CAPPED EQUITY SWAPS UNDER THE DOUBLE-JUMP STOCHASTIC VOLATILITY MODEL WITH STOCHASTIC INTEREST RATES

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This study proposes a double-jump stochastic volatility model with stochastic interest rates to price capped equity swaps and other multi-period derivative securities. Closed-form solutions for capped equity swaps with a fixed or variable notional principle are derived. In addition, numerical examples are employed to analyze comparative statics properties, counterparty risks, and the dynamics of the forward smile. © 2010 Wiley Periodicals, Inc. Jrl Fut Mark 31:340–370, 2011

INTRODUCTION

Substantial evidence exists in the empirical financial economic literature of the existence of both stochastic volatility and jumps in equity prices (Bakshi, Cao, & Chen, 1997; Broadie, Chernov, & Johannes, 2007). Yet, most existing pricing models of capped equity swaps do not provide a flexible framework to address the implications of these distributional traits. This study attempts

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to address that deficiency by proposing a new approach for pricing capped equity swaps using a double-jump stochastic volatility model with stochastic interest rates.

Because of the upper limit on the equity return payout, equity options are embedded in capped equity swaps and are called forward-start options. Forward-start options are common components of cliquet options, periodic caps, periodic floors, employee options, and equity-linked life insurance products with guaranteed return rates (Bacinello, 2003; Brennan & Schwartz, 1976). Two types of forward-start options are embedded in capped equity swaps: (1) one type is an absolute payoff and is considered in most of the existing literature (Broadie & Kaya, 2006; Guo & Hung, 2008; Kruse & Nögel, 2005; Rubinstein, 1991); (2) the other type is a relative payoff and is less explored. In this manuscript, a characteristic function-based approach is developed to derive closed-form solutions of capped equity swaps. This approach provides a way to apply Fubini's theorem to iterative expectation and contributes to the valuation of multi-period derivatives. The method is also applied to other existing models for the purpose of comparison. Moreover, numerical examples are employed to analyze comparative statics properties, counterparty risks, and the dynamics of the forward smile.

The remainder of this study is organized as follows. Section II describes the model, introduces the approach, and provides closed-form solutions for capped equity swaps. Section III discusses implementation problems, validates the results with a simulation, and investigates some comparative statics properties. Section IV gives a comparison with nested hybrid models and other option models. Section V analyzes the counterparty risk of capped equity swaps and forward smiles. Section VI concludes this paper.

THE MODEL

There are two categories of capped equity swaps classified by the notional principle. One has a fixed notional principle and the other has a variable notional principle. Because capped equity swap payoffs are homogeneous of degree one with respect to the notional principle, a \$1.00 notional principle is used to simplify the valuation.

Consider the example of a capped equity swap with a fixed notional principle defined as follows: (F1) a contract starts at time t_0, $t_0 \leq 0$; (F2) payoffs are paid at dates t_i, $0 < t_1 < t_2 < t_3 < \ldots < t_m$ with a fixed notional principle; (F3) after setting X as the cap rate over the swap life, the fixed rate payer pays a constant payment R and receives the minimum of the cap rate and the return on the underlying equity, $\min\{X, (S(t_i)/S(t_{i-1}) - 1)\}$ at date t_i.

The present value of a capped equity swap with a fixed notional principle is

$$PV_{cap}(t_0, t_m, X) = \sum_{i=1}^{m} E_0^Q\left[\exp\left(-\int_0^{t_i} r(s)ds\right)\left(\min\left\{X, \frac{S(t_i)}{S(t_{i-1})} - 1\right\} - R\right)\right] \quad (1)$$

where $r(t)$ is the spot interest rate. We assume the underlying equity, $S(t)$, the stochastic component of variance, $Y(t)$, and the T-maturity forward rate, $F(t, T)$, under the risk-neutral measure, Q, are driven by Equations (2)–(4):

$$\begin{aligned}\frac{dS(t)}{S(t)} &= r(t)dt + \delta_1 dW^F(t) + \sqrt{Y(t)}dW^S(t) \\ &\quad + (e^{x(t)} - 1)dq^S(t) - E^Q[(e^{x(t)} - 1)dq^S(t)]\end{aligned} \quad (2)$$

$$dY(t) = \kappa_Y(\overline{Y} - Y(t))dt + \sigma_Y\sqrt{Y(t)}dW^Y(t) + y(t)dq^Y(t) \quad (3)$$

$$dF(t, T) = -\gamma(t, T)a(t, T)dt + \gamma(t, T)dW^F(t) \quad (4)$$

where $a(t, T) = -\int_t^T \gamma(t, s)ds$ and $\gamma(t, T)$ may be any deterministic function. W^S, W^Y, and W^F represent three standard Brownian motions with the specification of $\mathrm{Cov}(W^S(t), W^F(t)) = \mathrm{Cov}(W^Y(t), W^F(t)) = 0$. Because $r(t) = F(t, t)$, δ_1 captures the correlation between interest rates and equity returns. We also specify that $\mathrm{Cov}(W^S(t), W^Y(t)) = \delta_2 t$ describes the correlation between variance rates and equity returns. $q^S(t)$ and $q^Y(t)$ are two simultaneous Poisson counters with $\lambda_{x,y}$ as their arrival rate. The volatility-jump amplitude, $y(t)$, follows an exponential distribution with mean θ_y. Given $y(t)$, the return-jump amplitude, $x(t)$, follows a conditional normal distribution with mean $\mu_0 + \mu_{x,y}y(t)$ and variance $\sigma_{x,y}^2$.

