Conditional Diagnosability Measures for Large Multiprocessor Systems

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Abstract—Diagnosability has played an important role in the reliability of an interconnection network. The classical problem of fault diagnosis is discussed widely and the diagnosability of many well-known networks have been explored. In this paper, we introduce a new measure of diagnosability, called conditional diagnosability, by restricting that any faulty set cannot contain all the neighbors of any vertex in the graph. Based on this requirement, the conditional diagnosability of the n-dimensional hypercube is shown to be $4(n-2)+1$, which is about four times as large as the classical diagnosability. Besides, we propose some useful conditions for verifying if a system is t-diagnosable and introduce a new concept, called a strongly t-diagnosable system, under the PMC model. Applying these concepts and conditions, we investigate some t-diagnosable networks which are also strongly t-diagnosable.

Index Terms—PMC model, diagnosability, t-diagnosable, strongly t-diagnosable, conditional faulty set, conditional diagnosability.

1 Introduction

High-performance signal processing architectures have become quite common with continuing advances in semiconductor technology. These architectures are used in several real-time applications and in high-performance large multiprocessor systems. However, the complexity of these systems can adversely affect the reliability. Therefore, the testing and diagnosis of these systems become an important aspect of system design.

The hypercube structure [24] is a well-known interconnection model for multiprocessor systems. Fault-tolerant computing for the hypercube structure has been of interest to many researchers. A hypercube of dimension n, denoted by Q_n, is an undirected graph consisting of 2^n vertices and $n2^{n-1}$ edges. The hypercube Q_1 is a complete graph K_2 with two vertices $\{0,1\}$. For $n \geq 2$, Q_n is constructed from two copies of Q_{n-1} by adding a perfect matching between them. Each vertex u of Q_n can be distinctly labeled by a binary n-bit string, $u_{n-1}u_{n-2}\dots u_1u_0$. There is an edge between two vertices if and only if their binary labels differ in exactly one bit position.

There are several variations of the hypercube, for example, the Crossed cube [6], the Twisted cube [13], and the Möbius cube [3]. For each of these cubes, an n-dimensional cube can be constructed from two copies of $(n-1)$-dimensional subcubes by adding a perfect matching between the two subcubes. The main difference is that each of these cubes has various perfect matching between its subcubes. An n-dimensional cube has 1) 2^n vertices, 2) connectivity n, and 3) each vertex has the same degree n (the two terms connectivity and degree will be defined subsequently). We define the *cube family* to include all such cubes which are constructed recursively by joining two subcubes with a perfect matching. For $n = 0$, 1, and 2, an n-dimensional cube is a single vertex, an edge, and a cycle of length four, respectively.

In this paper, we use the widely adopted PMC model [23] as the fault diagnosis model. In [11], Hakimi and Amin proved that a multiprocessor system is t-diagnosable if it is t-connected with at least $2t+1$ vertices. Besides, they gave a necessary and sufficient condition for verifying if a system is t-diagnosable under the PMC model. In this paper, we also propose a new necessary and sufficient condition, namely, Theorem 2, which will be useful from the graph theoretical point of view.

Reviewing the previous papers [1], [2], [9], [10], [11], [14], [15], [24], the Hypercube Q_n, the Crossed cube CQ_n, the Möbius cube MQ_n, and the Twisted cube TQ_n, all have diagnosability n under the PMC model. Moreover, we observe that they are almost $(n+1)$-diagnosable except for the case where all the neighbors of some vertex are faulty simultaneously. Closely related to this observation, we introduce the concept of a strongly t-diagnosable system and propose some conditions to assure which networks are strongly t-diagnosable.

The connectivity of a system is an important measure of fault tolerance. It is well-known that, for a system G, the connectivity of G is less than or equal to its minimum degree (this term will be defined subsequently). For example, the hypercube Q_n has connectivity n and this value n is equal to its minimum degree n. However, a scalable hypercube multiprocessor system can consist of thousands of processors. Under this complicated environment, more processors are likely to fail. To explore a more proper measure of fault tolerance, the conditional connectivity has been investigated in several research works [7], [12], [17], [22], [25].

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Under the classical PMC diagnosis model, only processors with direct connections are allowed to test one another. Given a system, if all the adjacent neighbors of a processor v are faulty simultaneously, it is not possible to determine whether processor v is fault-free or faulty. Hence, for most practical systems that are sparsely connected, only a small number of faulty processors can be recognized with the classical diagnosis model. So, it is an interesting problem to explore some measures for better reflecting fault patterns in a real system than the existing ones. For example, Das et al. [5] investigated fault diagnosis with local constraints.

In this paper, we propose a new measure of diagnosability, called conditional diagnosability, and study the conditional diagnosability of the hypercube. In classical measures of system-level diagnosability for multiprocessor systems, it has generally been assumed that any subset of processors can potentially fail at the same time. As a consequence, the diagnosability of a system is upper bounded by its minimum degree. We then consider these measures by restricting that, for each processor v in the network, all the processors which are directly connected to v do not fail at the same time. Under this condition, we show that the conditional diagnosability of Q_n is $4(n-2)+1$, which is about four times larger than that of the classical diagnosability of Q_n.

The rest of this paper is organized as follows: Section 2 provides terminology and preliminaries for diagnosing a system. Section 3 introduces the concept of a system being strongly *t*-diagnosable and proposes some necessary and sufficient conditions to check if a system is so. We then define conditional diagnosability and study the conditional diagnosability of Q_n in Section 4. Finally, our conclusions are given in Section 5.

2 Terminology and Preliminaries

A system or a network is usually represented by a graph. Throughout this paper, we follow [8] for the graph definition and focus on undirected graph without loops (simply abbreviated as graph).

Definition 1 [8]. *The components of a graph G are its maximal connected subgraphs. A component is trivial if it has no edges; otherwise, it is nontrivial.*

The degree of a vertex v in a graph G, written as $deg_G(v)$ or $deg(v)$, is the number of edges incident to v. The maximum degree is denoted by $\triangle(G)$, the minimum degree is $\delta(G)$, and G is regular if $\triangle(G) = \delta(G)$. It is *k*-regular if the common degree is k. The neighborhood of v, written $N_G(v)$ or $N(v)$, is the set of vertices adjacent to v. The connectivity $\kappa(G)$ of a graph $G(V,E)$ is the minimum number of vertices whose removal results in a disconnected or a trivial graph. A graph G is *k*-connected if its connectivity is at least k.

Let $G = G(V,E)$ (simply abbreviated as G) be a graph. For a set $S \subset V$, the notation $G - S$ represents the graph obtained by removing the vertices in S from G and deleting those edges with at least one end vertex in S simultaneously. If $G - S$ is disconnected, then S is called a vertex *cut* or a *separating set*. Let G_1, G_2 be two subgraphs of G; if there are ambiguities, we shall write the vertex set of G_1 as V_{G_1} or $V(G_1)$. The neighborhood set of the vertex set V_{G_1} is defined as $N(V_{G_1}) = \{y \in V(G)|$, there exists a vertex $x \in V_{G_1}$ such that $(x,y) \in E(G)\} - V_{G_1}$. The restricted neighborhood set of V_{G_1} in G_2 is defined as $N(V_{G_1}, G_2) = \{y \in V(G_2)|$, there exists a vertex $x \in V_{G_1}$ such that $(x,y) \in E(G)\} - V_{G_1}$. We use $|X|$ to denote the cardinality of set X. The restricted degree of a vertex v in a subgraph G_1 is defined as

$$deg_{G_1}(v) = |N(\{v\}, G_1)|.$$

A multiprocessor system is modeled as an undirected graph $G = G(V,E)$ whose vertices represent processors and edges represent communication links. Under the classical PMC model [23], adjacent processors are capable of performing tests on each other. For adjacent vertices $u, v \in V$, the ordered pair (u,v) represents the test performed by u on v. In this situation, u is called the *tester* and v is called the *tested vertex*. The outcome of a test (u,v) is 1 (respectively, 0) if u evaluates v as faulty (respectively, fault-free).

A *test assignment* for a system $G = G(V,E)$ is a collection of tests (u,v) for some adjacent pairs of vertices. It can be modeled as a directed graph $T = (V,L)$, where $(u,v) \in L$ implies that u and v are adjacent in G. Throughout this paper, we assume that each vertex tests the other whenever there is an edge between them and all these tests are gathered in test assignment.

The collection of all test results for a test assignment T is called a *syndrome*. Formally, a syndrome is a function $\sigma : L \rightarrow \{0,1\}$. The set of all faulty processors in the system is called a *faulty set*. This can be any subset of V. The process of identifying all the faulty vertices is called the *diagnosis* of the system. The maximum number of faulty vertices that the system G can guarantee to identify is called the *diagnosability* of G, written as $t(G)$.

For a given syndrome σ, a subset of vertices $F \subseteq V$ is said to be *consistent* with σ if syndrome σ can be produced from the situation that, for any $(u,v) \in L$ such that $u \in V - F, \sigma(u,v) = 1$ iff $v \in F$. Because a faulty tester can lead to an unreliable result, a given set F of faulty vertices may produce different syndromes. Let $\sigma(F)$ represent the set of all syndromes which could be produced if F is the set of faulty vertices.

