|_{p+1,\Omega}. \tag{4.21}$$

Proof. By (4.18), (4.19), and the approximation property (3.12), we have

$$
\begin{aligned}
\|U - U_h^{\mathrm{w}}\|_{1,\Omega} &\leqslant \|U - V_h\|_{1,\Omega} + \|U_h^{\mathrm{w}} - V_h\|_{1,\Omega}, \quad \forall V_h \in \mathscr{V}_h^{\mathrm{w}} \\
&= \|U - V_h\|_{1,\Omega} + \|\mathscr{G}(U - V_h)\|_{1,\Omega}, \quad \forall V_h \in \mathscr{V}_h^{\mathrm{w}} \\
&\leqslant \|U - V_h\|_{1,\Omega} + Ch^{-1}\|\mathscr{G}(U - V_h)\|_{0,\Omega}, \quad \forall V_h \in \mathscr{V}_h^{\mathrm{w}} \\
&\leqslant \|U - V_h\|_{1,\Omega} + Ch^{-1}\{\|U - V_h\|_{0,\Omega} + Ch\|U - V_h\|_{1,\Omega}\} \\
&\leqslant Ch^p \|U\|_{p+1,\Omega}. \quad \square
\end{aligned}
$$

5. Numerical experiments

We shall give a simple example which will be solved by using the SLSFEM (3.16). Consider the displacement–stress elasticity Eq. (2.13a) supplemented with the homogeneous displacement boundary condition (2.13b′). Taking $\Omega = (0,1) \times (0,1)$ and choosing

$$f_1 = (\alpha + \mu)\pi^2 \sin(\pi x) \sin(\pi y) - (\lambda + \mu)\pi^2 \cos(\pi x) \cos(\pi y),$$

$$f_2 = (\alpha + \mu)\pi^2 \sin(\pi x) \sin(\pi y) - (\lambda + \mu)\pi^2 \cos(\pi x) \cos(\pi y),$$

Table 1
The SLSFE approximations with $E = 2.5$ and $v = 0.25$

$1/h$	$\|e\|_{0,\Omega}$	RelErr	Conv. rate	$\|e\|_{a^s}$	RelErr	Conv. rate
2	0.69012	$2.1431 \cdot 10^{-1}$	—	11.5018	$3.68523 \cdot 10^{-1}$	—
4	0.19007	$5.90235 \cdot 10^{-2}$	1.86	5.86070	$1.87780 \cdot 10^{-1}$	0.97
8	0.05076	$1.57625 \cdot 10^{-2}$	1.90	2.96601	$9.50326 \cdot 10^{-2}$	0.98
16	0.01304	$4.05067 \cdot 10^{-3}$	1.96	1.48868	$4.76981 \cdot 10^{-2}$	0.99
32	0.00330	$1.02456 \cdot 10^{-3}$	1.98	0.74511	$2.38739 \cdot 10^{-2}$	1.00

Table 2
Rates of convergence in the a^s-norm with $E = 2.5$

$1/h$	$v = 0.05$	$v = 0.15$	$v = 0.35$	$v = 0.45$	$v = 0.49$	$v = 0.499$	$v = 0.4999$
2	—	—	—	—	—	—	—
4	0.98	0.98	0.96	0.93	0.90	0.90	0.89
8	0.99	0.99	0.98	0.96	0.95	0.95	0.95
16	1.00	1.00	0.99	0.99	0.98	0.98	0.98
32	1.00	1.00	1.00	1.00	0.99	0.99	0.99

the exact solution is then given by

$$\varphi_1 = \pi \cos(\pi x) \sin(\pi y), \quad \varphi_2 = \pi \sin(\pi x) \cos(\pi y),$$

$$\varphi_3 = \pi \sin(\pi x) \cos(\pi y), \quad \varphi_4 = \pi \cos(\pi x) \sin(\pi y),$$

$$u_1 = \sin(\pi x) \sin(\pi y), \quad u_2 = \sin(\pi x) \sin(\pi y).$$

To simplify the numerical implementation, we shall assume that the square domain Ω is uniformly partitioned into a set of $1/h^2$ square subdomains Ω_i^h with side-length h. Piecewise bilinear finite elements are used to approximate all components of the exact solution. For the case of Poisson's ratio $v = 0.25$ and Young's modulus $E = 2.5$, the results are collected in Table 1, where e denotes the exact error $U - U_h^s$ and RelErr denotes the relative error. Since the H^1-norm is equivalent to the a^s-norm for the standard least-squares case, Table 1 exhibits that the SLSFEM achieves optimal convergence both in the L^2-norm and in the H^1-norm for all the components.