The cash flow of a capped equity swap with a fixed notional principle at date t_i on the fixed rate payer side can be rearranged as Equation (5):

$$\begin{aligned}\min\left\{X, \frac{S(t_i)}{S(t_{i-1})} - 1\right\} - R &= \left[\left(\frac{S(t_i)}{S(t_{i-1})} - 1\right) - R\right] \\ &\quad - \max\left\{\frac{S(t_i)}{S(t_{i-1})} - (X + 1), 0\right\}.\end{aligned} \quad (5)$$

The payment is recognized as the payoff from the ordinary equity swap minus the payoff from the call struck at $K \equiv (X + 1)$. Therefore, this approach yields Equation (6):

$$PV_{cap}(t_0, t_m, X) = PV(t_0, t_m) - \sum_{i=1}^{m} E_0^Q\left[\frac{1}{B(0, t_i)}\left(\max\left\{\frac{S(t_i)}{S(t_{i-1})} - K, 0\right\}\right)\right] \quad (6)$$

where $PV(t_0, t_m)$[1] is the present value of the ordinary equity swap with a fixed notional principle and $B(t, T) = \exp\left\{\int_t^T r(s)ds\right\}$ is the money market account. Hence, a capped equity swap is similar to an ordinary equity swap coupled with a stream of short European call option positions. Strike prices of these options will be determined as permitted at future dates. They belong to the so-called forward-start options.

The time-t value of an absolute-payoff forward-start call option struck at $K(T_0) = KS(T_0)$ with $T_0 \leq T$ is defined by

$$\bar{c}(t; T_0, T, K) = E_t^Q\left[\frac{1}{B(t, T)}\max\{S(T) - KS(T_0), 0\}\right]. \tag{7}$$

Equation (7) can be solved by transforming the measure into the forward-neutral measure, Q^{T_0}, defined by the discount bond price $V(t, T_0) = \exp\left\{-\int_t^{T_0} F(t, s)ds\right\}$. Q^{T_0} is constructed by $\{Z_S^{T_0}, Z_Y^{T_0}, Z_F^{T_0}, x^{T_0}, y^{T_0}, q_S^{T_0}, q_Y^{T_0}\}$ where $dZ_S^{T_0} = dW^S$, $dZ_Y^{T_0} = dW^Y$, $dZ_F^{T_0} = dW^F - a(t, T_0)dt$, $x^{T_0} = x$, $y^{T_0} = y$, $q_S^{T_0} = q^S$, and $q_Y^{T_0} = q^Y$. Therefore, Equations (8)–(10) can be generated:

$$\begin{aligned}\frac{dS^{T_0}(t)}{S^{T_0}(t)} &= (\delta_1 - a(t, T_0))dZ_F^{T_0}(t) + \sqrt{Y(t)}dZ_S^{T_0}(t) \\ &\quad + (e^{x^{T_0}(t)} - 1)dq_S^{T_0}(t) - E_t^{T_0}[(e^{x^{T_0}(t)} - 1)dq_S^{T_0}(t)]\end{aligned} \tag{8}$$

$$dY(t) = \kappa_Y[\bar{Y} - Y(t)]dt + \sigma_Y\sqrt{Y(t)}dZ_Y^{T_0}(t) + y^{T_0}(t)dq_Y^{T_0}(t) \tag{9}$$

$$\frac{dV^{T_0}(t, T)}{V^{T_0}(t, T)} = (a(t, T) - a(t, T_0))dZ_F^{T_0}(t) \tag{10}$$

where $S^{T_0}(t) = \frac{S(t)}{V(t, T_0)}$ and $V^{T_0}(t, T) = \frac{V(t, T)}{V(t, T_0)}$ are forward prices defined by $V(t, T_0)$ for $S(t)$ and $V(t, T)$. Note that $S^{T_0}(t)$ and $V^{T_0}(t, T)$ are both martingales under the measure of Q^{T_0}. Given Equations (8)–(10), consider a tri-variant characteristic function under Q^{T_0} defined by

$$\begin{aligned}J(t, u; \phi_S, \phi_V, \phi_Y, T_0, T) \equiv E_t^{T_0}[&\exp[i\phi_S \ln[S^{T_0}(t + u)] \\ &+ i\phi_V \ln[V^{T_0}(t + u, T)] + i\phi_Y Y(t + u)]].\end{aligned} \tag{11}$$

[1] $PV(t_0, t_m) = \frac{S(0)}{S(t_0)} - V(0, t_m) - R\Sigma_{i=1}^m V(0, t_i)$, where $V(0, t)$ is the present value of the discount bond with maturity t (Kijima & Muromachi, 2001).