Two distinct sets $F_1, F_2 \subset V$ are said to be *indistinguishable* if $\sigma(F_1) \bigcap \sigma(F_2) \neq \emptyset$; otherwise, F_1, F_2 are said to be *distinguishable*. We say (F_1, F_2) is an *indistinguishable pair* if $\sigma(F_1) \bigcap \sigma(F_2) \neq \emptyset$, else, (F_1, F_2) is a *distinguishable pair*.

Some known results about the definition of a *t*-diagnosable system and related concepts are listed as follows. Some of these previous results are on directed graphs and others are on undirected graphs.

Definition 2 [23]. *A system of n units is t-diagnosable if all faulty units can be identified without replacement, provided that the number of faults presented does not exceed t.*

Let $F_1, F_2 \subset V$ be two distinct sets and let the symmetric difference $F_1 \triangle F_2 = (F_1 - F_2) \bigcup (F_2 - F_1)$. DahBura and Masson [4] proposed a polynomial time algorithm to check whether a system is *t*-diagnosable.

Lemma 1 [4]. *A system $G(V,E)$ is t-diagnosable if and only if, for each pair $F_1, F_2 \subset V$ with $|F_1|, |F_2| \leq t$ and $F_1 \neq F_2$, there is at least one test from $V - (F_1 \bigcup F_2)$ to $F_1 \triangle F_2$.*

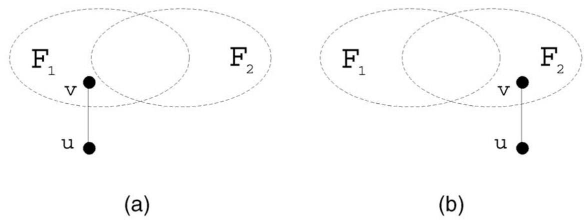

Fig. 1. Illustrations of a distinguishable pair (F_1, F_2).

The following two results related to t-diagnosable systems are due to Hakimi and Amin [11] and Preparata et al. [23], respectively.

Lemma 2 [23]. *Let $G(V,E)$ be the graph representation of a system G, with V representing the processors and E the interconnection among them. Let $|V| = n$. The following two conditions are necessary for G to be t-diagnosable:*

1. $n \geq 2t + 1$ *and*
2. *Each processor is tested by at least t other processors.*

Lemma 3 [11]. *The following two conditions are sufficient for a system G of n processors to be t-diagnosable:*

1. $n \geq 2t + 1$ *and*
2. $\kappa(G) \geq t$.

For a directed graph G and a vertex $v \in V(G)$, the notation $d_{in}(v)$ is used to denote the number of edges directed toward v in G. Let $\Gamma(v) = \{v_i | (v, v_i) \in E\}$ and $\Gamma(X) = \bigcup_{v \in X} \Gamma(v) - X,\ X \subset V$. Hakimi and Amin presented a necessary and sufficient condition for a system G to be t-diagnosable as follows:

Theorem 1 [11]. *Let $G(V,E)$ be the directed graph of a system G with n units. Then, G is t-diagnosable if and only if: 1) $n \geq 2t+1$, 2) $d_{in}(v) \geq t$ for all $v \in V$, and 3) for each integer p with $0 \leq p \leq t-1$ and each $X \subset V$ with $|X| = n - 2t + p$, $|\Gamma(X)| > p$.*

In this paper, we propose some new viewpoints on diagnosis and we will focus on undirected graph (simply abbreviated as graph). Let $G = G(V,E)$ be an undirected graph. The following lemma follows directly from Lemma 1.

Lemma 4. *For any two distinct sets F_1, $F_2 \subset V$, (F_1, F_2) is a distinguishable pair if and only if there exists a vertex $u \in V - (F_1 \bigcup F_2)$ and there exists a vertex $v \in F_1 \triangle F_2$ such that $(u, v) \in E$ (see Fig. 1).*

It follows from Definition 2 that the following lemma holds.

Lemma 5. *A system is t-diagnosable if and only if, for each distinct pair of sets F_1, $F_2 \subset V$ with $|F_1| \leq t$ and $|F_2| \leq t$, F_1 and F_2 are distinguishable.*

An equivalent way of stating the above lemma is the following:

Lemma 6. *A system is t-diagnosable if and only if, for each indistinguishable pair of sets F_1, $F_2 \subset V$, it implies that $|F_1| > t$ or $|F_2| > t$.*

By Lemma 2, a similar result for undirected graph is stated as follows.

Corollary 1 [23]. *Let $G(V,E)$ be an undirected graph. The following two conditions are necessary for G to be t-diagnosable:*

1. $n \geq 2t + 1$ *and*
2. $\delta(G) \geq t$.

For our discussion later, an alternative characterization of the t-diagnosable system is given below.

Theorem 2. *Let $G(V,E)$ be the graph of a system G. Then, G is t-diagnosable if and only if, for each vertex set $S \subset V$ with $|S| = p$, $0 \leq p \leq t-1$, every component C of $G - S$ satisfies $|V_C| \geq 2(t-p) + 1$.*

Proof. To prove that $|V_C| \geq 2(t-p) + 1$ is necessary, we show this by contradiction. Then, there exists a set of vertices $S \subset V$ with $|S| = p$, $0 \leq p \leq t-1$, such that one of the components $G - S$ has strictly less than $2(t-p) + 1$ vertices. Let C be such a component with $|V_C| \leq 2(t-p)$. We then arbitrarily partition V_C into two disjoint subsets, $V_C = A_1 \bigcup A_2$ with $|A_1| \leq t-p$ and $|A_2| \leq t-p$. Let $F_1 = A_1 \bigcup S$ and $F_2 = A_2 \bigcup S$. Then, $|F_1| \leq t$ and $|F_2| \leq t$. It is clear that there is no edge between $V - (F_1 \bigcup F_2)$ and $F_1 \triangle F_2$. By Lemma 4, F_1 and F_2 are indistinguishable. This contradicts the assumption that G is t-diagnosable.

To prove the sufficiency, suppose, on the contrary, that G is not t-diagnosable, i.e., there exists an indistinguishable pair (F_1, F_2) with $|F_i| \leq t$, $i = 1, 2$. By Lemma 4, there is no edge between $V - (F_1 \bigcup F_2)$ and $F_1 \triangle F_2$. Let $S = F_1 \bigcap F_2$. Thus, in $G - S$, $F_1 \triangle F_2$ is disconnected from other parts. We observe that $|F_1 \triangle F_2| \leq 2(t-p)$, where $|S| = p$ and $0 \leq p \leq t-1$. Therefore, there is at least one component C of $G - S$ with $|V_C| \leq 2(t-p)$, which is a contradiction. This completes the proof of the theorem. □

3 Strongly t-diagnosable Systems

The Hypercube Q_n, the Crossed cube CQ_n, the Möbius cube MQ_n, and the Twisted cube TQ_n are all known to be n-connected but not $(n+1)$-connected. For each of these cubes, every vertex cut of size n has a particular structure, as stated in the following lemma.

Lemma 7. *Let $n \geq 2$ and let XQ_n represent any n-dimensional cube which belongs to the cube family. For each set of vertices $S \subset V(XQ_n)$ with $|S| = n$, if $XQ_n - S$ is disconnected, there exists a vertex $v \in V(XQ_n)$ such that $N(v) = S$.*

Proof. We prove this lemma by induction on n. A two-dimensional cube XQ_2 is simply a cycle of length four. Clearly, this lemma is true for XQ_2. Assume it holds for some $n \geq 2$. We now show that it holds for $n+1$.

Let an $(n+1)$-dimensional cube XQ_{n+1} be obtained from two n-dimensional cubes XQ_n, denoted by XQ_n^L and XQ_n^R, by adding a perfect matching between them. Let $S \subset V(XQ_{n+1})$, $|S| = n+1$, and $S_L = V(XQ_n^L) \bigcap S$ and $S_R = V(XQ_n^R) \bigcap S$. In the remainder of this proof, we show that XQ_{n+1} satisfies one of the two conditions: 1) $XQ_{n+1} - S$ is connected,or 2) $XQ_{n+1} - S$ is disconnected and there is a vertex $v \in V(XQ_{n+1})$ such that $N(v) = S$.

We study three cases: 1) $|S_L| \le n-1$ and $|S_R| \le n-1$, 2) either $|S_L| = n$ or $|S_R| = n$, and 3) either $|S_L| = n+1$ or $|S_R| = n+1$.

Case 1: $|S_L| \le n-1$ and $|S_R| \le n-1$.

Since XQ_n is *n*-connected, both $XQ_n^L - S_L$ and $XQ_n^R - S_R$ are connected. For $n \ge 2$, we know that

$$|V(XQ_n^L) - S_L| \ge 2^n - (n-1) > n-1 \ge |S_R|$$

and

$$|V(XQ_n^R) - S_R| \ge 2^n - (n-1) > n-1 \ge |S_L|.$$

So, the subgraph $XQ_n^L - S_L$ is connected to the other subgraph $XQ_n^R - S_R$. Hence, $XQ_{n+1} - S$ is connected.

Case 2: Either $|S_L| = n$ or $|S_R| = n$.