The influence by the Poisson ratio v for the behavior of convergence is also examined. Tables 2 and 3 show that, except on very coarse meshes, the optimal convergence is still essentially insured for various Poisson ratios even for nearly incompressible elasticity. Table 4 shows that the convergence in the a^s-norm seems to be uniform in the Poisson ratio. It is not surprising since, roughly speaking, the a^s-norm can be viewed as the H^1-norm weighted appropriately by the Lamé coefficients λ and μ. However, we find that the situation is quite different for the full L^2-norm case since the values largely increase when v approaches to 0.5. Thus, one can conclude that in order to get optimal convergence in some Sobolev norm which is uniform in the Poisson ratio, the least-squares functionals need to be weighted appropriately by the Lamé coefficients (cf. [15, 17]).

Table 3
Rates of convergence in the L^2-norm with $E = 2.5$

1/h	$v = 0.05$	$v = 0.15$	$v = 0.35$	$v = 0.45$	$v = 0.49$	$v = 0.499$	$v = 0.4999$
2	—	—	—	—	—	—	—
4	1.82	1.83	1.85	1.57	1.47	1.45	1.45
8	1.90	1.91	1.86	1.77	1.74	1.73	1.73
16	1.96	1.96	1.95	1.91	1.89	1.89	1.89
32	1.98	1.98	1.98	1.97	1.97	1.96	1.96

Table 4
The values of $h^{-1}\|e\|_{a^s}\|U\|_{a^s}^{-1}$

1/h	$v = 0.05$	$v = 0.15$	$v = 0.35$	$v = 0.45$	$v = 0.49$	$v = 0.499$	$v = 0.4999$
2	0.758	0.748	0.735	0.763	0.789	0.797	0.797
4	0.767	0.759	0.756	0.803	0.844	0.856	0.856
8	0.774	0.767	0.768	0.824	0.874	0.889	0.890
16	0.777	0.769	0.772	0.832	0.885	0.900	0.902
32	0.777	0.770	0.773	0.834	0.888	0.904	0.905

6. Concluding remarks

In this paper, a new first-order displacement–stress formulation for the elasticity equations is introduced. Standard and weighted LSFEMs are proposed and analyzed. Convergence results for both methods are established in the natural norms associated with the least-squares bilinear forms. Furthermore, with the displacement boundary conditions, both the methods achieve optimal rates of convergence in the H^1-norm and in the L^2-norm for all the unknowns. Numerical experiments with various Poisson ratios are given to demonstrate the theoretical analysis.

Although it is interesting to note that the results of Tables 2 and 3 do not deteriorate as the Poisson ratio v tends to 0.5, we cannot say that the least-squares methods for the elasticity problem by using the new first-order system formulation avoid the locking phenomenon [2, 7, 8]. However, utilizing the techniques developed in the appendix in [42] and weighting the least-squares functionals by suitable parameters as that in [15, 17], a theoretical verification about possible improvement in regard to the locking problem based on the present first-order formulation appears to be promising. In this case, the auxiliary variables (2.3)–(2.6) may be replaced by $\varphi_1 = \partial u_1/\partial x$, $\varphi_2 = \partial u_1/\partial y$, $\varphi_3 = \partial u_2/\partial x$, and $p = -(v/1 - 2v)\nabla \cdot u$. This issue has become the subject of a separate investigation in progress.

While the basic convergence theory derived in Section 3 works well for the system (2.13a) with the displacement–stress boundary conditions (2.15) and (2.16), the error estimates developed in Section 4 may not cover this type of boundary conditions since the boundary-value problem (2.13a/b) with measure(Γ_1) > 0 does not satisfy the Lopatinski condition. However, it is possible to decompose the system (2.13a) into two subsystems. One is the stress system (2.7)–(2.10) and the other is the displacement system (2.11)–(2.12) or (2.3)–(2.6). The optimal convergence properties

for both methods may then be retained for the more general boundary conditions by means of the two-stage techniques [6, 18, 19].

Acknowledgements

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