Guo and Hung (2008) show that

$$J(t, u; \phi_S, \phi_V, \phi_Y, T_0, T) = \exp\left[\begin{array}{l} A(t, u; \phi_S, \phi_V, \phi_Y, T_0, T) + B(u; \phi_S, \phi_Y)Y(t) + \\ i\phi_S \log[S^{T_0}(t)] + i\phi_V \log[V^{T_0}(t, T)] + i\phi_Y Y(t) \end{array}\right]. \tag{12}$$

$A(t, u; \phi_S, \phi_V, \phi_Y, T_0, T)$ and $B(u; \phi_S, \phi_Y)$ are presented in Appendix A.

Theorem I: At the time of the determination of the strike, the price of an absolute-payoff forward-start call option is given by Equation (13):

$$\bar{c}(T_0; T_0, T, K) = S(T_0)\Pi(-i - v) - S(T_0)V(T_0, T)K\Pi(-v). \tag{13}$$

$\Pi(\phi_S)$ is presented in Appendix A. For proof, refer Guo and Hung (2008).

In addition, a relative-payoff forward-start option, $c(t; T_0, T, K)$, is defined by Equation (14):

$$c(t; T_0, T, K) = E_t^Q\left[\frac{1}{B(t, T)}\left(\max\left\{\frac{S(T) - KS(T_0)}{S(T_0)}, 0\right\}\right)\right]. \tag{14}$$

Note that at time T_0, $c(T_0; T_0, T, K) = \bar{c}(T_0; T_0, T, K)/S(T_0)$.

Proposition I: Relative-payoff (Type I) forward-start option

The present value of a relative-payoff forward-start option is given by

$$c(0; T_0, T, K) = V(0, T_0)\Phi(-i - v, v) - V(0, T)K\Phi(-v, -i + v). \tag{15}$$

$\Phi(\phi_S, \phi_V)$ is presented in Appendix B. For proof, see Appendix B.

Note that $c(0; T_0, T, K)$ does not depend on the underlying equity price, but on the discount bond prices with maturities T_0 and T, respectively. Because

$$E_0\left[\max\left\{\frac{S(t_1)}{S(t_0)} - K, 0\right\}\right] = \frac{S(0)}{S(t_0)}E_0\left[\max\left\{\frac{S(t_1)}{S(0)} - \frac{S(t_0)}{S(0)}K, 0\right\}\right] \tag{16}$$

a capped equity swap with a fixed notional principle can be written as

$$PV_{cap}(t_0, t_m, X) = PV(t_0, t_m) - \frac{S(0)}{S(t_0)}c\left(0; 0, t_1, \frac{S(t_0)}{S(0)}K\right) - \sum_{i=2}^{m} c(0; t_{i-1}, t_i, K). \tag{17}$$

With $PV_{cap}(0, t_m, X) = 0$, the swap rate R can be determined by

$$R = \frac{1 - V(0, t_m) - \sum_{i=1}^{m} c(0; t_{i-1}, t_i, K)}{\sum_{i=1}^{m} V(0, t_i)}. \tag{18}$$

Next, a capped equity swap with a variable notional principle is considered and defined as follows: (V1) a contract is initiated at time t_0 where the notional principle is 1; (V2) payoffs are paid at dates t_i with the notional principle equal to $S(t_{i-1})/S(t_0)$; (V3) after setting X as the cap rate over the swap life, the fixed rate payer makes a payment $RS(t_{i-1})/S(t_0)$ and receives the value of $(\min\{X, S(t_i)/S(t_{i-1}) - 1)\})(S(t_{i-1})/S(t_0))$ at date t_i. Hence, a capped equity swap with a variable notional principle can be written as

$$PV'_{cap}(t_0, t_m, X) = \sum_{i=1}^{m} E_0^Q\left[\exp\left(-\int_0^{t_i} r(s)ds\right)\left(\min\left\{X, \frac{S(t_i)}{S(t_{i-1})} - 1\right\} - R\right)\frac{S(t_{i-1})}{S(t_0)}\right]. \tag{19}$$

The cash flow at date t_i can be rearranged as presented in Equation (20):

$$\left(\min\left\{X, \frac{S(t_i)}{S(t_{i-1})} - 1\right\} - R\right)\frac{S(t_{i-1})}{S(t_0)} = \left[\left(\frac{S(t_i)}{S(t_{i-1})} - 1\right) - R\right]\frac{S(t_{i-1})}{S(t_0)} - \frac{1}{S(t_0)}\max\{S(t_i) - KS(t_{i-1}), 0\}. \tag{20}$$

The payment is recognized as the payoff from an equity swap with a variable notional principle minus the payoff from an absolute-payoff forward-start option. Hence, the present value of a capped equity swap with a variable notional principle is written as

$$PV'_{cap}(t_0, t_m, X) = PV'(t_0, t_m) - \frac{1}{S(t_0)}\sum_{i=1}^{m} E_0^Q\left[\frac{1}{B(0, t_i)}(\max\{S(t_i) - KS(t_{i-1}), 0\})\right] \tag{21}$$

where $PV'(t_0, t_m)$ is the present value of a variable notional principle equity swap.