Without loss of generality, suppose that $|S_L| = n$ and $|S_R| = 1$. Suppose $XQ_n^L - S_L$ is connected. Using a similar argument to that used in Case 1, we can prove that $XQ_{n+1} - S$ is connected. Otherwise, $XQ_n^L - S_L$ is disconnected. By induction hypothesis, there exists a vertex $v \in V(XQ_n^L)$ such that $N(\{v\}, XQ_n^L) = S_L$. Now, consider XQ_n^R and consider the matching neighbor u of v in XQ_n^R. Note that $XQ_n^R - S_R$ is connected for $n \ge 2$ and every vertex in XQ_n^R has a matching neighbor in XQ_n^L. Thus, $XQ_{n+1} - S$ is connected if $S_R \ne \{u\}$. If $S_R = \{u\}$, $XQ_{n+1} - S$ is disconnected and $S = N(v)$. This proves Case 2.

Case 3: Either $|S_L| = n+1$ or $|S_R| = n+1$.

Without loss of generality, suppose that $|S_L| = n+1$ and $|S_R| = 0$. Since there is one corresponding matched vertex for each vertex $v \in V(XQ_n^L - S_L)$ in $V(XQ_n^R)$, $XQ_{n+1} - S$ is connected.

Consequently, this lemma holds. □

Let F_1 and F_2 be two distinct sets of vertices of XQ_n with $|F_i| \le n+1$, $i = 1, 2$, and let $S = F_1 \bigcap F_2$. Then, $|S| \le n$. By the above lemma, either $XQ_n - S$ is connected or $XQ_n - S$ is disconnected and there is a vertex $v \in V(XQ_n)$ such that $S = N(v)$. If $XQ_n - S$ is connected, the two sets $V(XQ_n) - (F_1 \bigcup F_2)$ and $F_1 \triangle F_2$ both belong to the same component $XQ_n - S$. Thus, there exists one edge connecting $V(XQ_n) - (F_1 \bigcup F_2)$ and $F_1 \triangle F_2$. By Lemma 4, F_1 and F_2 are distinguishable. Therefore, if F_1 and F_2 are indistinguishable, $|F_i| \le n+1$, $i = 1, 2$, $XQ_n - S$ is disconnected, and there exists a vertex v such that $S = N(v)$. $S = F_1 \bigcap F_2$, so $N(v) \subseteq F_1$ and $N(v) \subseteq F_2$. We then propose the following concept.

Definition 3. *A system G is strongly t-diagnosable if the following two conditions hold:*

1. *G is t-diagnosable and*
2. *For any two distinct subsets F_1, $F_2 \subset V(G)$ with $|F_i| \le t+1$, $i = 1, 2$, either*

   a. *(F_1, F_2) is a distinguishable pair or*
   b. *(F_1, F_2) is an indistinguishable pair*

   and there exists a vertex $v \in V$ such that $N(v) \subseteq F_1$ and $N(v) \subseteq F_2$.

A $(t+1)$-diagnosable system is "stronger" than a t-diagnosable system and, of course, it is strongly t-diagnosable according to the above definition. However, among all those strongly t-diagnosable systems, we are interested in the one which is t-diagnosable but not $(t+1)$-diagnosable.

Following Lemma 3 and Definition 3, we propose a sufficient condition for verifying if a system G is strongly t-diagnosable.

Proposition 1. *A system $G(V, E)$ with n vertices is strongly t-diagnosable if the following three conditions hold:*

1. *$n \ge 2(t+1) + 1$,*
2. *$\kappa(G) \ge t$, and*
3. *for any vertex set $S \subset V$ with $|S| = t$, if $G - S$ is disconnected, there exists a vertex $v \in V$ such that $N(v) \subset S$.*

Proof. With conditions 1 and 2, by Lemma 3, G is t-diagnosable. Now, we want to prove condition 2 of Definition 3 holds. Let $F_1, F_2 \subset V$ be two distinct sets with $|F_i| \le t+1$, $i = 1, 2$, and $S = F_1 \bigcap F_2$. Suppose that $G - S$ is connected. Then, there exists one edge connecting $V - (F_1 \bigcup F_2)$ and $F_1 \triangle F_2$. By Lemma 4, F_1 and F_2 are distinguishable. That is, condition 2.a of Definition 3 holds.

Otherwise, $G - S$ is disconnected. By condition 2, the connectivity of G is at least t, and $0 \le |S| \le t$, so $|S| = t$. Then, by condition 3, there exists one vertex $v \in V$ such that $N(v) \subset S$. Therefore, $N(v) \subset F_1$ and $N(v) \subset F_2$. So, condition 2.b of Definition 3 holds. This completes the proof of this proposition. □

Next, we present a necessary and sufficient condition for a system G to be strongly t-diagnosable.

Lemma 8. *A system $G(V, E)$ with $|V| = n$ is strongly t-diagnosable if and only if the following three conditions hold:*

1. *$n \ge 2(t+1) + 1$,*
2. *$\delta(G) \ge t$, and*
3. *for any two distinct subsets F_1, $F_2 \subset V(G)$ with $|F_i| \le t+1$, $i = 1, 2$, the pair (F_1, F_2) satisfy condition 2.a or 2.b of Definition 3.*

Proof. We first prove the necessity. To prove condition 1, we show that the assumption $n \le 2(t+1)$ leads to a contradiction. Assume $n \le 2(t+1)$. We can partition V into two disjoint vertex sets V_1 and V_2, $V_1 \bigcap V_2 = \emptyset$ and $V = V_1 \bigcup V_2$, with $|V_i| \le t+1$, $i = 1, 2$. By Lemma 4, V_1 and V_2 are indistinguishable. Since G is strongly t-diagnosable, by Definition 3, $N(v) \subset V_1$ and $N(v) \subset V_2$, for some vertex $v \in V$, contradicting the assumption $V_1 \bigcap V_2 = \emptyset$.

To prove condition 2, since G is strongly t-diagnosable, it is t-diagnosable by definition. Then, by condition 2 of Corollary 1, $N(v) \ge t$ for each vertex $v \in V$. So, condition 2 is necessary. Condition 3 of this lemma is the same as condition 2 of Definition 3. This proves the necessity.

To prove the sufficiency of conditions 1, 2, and 3, we need only show that G is t-diagnosable. Suppose not, then there exists an indistinguishable pair of sets $F_1, F_2 \subset V$, $F_1 \ne F_2$, and $|F_i| \le t$, $i = 1, 2$. By condition 2.b of

Definition 3, there exists a vertex $v \in V$ such that $N(v) \subset F_1$ and $N(v) \subset F_2$. By condition 2, $|N(v)| \geq t$. However, $|F_1| \leq t$ and $|F_2| \leq t$. Hence, $F_1 = F_2 = N(v)$. This contradicts the fact that $F_1 \neq F_2$. The lemma follows. □

We now give another necessary and sufficient condition for checking whether a system is strongly t-diagnosable. The motivation of these conditions is as follows: Let $G(V, E)$ be a strongly t-diagnosable system. Suppose that G is $(t+1)$-diagnosable. Then, by Theorem 2, for every set $S \subset V$, $0 \leq p \leq t$, where $|S| = p$, each component C of $G - S$ satisfies $|V_C| \geq 2((t+1) - p) + 1$. Otherwise, G is t-diagnosable, but not $(t+1)$-diagnosable. Then, there exists an indistinguishable pair (F_1, F_2), $F_1 \neq F_2$, with $|F_i| \leq t+1$, $i = 1, 2$. By condition 2.b of Definition 3, there exists a vertex $v \in V$ such that $N(v) \subset F_1$ and $N(v) \subset F_2$, $i = 1, 2$. Note that $\delta(G) \geq t$ and, therefore, $|N(v)| \geq t$. It means that $\{v\}$ is a trivial component of $G - (F_1 \bigcap F_2)$. Setting $S = F_1 \bigcap F_2$ and $|S| = t$, $G - S$ has a trivial component.

Theorem 3. *A system $G = (V, E)$ is strongly t-diagnosable if and only if, for each vertex set $S \subset V$ with cardinality $|S| = p$, $0 \leq p \leq t$, the following two conditions are satisfied:*

1. *For $0 \leq p \leq t-1$, every component C of $G - S$ satisfies $|V_C| \geq 2((t+1) - p) + 1$ and*
2. *for $p = t$, either a) every component C of $G - S$ satisfies $|V_C| \geq 3$ or else b) $G - S$ contains at least one trivial component. (Remark: $2((t+1) - p) + 1 = 3$ as $p = t$.)*

Proof. We use Theorem 2 to prove the sufficiency of conditions 1 and 2. Let S be a set of vertexices with $|S| = p$, $0 \leq p \leq t-1$. By condition 1, every component C of $G - S$ satisfies $|V_C| \geq 2((t+1) - p) + 1 \geq 2(t - p) + 1$. Then, by Theorem 2, G is t-diagnosable.