Theorem II: Absolute-payoff (Type II) Forward-start Option

The present value of an absolute-payoff forward-start option is given by

$$\bar{c}(0; T_0, T, K) = S(0)\overline{\Pi}(-i - v, v) - S(0)K\frac{V(0, T)}{V(0, T_0)}\zeta(0, T_0, T)\overline{\Pi}(-v, -i + v). \tag{22}$$

$\zeta(0, T_0, T)$ and $\overline{\Pi}(\phi_S, \phi_V)$ are presented in Appendix C. For proof, refer Guo and Hung (2008).

Proposition II: The present value of a variable notional principle equity swap is given by

$$PV'(t_0, t_m) = \frac{S(0)}{S(t_0)}\left[m - (1 + R) \sum_{i=2}^{m} \frac{V(0, t_i)}{V(0, t_{i-1})} \times \zeta(0, t_{i-1}, t_i) \right] - (1 + R)V(0, t_1). \tag{23}$$

For proof, see Appendix D.

Additionally, note that

$$E_0[\max\{S(t_1) - KS(t_0), 0\}] = E_0\left[\max\left\{ S(t_1) - \frac{S(t_0)}{S(0)} KS(0), 0 \right\} \right]. \tag{24}$$

According to Theorem II, Proposition II, and Equation (24), the present value of a capped equity swap with a variable notional principle is given by

$$PV'_{cap}(t_0, t_m, X) = PV'(t_0, t_m) - \frac{1}{S(t_0)} \bar{c}\left(0; 0, t_1, \frac{S(t_0)}{S(0)} K \right) - \frac{1}{S(t_0)} \sum_{i=2}^{m} \bar{c}(0; t_{i-1}, t_i, K). \tag{25}$$

With $PV'_{cap}(0, t_m, X) = 0$, the swap rate R can be determined by

$$R = \frac{m - \frac{1}{S(0)} \sum_{i=1}^{m} \bar{c}(0; t_{i-1}, t_i, K)}{\sum_{i=1}^{m} \frac{V(0, t_i)}{V(0, t_{i-1})} \zeta(0, t_{i-1}, t_i)} - 1. \tag{26}$$

IMPLEMENTATION AND COMPARATIVE STATICS PROPERTY

Given Equations (17) and (25), the primary problems with this approach are the complex logarithmic function and the oscillating integrand in the Heston option formula. As described in Kahl and Jäckel (2005), because most software packages and programming library routines restrict the complex logarithmic function to its principal branch, they may encounter the difficulty known as the Heston discontinuity. However, Albrecher, Mayer, Schoutens, and Tistaert (2007) provide two specifications for the characteristic function satisfying the same Riccati equation. For the first specification found in Heston (1993) or in Kahl and Jäckel (2005), Albrecher et al. (2007) show that instabilities will occur, even for typical Heston parameters, when the time to maturity is greater than the "threshold" except for $\kappa_Y \bar{Y} = n\sigma_Y^2$ where $n \in Z$ and prove that the stability of the second specification used in Schoutens, Simons, and Tistaert (2004) and in Gatheral (2005) can be guaranteed for all levels of Heston parameters. The second specification of the characteristic function is employed

TABLE I
Capped Equity Swaps—Exact Simulation vs. Closed-Form Solution

Notional Principle	*Cap Rate (%)*	*Exact Simulation ± Standard Deviation[a] (10,000 trials) (%)*	*Closed-Form Solution[a] (%)*
Fixed	15	−58.60 ± 2.58	−58.44
	20	−51.30 ± 2.92	−51.10
	25	−44.97 ± 3.25	−44.73
	30	−39.54 ± 3.57	−39.26
	35	−34.94 ± 3.7	−34.61
Variable	15	−76.77 ± 1.18	−76.80
	20	−66.81 ± 1.64	−66.93
	25	−58.15 ± 2.27	−58.34
	30	−50.72 ± 2.93	−50.97
	35	−44.39 ± 3.56	−44.71

[a]Numerical results for "exact" simulations and for closed-form solutions are obtained from the valuation of capped equity swaps with swap rate $R = 20\%$ and model parameter specifications: $m = 5$, $t_0 = -1$, $t_m = 5$, $S(t_0) = S(0) = 1$, $Y(0) = 0.04$, $\sigma_Y = 0.61$, $\overline{Y} = 0.012$, $\kappa_Y = 5.06$, $\delta_1 = 0.1$, $\delta_2 = -0.1$, $\lambda_{x,y} = 1.64$, $\mu_0 = -0.03$, $\mu_{x,y} = -7.87$, $\sigma_{x,y} = 0.22$, $\theta_y = 0.0036$, $\gamma(t, T) = \gamma e^{-\kappa(T-t)}$, $\gamma = 0.01$, $\kappa = 0.05$, and $F(0, t) = 0.15 + 0.005t - 0.0002t^2$.