To show that G is strongly t-diagnosable, we need to prove that condition 2 of Definition 3 holds. Suppose that conditions 1 and 2.a are both satisfied. Then, by Theorem 2, G is $(t+1)$-diagnosable. Now, consider the case that G is not $(t+1)$-diagnosable. Let (F_1, F_2) be an indistinguishable pair, $F_1 \neq F_2$, with $|F_1| \leq t+1$ and $|F_2| \leq t+1$. We let $S = F_1 \bigcap F_2$ and $X = V - (F_1 \bigcup F_2)$, then $0 \leq p \leq t$, where $|S| = p$. Since F_1 and F_2 are indistinguishable, by Lemma 4, there is no edge between X and $F_1 \triangle F_2$. Therefore, in $G - S$, $F_1 \triangle F_2$ is disconnected from the other components. Observe that $|F_1 \triangle F_2| \leq 2((t+1) - p)$, by condition 1, p cannot be in the range from 0 to $t-1$. So, $p = t$ and $|F_1 \triangle F_2| \leq 2((t+1) - p) = 2((t+1) - t) = 2$. Then, by condition 2.b, $G - S$ must have a trivial component $\{v\}$. So, $N(v) \subset S$. G is t-diagnosable by condition 2 of Corollary 1, $|N(v)| \geq t$. Hence, $S = N(v)$. Since $S = F_1 \bigcap F_2$, $N(v) \subset F_1$, and $N(v) \subset F_2$. Therefore, G is strongly t-diagnosable.

This proves the sufficiency. Next, we show that conditions 1 and 2 are also necessary.

To show condition 1, suppose on the contrary that there exists a set of vertices $S \subset V$ with $|S| = p$, $0 \leq p \leq t-1$, such that $G - S$ has a component with strictly less than $2((t+1) - p) + 1$ vertices. Let C be such a component with $|V_C| \leq 2((t+1) - p)$. We can partition V_C into two disjoint subsets A_1 and A_2, $A_1 \bigcup A_2 = V_C$ and $A_1 \bigcap A_2 = \emptyset$, with $|A_i| \leq (t+1) - p$, $i = 1, 2$. Let $F_1 = A_1 \bigcup S$ and $F_2 = A_2 \bigcup S$. Then, $|F_i| \leq t+1$, $i = 1, 2$, and F_1 and F_2 are indistinguishable by Lemma 4. Since G is strongly t-diagnosable, by condition 2.b of Definition 3, there exists a vertex v such that $N(v) \subset F_1$ and $N(v) \subset F_2$. G is t-diagnosable, by Corollary 1, each vertex of G has degree at least t. So, $|N(v)| \geq t$. However, $N(v) \subset F_1 \bigcap F_2 = S$ and $|S| = p \leq t-1$; this is a contradiction. Thus, condition 1 is necessary.

Now, we prove that condition 2 is necessary. Let S be a set of vertex with $|S| = p$ and $p = t$. Suppose that G is $(t+1)$-diagnosable. By Theorem 2, for $p = t$, every component C of $G - S$ satisfies $|V_C| \geq 2((t+1) - t) + 1 = 3$. That is, condition 2.a holds if G is $(t+1)$-diagnosable. Otherwise, G is not $(t+1)$-diagnosable and there exists a component C in $G - S$ with strictly less than three vertices, $|V_C| \leq 2$. We have to show that there is a trivial component in $G - S$. If $|V_C| = 1$, we are done. Assume that $|V_C| = 2$, say, $V_C = \{v_1, v_2\}$. Let $F_1 = S \bigcup \{v_1\}$ and $F_2 = S \bigcup \{v_2\}$. Then, $|F_1| = t+1$, $|F_2| = t+1$, and F_1 and F_2 are indistinguishable. Since G is strongly t-diagnosable by condition 2.b of Definition 3, there exists a vertex v such that $N(v) \subset F_1$ and $N(v) \subset F_2$. We have $S = F_1 \bigcap F_2$ and $N(v) \subset S$. Therefore, $\{v\}$ is a trivial component in $G - S$; this proves condition 2.b.

Consequently, the theorem holds. □

The above theorem again states that a strongly t-diagnosable system is almost $(t+1)$-diagnosable, if it is not so. The only case that stops it from being $(t+1)$-diagnosable occurs in the following situation: All the neighboring vertices $N(v)$ of some vertex v are faulty simultaneously.

In previous studies, the diagnosability of many practical interconnection networks has been explored. Actually, some of them are not only n-diagnosable, but also strongly n-diagnosable, for example, the Hypercube Q_n, the Crossed cube CQ_n, the Möbius cube MQ_n, and the Twisted cube TQ_n are so. In the following, we shall prove that all members in the cube family are strongly n-diagnosable for $n \geq 4$.

A family of interconnection networks, called the *Matching Composition Networks* (*MCN*) [18], which can be constructed from two graphs G_1 and G_2 with the same number of vertices by adding a perfect matching M between the vertices of G_1 and G_2. We shall call these two graphs G_1 and G_2 the *M-components* of MCN. Formally, we use the notation $G_1 \bigoplus_M G_2$ to denote an MCN, which has vertex set $V(G_1 \bigoplus_M G_2) = V(G_1) \bigcup V(G_2)$ and edge set $E(G_1 \bigoplus_M G_2) = E(G_1) \bigcup E(G_2) \bigcup M$. MCN includes many well-known interconnection networks as special cases, such as the Hypercube Q_n, the Crossed cube CQ_n, the Twisted cube TQ_n, and the Möbius cube MQ_n.

Under the comparison model [19], [20], it is proven that a MCN with two t-connected and t-diagnosable M-components is $(t+1)$-diagnosable in [18]. In the following theorem, we shall show that an MCN with two t-diagnosable M-components is strongly $(t+1)$-diagnosable under the PMC model.

Theorem 4. *Let $G_1(V_1, E_1)$, $G_2(V_2, E_2)$ be two t-diagnosable systems with the same number of vertices, where $t \geq 2$. Then, MCN $G = G_1 \bigoplus_M G_2$ is strongly $(t+1)$-diagnosable.*

Proof. We use Theorem 3 to prove it. Let $G = G(V, E) = G_1 \bigoplus_M G_2$ and $S \subset V$ with $|S| = p$, $0 \leq p \leq t+1$. Let $S_1 = S \bigcap V_1$, $S_2 = S \bigcap V_2$, $|S_1| = p_1$, and $|S_2| = p_2$. In the following proof, we consider two cases: 1) $S_1 = \emptyset$ or $S_2 = \emptyset$ and 2) $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$. We shall prove that: i) $|V_C| \geq 2((t+2) - p) + 1$ for every component C of $G - S$ as $0 \leq p \leq t$ and ii) for $p = t+1$, either a) every component C of $G - S$ satisfies $|V_C| \geq 3$ or else b) $G - S$ contains at least one trivial component. Then, by Theorem 3, G is strongly $(t+1)$-diagnosable.

Case 1: $S_1 = \emptyset$ or $S_2 = \emptyset$.

Without loss of generality, assume $S_1 = \emptyset$ and $S_2 = S$. We know that each vertex of V_2 has an adjacent neighbor in V_1, so $G - S$ is connected. The only component C of $G - S$ is $G - S$ itself. Hence, $|V_C| = |V - S| = |V_1| + |V_2| - p$. G_i is t-diagnosable, $i = 1, 2$, by Corollary 1, $|V_i| \geq 2t + 1$. So, $|V_C| \geq 2(2t+1) - p \geq 2((t+2) - p) + 1$ for $t \geq 2$. That is, conditions 1 and 2.a of Theorem 3 are satisfied.

Case 2: $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$.

$S_1 \neq \emptyset$ and $S_2 \neq \emptyset$, it implies $p_1 \geq 1$ and $p_2 \geq 1$. Then, we divide the case into two subcases: 2.a) both $p_1 \leq t-1$ and $p_2 \leq t-1$ and 2.b) either $p_1 = t$ or $p_2 = t$. Note that $0 \leq p \leq t+1$ and $p = p_1 + p_2$. For subcase 2.a), $1 \leq p_1 \leq t-1$ and $1 \leq p_2 \leq t-1$ and, for subcase 2.b), either $p_1 = t$ and $p_2 = 1$ or $p_2 = t$ and $p_1 = 1$.

Subcase 2.a: $1 \leq p_1 \leq t-1$ and $1 \leq p_2 \leq t-1$.

Let C_1 be a component of $G_1 - S_1$. G_1 is t-diagnosable by Theorem 2, $|V_{C_1}| \geq 2(t - p_1) + 1$. We claim that $2(t - p_1) + 1 \geq p_2 + 1$. Since $p = p_1 + p_2$, $2(t - p_1) + 1 = 2(t - (p - p_2)) + 1 = 2p_2 + 2(t - p) + 1$. Suppose $p \leq t$, $|V_{C_1}| \geq 2p_2 + 1$. Otherwise, $p = t+1$. Since $p_1 \leq t-1$, $p_2 \geq 2$ and $2p_2 + 2(t-p) + 1 \geq p_2 + 1$. Hence, $|V_{C_1}| \geq 2(t - p_1) + 1 \geq p_2 + 1$. That is, V_{C_1} has at least one adjacent neighbor $v \in V_2$ and $v \notin S_2$. G_2 is t-diagnosable by Theorem 2, every component of $G_2 - S_2$ has at least $2(t - p_2) + 1$ vertices. Let C_2 be the component of $G_2 - S_2$ such that $v \in V_{C_2}$ and let C be the component of $G - S$ such that $V_{C_1} \bigcup V_{C_2} \subset V_C$. Then,

$$\begin{aligned} |V_C| &\geq |V_{C_1}| + |V_{C_2}| \geq (2(t - p_1) + 1) + (2(t - p_2) + 1) \\ &= 2(2t - p + 1) \geq 2((t+2) - p) + 1 \end{aligned}$$

as $t \geq 2$. So, every component of $G - S$ has at least $2((t+2) - p) + 1$ vertices in this subcase. It means that conditions 1 and 2.a of Theorem 3 are satisfied.