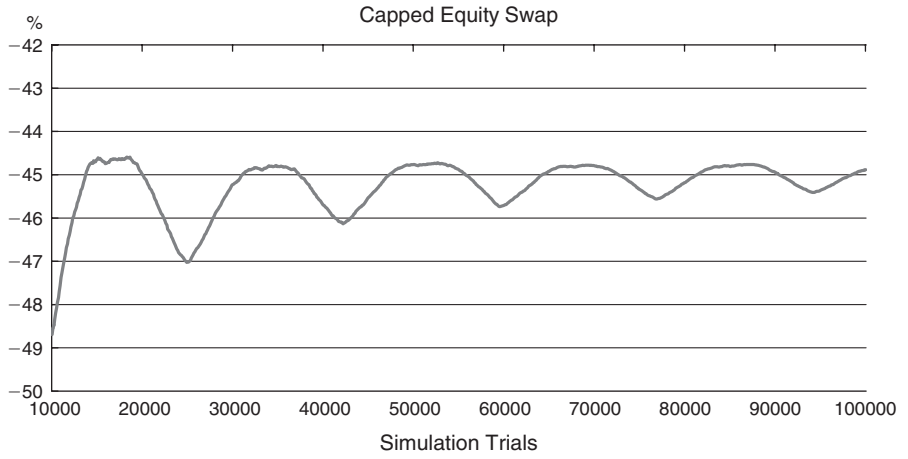


FIGURE 1
"Exact" simulation. Simulation results for the "Exact" method are obtained from the valuation of fixed notional principle capped equity swaps with swap rate $R = 20\%$, cap rate $X = 25\%$, and model parameter specifications: $m = 5$, $t_0 = -1$, $t_m = 5$, $S(t_0) = S(0) = 1$, $Y(0) = 0.04$, $\sigma_Y = 0.61$, $\overline{Y} = 0.012$, $\kappa_Y = 5.06$, $\delta_1 = 0.1$, $\delta_2 = -0.1$, $\lambda_{x,y} = 1.64$, $\mu_0 = -0.03$, $\mu_{x,y} = -7.87$, $\sigma_{x,y} = 0.22$. $\theta_y = 0.0036$, $\gamma(T, t) = \gamma e^{-\kappa(T-t)}$, $\gamma = 0.01$, $\kappa = 0.05$, and $F(0, t) = 0.15 + 0.005t - 0.0002t^2$. The closed-form solution given in Table I is −44.73%.

here to ensure robustness. For the oscillating integrand problem, the adaptive Gauss-Lobatto algorithm of Gander and Gautschi (1998) is applied.

Table I shows that our results are consistent with those of Broadie and Kaya (2006). Figure 1 shows that the simulation result converges to the closed-form

TABLE II
Swap Rates for Capped Equity Swaps

Notional Principle	*Cap Rate (%)*	*Swap Maturity (years)*[a]					
		1	*2*	*3*	*5*	*7*	*10*
Fixed	25	5.31	5.62	5.79	6.00	6.13	6.26
	50	11.48	11.80	12.00	12.29	12.50	12.69
	100	15.32	15.59	15.81	16.16	16.41	16.67
	200	16.36	16.61	16.83	17.20	17.47	17.74
	∞	16.47	16.71	16.93	17.30	17.57	17.85
Variable	25	5.31	5.69	5.92	6.27	6.55	6.89
	50	11.48	11.88	12.16	12.64	13.05	13.55
	100	15.32	15.67	15.98	16.54	17.04	17.67
	200	16.36	16.69	17.00	17.59	18.11	18.78
	∞	16.47	16.79	17.11	17.69	18.21	18.89

[a]The term structure of capped equity swap rates in the fixed and variable notional principle cases with the following model parameter specification: $t_0 = 0$, $S(0) = 1$, $Y(0) = 0.04$, $\sigma_Y = 0.61$, $\overline{Y} = 0.012$, $\kappa_Y = 5.06$, $\delta_1 = 0.1$, $\delta_2 = -0.1$, $\lambda_{x,y} = 1.64$, $\mu_0 = -0.03$, $\mu_{x,y} = -7.87$, $\sigma_{x,y} = 0.22$, $\theta_y = 0.0036$, $\gamma(t, T) = \gamma e^{-\kappa(T-t)}$, $\gamma = 0.01$, $\kappa = 0.05$, and $F(0, t) = 0.15 + 0.005t - 0.0002t^2$.

solution as simulation trials increase. The average computing time of one simulation trial in this case is approximately seven seconds. These computational results were obtained using a desktop PC running Windows XP Professional, with an Intel Pentium D 3.4 GHz processor and 1 GB of RAM. The codes were written using VC++ software.

Table II shows the term structure of swap rates for capped equity swaps in which the time period is one year. Note that no difference exists between capped equity swaps with a fixed notional principle and those with a variable notional principle if they are single-period swaps. Hence, when the swap matures in one year, swap rates of capped equity swaps with a fixed notional principle are the same as those with a variable notional principle. Table II further shows that swap rates increase and approach those of equity swaps as cap rates increase.

The impact of δ_1 on swap rates of capped equity swaps with a fixed notional principle is very different from the impact of δ_1 on swap rates of capped equity swaps with a variable notional principle (Table III). δ_1 becomes the impact parameter because of relative-payoff forward-start options. With a variable notional principle, δ_1 becomes the impact parameter because of variable notional principle equity swaps and absolute-payoff forward-start options. A low cap rate results in a high forward-start option value. Hence, the impact of δ_1 on forward-start options determines the impact on capped equity swap rates when cap rates are low. This observation explains the phenomenon that, when cap rates are low, the impact of δ_1 on the swap rates of capped equity swaps with a fixed notional principle is similar to the impact of δ_1 on the swap rates of capped

TABLE III
Swap Rates for Capped Equity Swaps

Notional Principle	*Cap Rate (%)*	δ_1(%)[a]				
		−20	*−10*	*0*	*10*	*20*
Fixed	25	3.87	6.17	7.07	6.00	3.59
	50	10.62	12.41	12.95	12.29	10.38
	100	15.56	16.19	16.36	16.16	15.46
	200	17.12	17.20	17.21	17.20	17.10
	∞	17.30	17.30	17.30	17.30	17.30
Variable	25	3.81	6.23	7.25	6.27	3.94
	50	10.50	12.44	13.14	12.64	10.85
	100	15.36	16.18	16.55	16.54	16.03
	200	16.88	17.17	17.39	17.59	17.70
	∞	17.05	17.26	17.48	17.69	17.90

[a]The impact of the correlation between interest rates and equity returns on capped equity swap rates in the fixed and variable notional principle cases with the following model parameter specification: $m = 5$, $t_0 = 0$, $t_m = 5$, $S(0) = 1$, $Y(0) = 0.04$, $\sigma_Y = 0.61$, $\overline{Y} = 0.012$, $\kappa_Y = 5.06$, $\delta_2 = -0.1$, $\lambda_{x,y} = 1.64$, $\mu_0 = -0.03$, $\mu_{x,y} = -7.87$, $\sigma_{x,y} = 0.22$, $\theta_y = 0.0036$, $\gamma(t, T) = \gamma e^{-\kappa(T-t)}$, $\gamma = 0.01$, $\kappa = 0.05$, $F(0, t) = 0.15 + 0.005t - 0.0002t^2$.

TABLE IV
Swap Rates for Capped Equity Swaps

Notional Principle	*Cap Rate (%)*	δ_2 (%)[a]				
		−20	*−10*	*0*	*10*	*20*
Fixed	25	6.01	6.00	5.98	5.97	5.97
	50	12.32	12.29	12.26	12.23	12.20
	100	16.17	16.16	16.14	16.12	16.10
	200	17.20	17.20	17.19	17.19	17.19
	∞	17.30	17.30	17.30	17.30	17.30
Variable	25	6.28	6.27	6.25	6.24	6.23
	50	12.67	12.64	12.61	12.58	12.55
	100	16.56	16.54	16.52	16.51	16.49
	200	17.59	17.59	17.58	17.58	17.58
	∞	17.69	17.69	17.69	17.69	17.69

[a]The impact of the correlation between variance rates and equity returns on capped equity swap rates in the fixed and variable notional principle cases with the following model parameter specification: $m = 5$, $t_0 = 0$, $t_m = 5$, $S(0) = 1$, $Y(0) = 0.04$, $\sigma_Y = 0.61$, $\overline{Y} = 0.012$, $\kappa_Y = 5.06$, $\delta_1 = 0.1$, $\lambda_{x,y} = 1.64$, $\mu_0 = -0.03$, $\mu_{x,y} = 7.87$, $\sigma_{x,y} = 0.22$, $\theta_y = 0.0036$, $\gamma(t, T) = \gamma e^{-\kappa(T-t)}$, $\gamma = 0.01$, $\kappa = 0.05$, and $F(0, t) = 0.15 + 0.005t - 0.0002t^2$.

equity swaps with a variable notional principle. When cap rates are high, forward-start option values approach zero and δ_1 exhibits no impact on the swap rates of capped equity swaps with a fixed notional principle; however, δ_1 impacts the swap rates of capped equity swaps with a variable notional principle.

Table IV shows that the impact of δ_2 on swap rates is different from that of δ_1. First, δ_2 becomes the impact parameter of capped equity swaps via

TABLE V
Swap Rates for Capped Equity Swaps

Notional Principle	*Cap Rate (%)*	$\mu_{x,y}$[a]				
		−15.74	*−7.87*	*0*	*7.87*	*15.74*
Fixed	25	5.60	6.00	6.06	5.72	4.96
	50	12.27	12.29	12.07	11.55	10.71
	100	16.23	16.16	15.99	15.68	15.16
	200	17.21	17.20	17.16	17.09	16.95
	∞	17.30	17.30	17.30	17.30	17.30
Variable	25	5.90	6.27	6.29	5.91	5.12
	50	12.65	12.64	12.38	11.83	10.96
	100	16.63	16.54	16.36	16.03	15.50
	200	17.61	17.59	17.55	17.47	17.33
	∞	17.69	17.69	17.69	17.69	17.69