Subcase 2.b: Either $p_1 = t$ and $p_2 = 1$ or $p_2 = t$ and $p_1 = 1$.

Without loss of generality, assume $p_2 = t$ and $p_1 = 1$. Since $p = p_1 + p_2 = t+1$, we need only to prove either condition 2.a or 2.b of Theorem 3 holds. Let C_1 be a component of $G_1 - S_1$. G_1 is t-diagnosable by Theorem 2, $|V_{C_1}| \geq 2(t - p_1) + 1 = 2(t-1) + 1$. Since $t \geq 2$, $|V_{C_1}| \geq 2(t-1) + 1 \geq 3$. So, the component of $G - S$ containing the vertex set V_{C_1} has at least three vertices.

Let C_2 be a component of $G_2 - S_2$, $N(V_{C_2}, V_2) \subset S_2$. If V_{C_2} has some adjacent neighbor $v_1 \in V_1$ and vertex v_1 belongs to some component C_1 of $G_1 - S_1$, then the component C containing the two vertex sets V_{C_1} and V_{C_2} has at least four vertices. Thus, condition 2.a of Theorem 3 holds. Otherwise, $N(V_{C_2}, V_1) \subset S_1$. Since $|S_1| = p_1 = 1$, $|N(V_{C_2}, V_1)| = 1$. That is, $|V_{C_2}| = 1$ and $N(V_{C_2}) \subset S_1 \bigcup S_2$. Hence, C_2 is a trivial component of $G - S$ and, therefore, condition 2.b of Theorem 3 holds. Consequently, the theorem follows. □

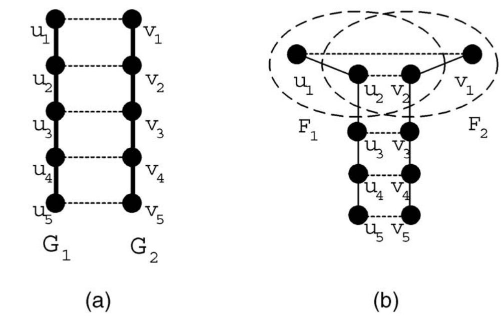

Fig. 2. An example of nonstrongly $(t+1)$-diagnosable as $t = 1$.

For $t = 1$, the above result is not necessarily true; we give an example shown in Fig. 2. Let G_1 and G_2 be two path graphs of length four with vertex sets $\{u_1, u_2, u_3, u_4, u_5\}$ and $\{v_1, v_2, v_3, v_4, v_5\}$, respectively. Let G be the Matching Composition Network constructed by adding a perfect matching (the dashed lines in Fig. 2a) between G_1 and G_2. By Lemma 3, both G_1 and G_2 are 1-diagnosable and G is 2-diagnosable. See Fig. 2b, let $F_1 = \{u_1, u_2, v_2\}$ and $F_2 = \{v_1, v_2, u_2\}$. By Lemma 4, F_1 and F_2 are indistinguishable, but there no vertex exists $v \in V(G_i)$, $i = 1, 2$, such that $N(v) \subset F_1$ and $N(v) \subset F_2$. So, G is not strongly 2-diagnosable.

It follows from Theorem 4 and Definition 3 that the following corollary holds.

Corollary 2. *Let $G_1(V_1, E_1)$, $G_2(V_2, E_2)$ be two t-diagnosable systems with the same number of vertices, where $t \geq 2$. Then, MCN $G = G_1 \bigoplus_M G_2$ is $(t+1)$-diagnosable.*

Applying Theorem 4, all systems in the cube family are strongly $(t+1)$-diagnosable if their subcubes are t-diagnosable for $t \geq 2$. The Hypercube Q_n, the Crossed cube CQ_n, the Twisted cube TQ_n, and the Möbius cube MQ_n are well-known members in the cube family. For $n = 2$, these cubes are all isomorphic to the cycle of length four; they are 1-diagnosable, but not 2-diagnosable. For $n = 3$, these cubes are all 3-connected, by Lemma 3, they are 3-diagnosable. So, we have the following corollary.

Corollary 3. *The Hypercube Q_n, the Crossed cube CQ_n, the Möbius cube MQ_n, and the Twisted cube TQ_n are all strongly n-diagnosable for $n \geq 4$.*

We now give some examples which are not strongly t-diagnosable. Consider the three-dimensional hypercube Q_3, it is 3-diagnosable, but not strongly 3-diagnosable due to the fact that $|V(Q_3)| = 8 \leq 2(t+1) + 1$ as $t = 3$, which contradicts condition 1 of Lemma 8. Let C_n be a cycle of length n, $n \geq 7$. By Lemma 3, C_n is 2-diagnosable, but it is not strongly 2-diagnosable. Another nontrivial example is presented in Fig. 3. This graph G is 3-regular, 2-connected and, by Theorem 2, it is 3-diagnosable. As shown in Fig. 3, $F_1 = \{1, 2, 5, 6\}$ and $F_2 = \{3, 4, 5, 6\}$. (F_1, F_2) is an indistinguishable pair, but there does not exist any vertex v in $V(G)$

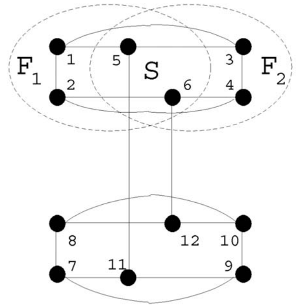

Fig. 3. An example of a nonstrongly 3-diagnosable system.

such that $N(v) \subset F_1$ and $N(v) \subset F_2$. By Definition 3, the graph is not strongly 3-diagnosable.

4 Conditional Diagnosability of Q_n

Consider a system G with diagnosability $t(G) = t$; so G is t-diagnosable but not $(t+1)$-diagnosable. In previous research on diagnosability, the investigated networks are often strongly t-diagnosable, for example, members in the cube family are so. Given a system G, suppose that it is strongly t-diagnosable but not $(t+1)$-diagnosable. As we mentioned before, the only case that stops it from being $(t+1)$-diagnosable is that there exists a vertex v whose neighboring vertices are faulty simultaneously. We are, therefore, led to the following question: How large can the maximum value of t be such that G remains t-diagnosable under the condition that every faulty set F satisfies $N(v) \not\subseteq F$ for each vertex $v \in V$?

For classical measurement of diagnosability, it is usually assumed that processor failures are statically independent. It does not reflect the total number of processors in the system and the probabilities of processor failures. In [21], Najjar and Gaudiot have proposed *fault resilience* as the maximum number of failures that can be sustained while the network remains connected with a reasonably high probability. For hypercube, the fault resilience is shown as 25 percent for the four-dimensional cube Q_4 and it increases to 33 percent for the 10-dimensional cube Q_{10}. More particularly, for the 10-dimensional cube Q_{10}, 33 percent of processors can fail and the network still remains connected with a probability of 99 percent. They also gave a conclusion that large-scale systems with a constant degree are more susceptible to failures by disconnection than smaller networks. With the observation of Lemma 4, a connected network gives higher probability to diagnosis faulty processors and has better ability to distinguish any two sets of processors.

Motivated by the deficiency of the classical measurement of diagnosability and the broadness of a system being strongly t-diagnosable, we introduce a measure of conditional diagnosability by claiming the property that any faulty set cannot contain all neighbors of any processor. We formally introduce some terms related to the conditional diagnosability. A faulty set $F \subset V$ is called a *conditional faulty set* if $N(v) \not\subseteq F$ for any vertex $v \in V$. A system $G(V,E)$ is *conditionally* t-diagnosable if F_1 and F_2 are distinguishable, for each pair of conditional faulty sets F_1, $F_2 \subset V$, and $F_1 \neq F_2$, with $|F_1| \leq t$ and $|F_2| \leq t$. The *conditional diagnosability* of a system G, written as $t_c(G)$, is defined to be the maximum value of t such that G is conditionally t-diagnosable. It is clear that $t_c(G) \geq t(G)$.

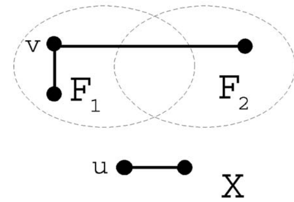

Fig. 4. An illustration of Lemma 11.

Lemma 9. *Let G be a network system. Then, $t_c(G) \geq t(G)$.*

Let $F_1, F_2 \subset V$ and $F_1 \neq F_2$. We say (F_1, F_2) is a *distinguishable conditional-pair* (an indistinguishable conditional-pair, respectively) if F_1 and F_2 are conditional faulty sets and are distinguishable (indistinguishable, respectively).

It follows from the definition that a strongly t-diagnosable system is clearly conditionally $(t+1)$-diagnosable. However, the conditional diagnosability of some strongly t-diagnosable systems can be far greater than $t+1$. This motivates us to study the conditional diagnosability of the hypercube.

Lemma 10. *Let G be a strongly t-diagnosable system. Then, G is conditionally $(t+1)$-diagnosable.*

Before discussing the conditional diagnosability, we have some observations as follows: Let $F_1, F_2 \subset V$ be an indistinguishable conditional-pair. Let $X = V - (F_1 \bigcup F_2)$. Then, there is no edge between X and $F_1 \triangle F_2$. So, $N(F_1 \triangle F_2, X) = \phi$ and $N(X, F_1 \triangle F_2) = \phi$. Let vertex $v \in F_1 - F_2$ (or $v \in F_2 - F_1$). Then, $N(v) \subset (F_1 \bigcup F_2)$. F_1 is a conditional faulty set, so $N(v) \not\subseteq F_1$ and $N(v) \bigcap (F_2 - F_1) \neq \phi$. Similarly, F_2 is a conditional faulty set, $N(v) \not\subseteq F_2$ and $N(v) \bigcap (F_1 - F_2) \neq \phi$. So, $|N(v) \bigcap (F_1 - F_2)| \geq 1$ and $|N(v) \bigcap (F_2 - F_1)| \geq 1$ for every vertex $v \in F_1 \triangle F_2$. Now, consider a vertex $u \in X = V - (F_1 \bigcup F_2)$. Since F_1 and F_2 are an indistinguishable conditional-pair, $N(u) \bigcap (F_1 \triangle F_2) = \phi$, $N(u) \not\subseteq F_1$ and $N(u) \not\subseteq F_2$. So, $N(u) \not\subseteq (F_1 \bigcup F_2)$. Therefore, every vertex $u \in X$ has at least one neighbor in X (see Fig. 4). We state this fact in the following lemma.

Lemma 11. *Let $G(V,E)$ be a system. Given an indistinguishable conditional-pair (F_1, F_2), $F_1 \neq F_2$, the following two conditions hold:*

1. $|N(u) \bigcap (V - (F_1 \bigcup F_2))| \geq 1$ *for* $u \in (V - (F_1 \bigcup F_2))$ *and*
2. $|N(v) \bigcap (F_1 - F_2)| \geq 1$ *and* $|N(v) \bigcap (F_2 - F_1)| \geq 1$ *for* $v \in F_1 \triangle F_2$.

Let (F_1, F_2) be an indistinguishable conditional-pair and let $S = F_1 \bigcap F_2$. By the above observations, every component of $G - S$ is nontrivial. Moreover, for each component C_1 of $G - S$, if $V_{C_1} \bigcap (F_1 \triangle F_2) = \phi$, $deg_{C_1}(v) \geq 1$ for $v \in V_{C_1}$; for each component C_2 of $G - S$, if $V_{C_2} \bigcap (F_1 \triangle F_2) \neq \phi$, $deg_{C_2}(v) \geq 2$ for $v \in V_{C_2}$. To find the conditional diagnosability of the hypercube Q_n, we need to study the cardinality of the set S.

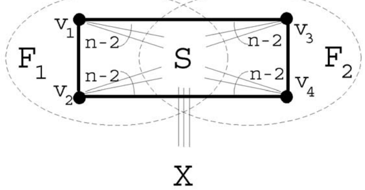

Fig. 5. An indistinguishable conditional-pair (F_1, F_2), where $|F_1| = |F_2| = 4(n-2)+2$.

First, we give an example to show that the conditional diagnosability of the hypercube Q_n is no greater than $4(n-2)+1$. As shown in Fig. 5, we take a cycle of length four in Q_n, let $\{v_1, v_2, v_3, v_4\}$ be the four consecutive vertices on this cycle, and let $F_1 = N(\{v_1, v_2, v_3, v_4\}) \bigcup \{v_1, v_2\}$ and $F_2 = N(\{v_1, v_2, v_3, v_4\}) \bigcup \{v_3, v_4\}$. It is a simple matter to check that (F_1, F_2) is an indistinguishable conditional-pair. Note that the hypercube Q_n has no triangle and any two vertices have at most two common neighbors. As we can see, $|F_1 - F_2| = |F_2 - F_1| = 2$ and $|F_1 \bigcap F_2| = 4(n-2)$. Hence, Q_n is not conditionally $(4(n-2)+2)$-diagnosable and $t_c(Q_n) \le 4(n-2)+1$. Then, we shall show that Q_n is, in fact, conditionally t-diagnosable, where $t = 4(n-2)+1$.

Lemma 12. $t_c(Q_n) \le 4(n-2)+1$ *for* $n \ge 3$.

Let S be a set of vertices, $S \subset V(Q_n)$. Suppose that $Q_n - S$ is disconnected and C is a component of $Q_n - S$. We need some results on the cardinalities of S and V_C under some restricted conditions. The results are listed in Lemmas 13 and 14.

These two lemmas are both proven by dividing Q_n into two Q_{n-1}s, denoted by Q^L_{n-1} and Q^R_{n-1}. To simplify the explanation, we define some symbols as follows: $V_L = V(Q^L_{n-1})$, $V_R = V(Q^R_{n-1})$, $C_L = Q^L_{n-1} \bigcap C$, $C_R = Q^R_{n-1} \bigcap C$, $V_{C_L} = V(C_L)$, $V_{C_R} = V(C_R)$, $S_L = V_L \bigcap S$, and $S_R = V_R \bigcap S$.

The following result is also implicit in [16].

Lemma 13. *Let Q_n be the n-dimensional hypercube, $n \ge 3$, and let S be a set of vertices $S \subset V(Q_n)$. Suppose that $Q_n - S$ is disconnected. Then the following two conditions hold:*

1. $|S| \ge n$ *and*
2. *If $n \le |S| \le 2(n-1)-1$, then $Q_n - S$ has exactly two components, one is trivial and the other is nontrivial. The nontrivial component of $Q_n - S$ contains $2^n - |S| - 1$ vertices.*

Proof. Since $\kappa(Q_n) = n$ [24], condition 1 holds. We need only to prove condition 2 is true. Because $Q_n - S$ is disconnected, there are at least two components in $Q_n - S$. We consider three cases: 1) $Q_n - S$ contains at least two trivial components, 2) $Q_n - S$ has at least two nontrivial components, 3) there are exactly one trivial component and one nontrivial component in $Q_n - S$. In cases 1) and 2), we shall prove that $|S| \ge 2(n-1)$. Then, $n \le |S| \le 2(n-1)-1$ implies $Q_n - S$ belongs to case 3).

Case 1: $Q_n - S$ contains at least two trivial components.

Let $v_i \in V$, $i = 1, 2$ and $\{v_1\}, \{v_2\} \subset V(Q_n)$ be two trivial components of $Q_n - S$. It means that $N(v_1) \subset S$ and $N(v_2) \subset S$. For Q_n, it is not difficult to see that any two vertices have at most two common neighbors. That is, $|N(v_1) \bigcap N(v_2)| \le 2$. Hence,

$$|S| \ge |N(v_1) \bigcup N(v_2)| = |N(v_1)| + |N(v_2)| - |N(v_1) \bigcap N(v_2)| \ge 2n - 2 = 2(n-1).$$

Case 2: $Q_n - S$ has at least two nontrivial components.

We prove, by induction on n, that $|S| \ge 2(n-1)$. For $n = 3$, suppose $n \le |S| \le 2(n-1)-1$, which implies that $|S| = 3$. The connectivity of Q_3 is 3. By Lemma 7, the only vertex cut S with $|S| = 3$ in Q_3 is $S = N(v)$ for some vertex $v \in V(Q_3)$. It follows that $Q_3 - S$ has exactly two components, one is trivial and the other is nontrivial. Therefore, if $Q_3 - S$ has at least two nontrivial components, $|S| \ge 2(n-1)$, where $n = 3$. Assume the case holds for some $n-1$, $n-1 \ge 3$. We now show that it holds for n.

Let C and C' be two nontrivial component of $Q_n - S$. So, $|V_C| \ge 2$. It is feasible to divide Q_n into the two disjoint Q_{n-1}s, denoted by Q^L_{n-1} and Q^R_{n-1}, such that $|V_{C_L}| \ge 1$ and $|V_{C_R}| \ge 1$. There is another component C' of $Q_n - S$, so at least one of the two graphs $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ is disconnected.

Suppose that both $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ are disconnected. Since $\kappa(Q_{n-1}) = n-1$, $|S_L| \ge n-1$ and $|S_R| \ge n-1$. Then, $|S| = |S_L| + |S_R| \ge 2(n-1)$. Otherwise, one of the two subgraphs $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ is connected. Without loss of generality, assume that $Q^L_{n-1} - S_L$ is connected and $Q^R_{n-1} - S_R$ is disconnected. Then, $V_L = V_{C_L} \bigcup S_L$ and the other nontrivial component C' of $Q_n - S$ is completely contained in $Q^R_{n-1} - S_R$. Since $V_{C'}$ is disconnected from V_{C_L}, the corresponding matched vertices of $V_{C'}$ in Q^L_{n-1} are in S_L. That is, $N(V_{C'}, Q^L_{n-1}) \subseteq S_L$. Hence, $|S_L| \ge |V_{C'}| \ge 2$.