[a]The impact of the correlation between level jumps and volatility jumps on capped equity swap rates in the fixed and variable notional principle cases with the following model parameter specification: $m = 5$, $t_0 = 0$, $t_m = 5$, $S(0) = 1$, $Y(0) = 0.04$, $\sigma_Y = 0.61$, $\overline{Y} = 0.012$, $\kappa_Y = 5.06$, $\delta_1 = 0.1$, $\delta_2 = -0.1$, $\lambda_{x,y} = 1.64$, $\mu_0 = -0.03$, $\sigma_{x,y} = 0.22$, $\theta_y = 0.0036$, $\gamma(t, T) = \gamma e^{-\kappa(T-t)}$, $\gamma = 0.01$, $\kappa = 0.05$, and $F(0, t) = 0.15 + 0.005t - 0.0002t^2$.

forward-start options. In the fixed notional principle case, δ_2 influences swap rates via relative-payoff forward-start options. In the variable notional principle case, δ_2 influences swap rates via absolute-payoff forward-start options. When cap rates are high, forward-start option values approach zero. Hence, in each case, δ_2 has no impact on swap rates. Second, forward-start option values increase as δ_2 increases and swap rates decrease as δ_2 increases. However, Table III shows that when cap rates are low, forward-start option values approach the minimum value and swap rates approach their maximum values as δ_1 approaches zero.

Table V substantiates the impact of the jump-amplitude correlation between level jumps and volatility jumps on capped equity swap rates. The parameter $\mu_{x,y}$ describes the jump-amplitude correlation between level jumps and volatility jumps and is often observed to be negative in the equity market. A negative value for $\mu_{x,y}$ implies that high volatility always accompanies bad news. However, the impact of $\mu_{x,y}$ on swap rates is somewhat complicated. As $\mu_{x,y}$ increases, swap rates gradually increase to approach the maximum value and subsequently decrease more rapidly in both the fixed and variable notional principle cases. When cap rates are low, the maximum swap rate appears near the specification of $\mu_{x,y} = 0$. As cap rates increase, the maximum swap rate gradually moves to the left. The impact of the jump frequency on capped equity swap rates is shown in Table VI. As $\lambda_{x,y}$ increases, swap rates decrease to approach zero in both the fixed and variable notional principle cases. When cap

TABLE VI
Swap Rates for Capped Equity Swaps

Notional Principle	*Cap Rate (%)*	$\lambda_{x,y}$[a]				
		0	*0.82*	*1.64*	*2.46*	*3.28*
Fixed	25	13.15	9.25	6.00	3.22	0.79
	50	16.76	14.58	12.29	10.05	7.92
	100	17.29	16.88	16.16	15.20	14.08
	200	17.30	17.28	17.20	17.03	16.77
	∞	17.30	17.30	17.30	17.30	17.30
Variable	25	13.44	9.53	6.27	3.48	1.05
	50	17.14	14.94	12.64	10.39	8.25
	100	17.68	17.26	16.54	15.58	14.46
	200	17.69	17.67	17.59	17.42	17.16
	∞	17.69	17.69	17.69	17.69	17.69

[a]The impact of the jump frequency on capped equity swap rates in the fixed and variable notional principle cases with the following model parameter specification: $m = 5$, $t_0 = 0$, $t_m = 5$, $S(0) = 1$, $Y(0) = 0.04$, $\sigma_Y = 0.61$, $\overline{Y} = 0.012$, $\kappa_Y = 5.06$, $\delta_1 = 0.1$, $\delta_2 = -0.1$, $\mu_0 = -0.03$, $\mu_{x,y} = -7.87$, $\sigma_{x,y} = 0.22$, $\theta_y = 0.0036$, $\gamma(t, T) = \gamma e^{-\kappa(T-t)}$, $\gamma = 0.01$, $\kappa = 0.05$, and $F(0, t) = 0.15 + 0.005t - 0.0002t^2$.

rates are low, swap rates rapidly decrease as $\lambda_{x,y}$ increases. When cap rates are high, swap rates gradually decrease as $\lambda_{x,y}$ increases.

NESTED HYBRID MODELS AND OTHER SPECIAL MODELS

The forward rate model of Heath, Jarrow, and Morton (HJM) (1992) nests several existing short rate models. According to Inui and Kijima (1998), the specification of $\gamma(t, T) = \gamma \exp\left\{-\int_t^T \kappa(s)ds\right\}$ for a constant parameter $\gamma(>0)$ and the deterministic function $\kappa(t)(\geq 0)$ reduces the HJM model to the short-rate model of Hull and White (HW) (1990): $dr(t) = [\theta(t) - \kappa(t)r(t)]dt + \gamma dW^F(t)$ where $\theta(t) = \kappa(t)f(0, t) + f_t(0, t) + \gamma^2 \int_0^t e^{-2\int_s^t \kappa(u)du} ds$. The model proposed by Ho and Lee (HL) (1986) is a reduced case of the HW model in which $\kappa(t) = 0$.