If $|S_R| \ge 2(n-2)$, then

$$|S| = |S_L| + |S_R| \ge 2 + 2(n-2) = 2(n-1).$$

Otherwise, $n-1 \le |S_R| \le 2(n-2)-1$, by induction hypothesis that $Q^R_{n-1} - S_R$ cannot have two nontrivial components and, by the result of Case 1, $Q^R_{n-1} - S_R$ has exactly two components, one is trivial and the other is nontrivial. We know that $Q^R_{n-1} - S_R$ has C_R and C' as its components and C' is a nontrivial component. So, C_R must be a trivial component of $Q^R_{n-1} - S_R$ and $|V_{C'}| = 2^{n-1} - |S_R| - 1$. Note that $N(V_{C'}, Q^L_{n-1}) \subseteq S_L$. Then, $|S| = |S_L| + |S_R| \ge |V_{C'}| + |S_R| = 2^{n-1} - |S_R| - 1 + |S_R| = 2^{n-1} - 1 \ge 2(n-1)$ for $n \ge 4$.

Consequently, condition 2 is true and the lemma holds. □

Suppose that $Q_n - S$ is disconnected, every component of $Q_n - S$ is nontrivial, and there exists one component C of $Q_n - S$ such that $deg_C(v) \ge 2$ for every vertex v in C. In view of the example given in Fig. 4 and Lemma 11, we shall prove that either $|S|$ is sufficiently large or else $|V_C|$ is large, as stated in the following lemma.

Lemma 14. *Let Q_n be the n-dimensional hypercube and $n \ge 5$ and let S be a vertex set $S \subseteq V(Q_n)$. Suppose that $Q_n - S$ is disconnected and every component of $Q_n - S$ is nontrivial and suppose that there exists one component C of $Q_n - S$ such that*

$deg_C(v) \geq 2$ for every vertex v in C. Then, one of the following two conditions holds:

1. $|S| \geq 4(n-2)$ *or*
2. $|V_C| \geq 4(n-2)-1$.

Proof. Since $deg_C(v) \geq 2$ for every vertex v in C, it is feasible to divide Q_n into two disjoint Q_{n-1}s, denoted by Q^L_{n-1} and Q^R_{n-1}, such that $V(Q^L_{n-1} \bigcap C) \neq \phi$ and $V(Q^R_{n-1} \bigcap C) \neq \phi$. Let $C_L = Q^L_{n-1} \bigcap C$ and $C_R = Q^R_{n-1} \bigcap C$. For each vertex x in C_L (y in C_R, respectively), it has at most one neighbor in C_R (C_L, respectively). Hence, $deg_{C_L}(x) \geq 1$ and $deg_{C_R}(y) \geq 1$ for $x \in V_{C_L}$ and $y \in V_{C_R}$, respectively.

$Q_n - S$ is disconnected, there are at least two components in $Q_n - S$. Let $S_L = V_L \bigcap S$ and $S_R = V_R \bigcap S$. Note that both Q^L_{n-1} and Q^R_{n-1} contain some nonempty part of the component C. So, at least one of the two subgraphs $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ is disconnected. In the following proof, we investigate two cases: 1) One of $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ is connected, 2) both $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ are disconnected.

Case 1: One of $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ is connected and the other is disconnected.

Without loss of generality, assume $Q^L_{n-1} - S_L$ is connected and $Q^R_{n-1} - S_R$ is disconnected. Let C' be another component of $Q_n - S$ other than C. Then, $V_L = S_L \bigcup V_{C_L}$ and the component C' of $Q_n - S$ is in $Q^R_{n-1} - S_R - V_{C_R}$. Since C_R and C' are both nontrivial components, by Lemma 13, $|S_R| \geq 2(n-2)$. If $|S_L| \geq 2(n-2)$, then $|S| = |S_L| + |S_R| \geq 4(n-2)$ and condition 1 holds. Otherwise, $|S_L| \leq 2(n-2)-1$. Then, $|V_{C_L}| = 2^{n-1} - |S_L| \geq 2^{n-1} - 2(n-2) + 1$. That is, $|V_C| = |V_{C_L}| + |V_{C_R}| \geq (2^{n-1} - 2(n-2) + 1) + 2 = 2^{n-1} - 2(n-2) + 3 \geq 4(n-2) - 1$ for $n \geq 4$ and condition 2 holds.

Case 2: Both $Q^L_{n-1} - S_L$ and $Q^R_{n-1} - S_R$ are disconnected.

By Lemma 13, we consider the following three subcases:

- 2a. $|S_L| \geq 2(n-2)$ and $|S_R| \geq 2(n-2)$,
- 2b. $n-1 \leq |S_L| \leq 2(n-2)-1$ and $n-1 \leq |S_R| \leq 2(n-2)-1$, and
- 2c. either $|S_L| \geq 2(n-2)$, $n-1 \leq |S_R| \leq 2(n-2)-1$ or $|S_R| \geq 2(n-2)$, $n-1 \leq |S_L| \leq 2(n-2)-1$.

Subcase 2.a: $|S_L| \geq 2(n-2)$ and $|S_R| \geq 2(n-2)$.

Since $|S_L| \geq 2(n-2)$ and $|S_R| \geq 2(n-2)$,

$$|S| = |S_L| + |S_R| \geq 4(n-2).$$

Hence, condition 1 holds.

Subcase 2.b: $n-1 \leq |S_L| \leq 2(n-2)-1$ and $n-1 \leq |S_R| \leq 2(n-2)-1$.

In this subcase, $|V_{C_L}| = 2^{n-1} - |S_L| - 1$ and $|V_{C_R}| = 2^{n-1} - |S_R| - 1$. So,

$$|V_C| = |V_{C_L}| + |V_{C_R}| = 2^n - |S| - 2.$$

Suppose $|S| \geq 4(n-2)$. Then, condition 1 holds. Otherwise, $|S| \leq 4(n-2)-1$. Then, $|V_C| = 2^n - |S| - 2 \geq 2^n - (4(n-2)-1) - 2 = 2^n - 4(n-2) - 1 \geq 4(n-2) - 1$ for $n \geq 4$. Hence, condition 2 holds.

Subcase 2.c: Either $|S_L| \geq 2(n-2)$, $n-1 \leq |S_R| \leq 2(n-2)-1$ or $|S_R| \geq 2(n-2)$,

$$n-1 \leq |S_L| \leq 2(n-2)-1.$$

Without loss of generality, assume that $|S_L| \geq 2(n-2)$, $n-1 \leq |S_R| \leq 2(n-2)-1$. Then, $|V_{C_R}| = 2^{n-1} - |S_R| - 1 \geq 2^{n-1} - 2(n-2)$. Since

$$deg_{C_L}(x) \geq 1,$$

for each vertex $x \in V_{C_L}$, we have $|V_{C_L}| \geq 2$. Thus, $|V_C| = |V_{C_L}| + |V_{C_R}| \geq 2 + (2^{n-1} - 2(n-2)) = 2^{n-1} - 2(n-2) + 2 \geq 4(n-2) - 1$ for $n \geq 5$.

This completes the proof of the lemma. □

We are now ready to show the conditional diagnosability of Q_n is $4(n-2)+1$ for $n \geq 5$. Let $F_1, F_2 \subset V(Q_n)$ be an indistinguishable conditional-pair, $n \geq 5$. We shall show our result by proving that either $|F_1| \geq 4(n-2)+2$ or $|F_2| \geq 4(n-2)+2$. Let $S = F_1 \bigcap F_2$. We consider two cases: 1) $Q_n - S$ is connected and 2) $Q_n - S$ is disconnected.

Lemma 15. *Let Q_n be the n-dimensional hypercube, $n \geq 5$. Let $F_1, F_2 \subset V(Q_n)$, $F_1 \neq F_2$, be an indistinguishable conditional-pair and $S = F_1 \bigcap F_2$. Then, either $|F_1| \geq 4(n-2)+2$ or $|F_2| \geq 4(n-2)+2$.*

Proof. Suppose that $Q_n - S$ is connected. Then, $F_1 \triangle F_2 = V(Q_n - S)$ and $V(Q_n) = F_1 \bigcup F_2$. Suppose, on the contrary, that $|F_1| \leq 4(n-2)+1$ and $|F_2| \leq 4(n-2)+1$. Then,

$$\begin{aligned} 2^n &= |F_1| + |F_2| - |F_1 \textstyle\bigcap F_2| \\ &\leq (4(n-2)+1) + (4(n-2)+1) - 0 = 8(n-2)+2. \end{aligned}$$

This contradicts the fact that $2^n > 8(n-2)+2$ for $n \geq 5$. Hence, the result holds as $Q_n - S$ is connected.

Now, we consider the case that $Q_n - S$ is disconnected, by Lemma 11, $Q_n - S$ has a component C with $deg_C(v) \geq 2$ for every vertex $v \in V_C$. By Lemma 14, we have $|S| \geq 4(n-2)$ or $|V_C| \geq 4(n-2)-1$.

Suppose $|S| \geq 4(n-2)$. Since $deg_C(v) \geq 2$ for every vertex v in C and Q_n does not contain any cycle of length three, so $|V_C| \geq 4$. With the observation that $V_C \subset F_1 \triangle F_2$, we conclude that either $(F_1 - F_2) \geq \lceil \frac{|V_C|}{2} \rceil \geq 2$ or $(F_2 - F_1) \geq \lceil \frac{|V_C|}{2} \rceil \geq 2$. Therefore, either $|F_1| = |S| + |F_1 - F_2| \geq 4(n-2)+2$ or $|F_2| = |S| + |F_2 - F_1| \geq 4(n-2)+2$.

Otherwise, $|V_C| \geq 4(n-2)-1$. Then, either $(F_1 - F_2) \geq \lceil \frac{|V_C|}{2} \rceil \geq 2(n-2)$ or $(F_2 - F_1) \geq \lceil \frac{|V_C|}{2} \rceil \geq 2(n-2)$. Because there are at least two nontrivial components in $Q_n - S$, by Lemma 13, $|S| \geq 2(n-1)$. Hence, $|F_1| = |S| + |F_1 - F_2| \geq 4(n-2)+2$ or

$$|F_2| = |S| + |F_2 - F_1| \geq 4(n-2)+2.$$

Therefore, for any indistinguishable conditional-pair $F_1, F_2 \subset V(Q_n)$, it implies that $|F_1| \geq 4(n-2)+2$ or $|F_2| \geq 4(n-2)+2$. This proves the lemma. □

By Lemma 12, $t_c(Q_n) \leq 4(n-2)+1$, and by Lemmas 6 and 15, Q_n is conditionally $(4(n-2)+1)$-diagnosable for $n \geq 5$. Hence, $t_c(Q_n) = 4(n-2)+1$ for $n \geq 5$. For Q_3 and Q_4, we observe that Q_3 is not conditionally 4-diagnosable

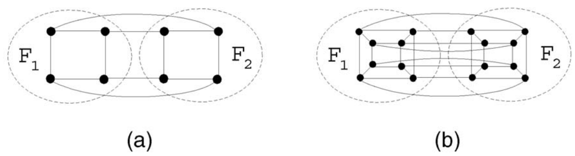

Fig. 6. Two indistinguishable conditional-pairs for Q_3 and Q_4.

and Q_4 is not conditionally 8-diagnosable, as shown in Fig. 6a and Fig. 6b. So, $t_c(Q_3) \leq 3$ and $t_c(Q_4) \leq 7$. Hence, the conditional diagnosabilities of Q_3 and Q_4 are both strictly less than $4(n-2)+1$.

Q_3 is 3-diagnosable and it is not conditionally 4-diagnosable. It follows from Lemma 9 that $t_c(Q_3) = 3$. For Q_4, we prove that $t_c(Q_4) = 7$ in the following lemma.

Lemma 16. $t_c(Q_4) = 7$.

Proof. We already know $t_c(Q_4) \leq 7$. Suppose, on the contrary, that Q_4 is not conditionally 7-diagnosable. Let $F_1, F_2 \subset V(Q_4)$ be an indistinguishable conditional-pair with $|F_i| \leq 7$, $i = 1, 2$, and let $S = F_1 \bigcap F_2$. It follows from Lemmas 11 and 13 that $|S| \geq 2(n-1) = 6$ for $n = 4$. Furthermore, $|F_1 - F_2| \geq 2$ and $|F_2 - F_1| \geq 2$. Then, $|F_1| \geq 8$ and $|F_2| \geq 8$, which is a contradiction. So, $t_c(Q_4) = 7$. □

Finally, the conditional diagnosability of hypercube Q_n is stated as follows:

Theorem 5. *The conditional diagnosability of* Q_n *is* $t_c(Q_n) = 4(n-2)+1$ *for* $n \geq 5$, $t_c(Q_3) = 3$, *and* $t_c(Q_4) = 7$.

5 Conclusions

In probabilistic models of multiprocessor systems, processors fail independently, but with different probabilities. The probability that all faulty processors are neighbors of one processor is very small. In this paper, we propose the concept of a strongly t-diagnosable system and derive some conditions for verifying whether a system is strongly t-diagnosable. To grant more accurate measurement of diagnosability for a large-scale processing system, we also introduce the conditional diagnosability of a system under the PMC model. The conditional diagnosability of the hypercube Q_n is demonstrated to be $4(n-2)+1$.

In the area of diagnosability, the comparison model is another well-known and widely chosen fault diagnosis model. Hence, it is worth investigating the issue of a system being strongly t-diagnosable and determining the conditional diagnosability of a system under the comparison model.

The classical diagnosability of a system is small owing to the fact that it ignores the unlikelihood of the corresponding processors failing at the same time. Therefore, it is attractive work to develop more different measures of diagnosability based on application environment, network topology, network reliability, and statistics related to fault patterns.

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References

[1] J.R. Armstrong and F.G. Gray, "Fault Diagnosis in a Boolean n Cube Array of Multiprocessors," *IEEE Trans. Computers,* vol. 30, no. 8, pp. 587-590, Aug. 1981.

[2] C.P. Chang, J.N. Wang, and L.H. Hsu, "Topological Properties of Twisted Cubes," *Information Sciences,* vol. 113, nos. 1-2, pp. 147-167, Jan. 1999.

[3] P. Cull and S.M. Larson, "The Möbius Cubes," *IEEE Trans. Computers,* vol. 44, no. 5, pp. 647-659, May 1995.

[4] A.T. Dahbura and G.M. Masson, "An $O(n^{2.5})$ Fault Identification Algorithm for Diagnosable Systems," *IEEE Trans. Computers,* vol. 33, no. 6, pp. 486-492, June 1984.

[5] A. Das, K. Thulasiraman, V.K. Agarwal, and K.B. Lakshmanan, "Multiprocessor Fault Diagnosis under Local Constraints," *IEEE Trans. Computers,* vol. 42, no. 8, pp. 984-988, Aug. 1993.

[6] K. Efe, "A Variation on the Hypercube with Lower Diameter," *IEEE Trans. Computers,* vol. 40, no. 11, pp. 1312-1316, Nov. 1991.

[7] A.H. Esfahanian, "Generalized Measures of Fault Tolerance with Application to N-Cube Networks," *IEEE Trans. Computers,* vol. 38, no. 11, pp. 1586-1591, Nov. 1989.

[8] D.B. West, *Introduction to Graph Theory.* Prentice Hall, 2001.

[9] J. Fan, "Diagnosability of Crossed Cubes under the Two Strategies," *Chinese J. Computers,* vol. 21, no. 5, pp. 456-462, May 1998.

[10] J. Fan, "Diagnosability of the Möbius Cubes," *IEEE Trans. Parallel and Distributed Systems,* vol. 9, no. 9, pp. 923-928, Sept. 1998.

[11] S.L. Hakimi and A.T. Amin, "Characterization of Connection Assigment of Diagnosable Systems," *IEEE Trans. Computers,* vol. 23, no. 1, pp. 86-88, Jan. 1974.

[12] F. Harary, "Conditional Connectivity," *Networks,* vol. 13, pp. 346-357, 1983.

[13] P.A.J. Hilbers, M.R.J. Koopman, and J.L.A. van de Snepscheut, "The Twisted Cube," *Parallel Architectures and Languages Europe,* pp. 152-159, June 1987.

[14] A. Kavianpour and K.H. Kim, "Diagnosability of Hypercube under the Pessimistic One-Step Diagnosis Strategy," *IEEE Trans. Computers,* vol. 40, no. 2, pp. 232-237, Feb. 1991.

[15] P. Kulasinghe, "Connectivity of the Crossed Cube," *Information Processing Letters,* vol. 61, no. 4, pp. 221-226, Feb. 1997.

[16] S. Latifi, "Combinatorial Analysis of the Fault-Diameter of the n-Cube," *IEEE Trans. Computers,* vol. 42, no. 1, pp. 27-33, Jan. 1993.

[17] S. Latifi, M. Hegde, and M. Naraghi-Pour, "Conditional Connectivity Measures for Large Multiprocessor Systems," *IEEE Trans. Computers,* vol. 43, no. 2, pp. 218-222, Feb. 1994.

[18] P.L. Lai, J.J.M. Tan, C.H. Tsai, and L.H. Hsu, " The Diagnosability of Matching Composition Network under the Comparison Diagnosis Model" *IEEE Trans. Computers,* vol. 53, no. 8, pp. 1064-1069, Aug. 2004.

[19] J. Maeng and M. Malek, "A Comparison Connection Assignment for Self-Diagnosis of Multiprocessors Systems," *Proc. 11th Int'l Symp. Fault-Tolerant Computing,* pp. 173-175, 1981.

[20] M. Malek, "A Comparison Connection Assignment for Diagnosis of Multiprocessor Systems," *Proc. Seventh Int'l Symp. Computer Architecture,* pp. 31-35, 1980.

[21] W. Najjar and J.L. Gaudiot, "Network Resilience: A Measure of Network Fault Tolerance," *IEEE Trans. Computers,* vol. 39, no. 2, pp. 174-181, Feb. 1990.

[22] A.D. Oh and H.A. Choi, "Generalized Measures of Fault Tolerance in n-Cube Networks," *IEEE Trans. Parallel and Distributed Systems,* vol. 4, no. 6, pp. 702-703, June 1993.

[23] F.P. Preparata, G. Metze, and R.T. Chien, "On the Connection Assignment Problem of Diagnosis Systems," *IEEE Trans. Electronic Computers,* vol. 16, no. 12, pp. 848-854, Dec. 1967.

[24] Y. Saad and M.H. Schultz, "Topological Properties of Hypercubes," *IEEE Trans. Computers,* vol. 37, no 7, pp. 867-872, July 1988.

[25] J. Xu, *Topological Structure and Analysis of Interconnection Networks.* Kluwer Academic, 2001.

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