Note that the forward rate volatility of the HW model is a declining function of maturity, whereas the forward rate volatility of the HL model is the same for all maturities. However, empirical evidence from interest rate caps shows that the volatility structure of forward rates may have a humped shape. Figure 2 shows the volatility of the three-month forward rate as a function of maturity for the HJM, HW, and HL models. The value of $\gamma(t, T)$ in the HJM model can be selected to produce a maximum (hump) as a function of maturity that is consistent with empirical evidence.

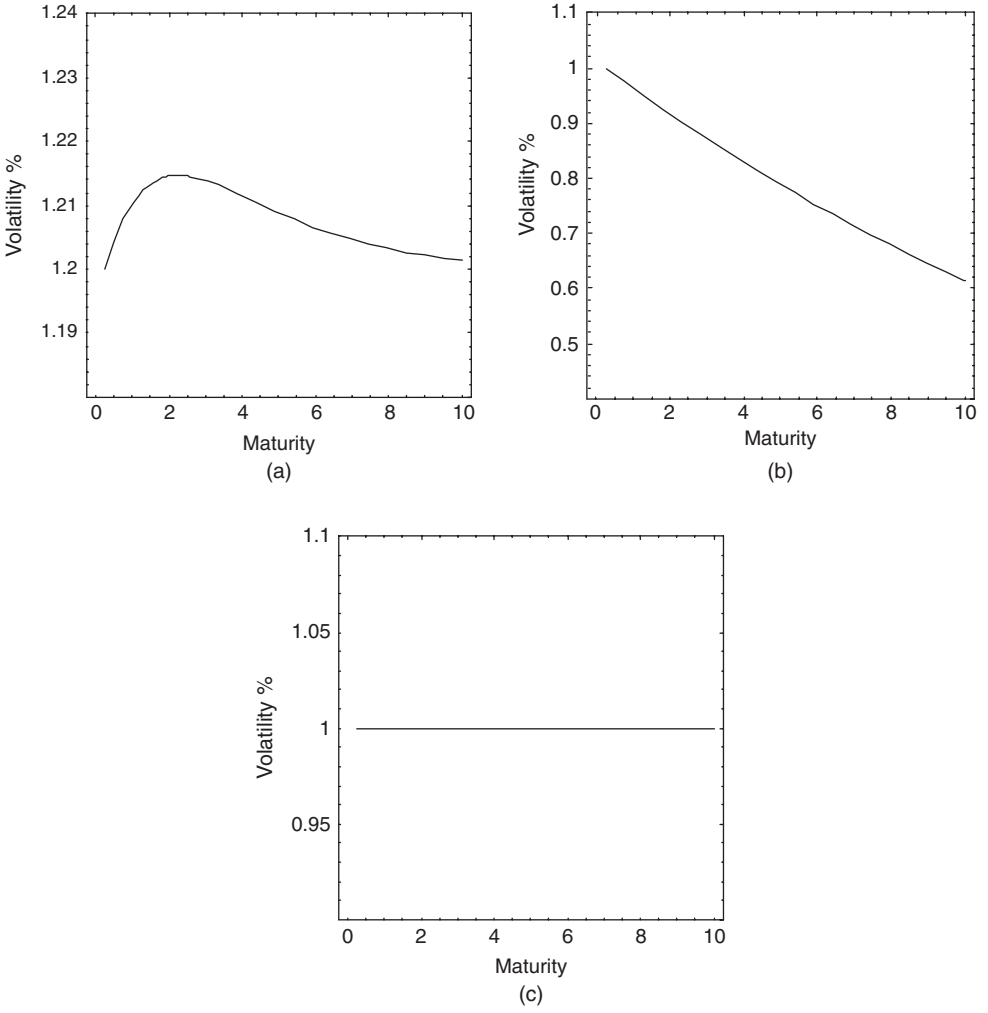


FIGURE 2
Forward rate volatility. Volatility of the three-month forward rate for (a) the HJM model, (b) the HW model, and (c) the HL model as a function of maturity.

In Table VII, the value of $\gamma(t, T)$ in the HJM model is chosen to be an increasing function of maturity, $\gamma(t, T) = \gamma \exp\left\{\int_t^T \kappa(s)ds\right\}$, to differentiate it from the HW model. In addition, if the jump arrival rate ($\lambda_{x,y}$) is set to zero, Equations (2) and (3) will reduce to Heston's stochastic volatility model (1993). Hence, several hybrid models are nested in the proposed model (DPS-HJM), including the DPS-HW, Heston-HJM, and Heston-HW models; DPS denotes the double-jump stochastic volatility model of Duffie, Pan, and Singleton (2000). Table VII provides a numerical comparison between these

TABLE VII
Capped Equity Swaps of Hybrid Models

Notional Principle	*Cap Rate (%)*	*DPS-HJM (%)*	*DPS-HW (%)*	*Heston-HJM (%)*	*Heston-HW (%)*
Fixed	15	−82.39	−74.22	−68.58	−58.68
	20	−75.85	−67.47	−61.85	−51.73
	25	−69.88	−61.36	−55.85	−45.63
	30	−64.44	−55.86	−50.54	−40.32
	35	−59.51	−50.90	−45.85	−35.72
Variable	15	−67.46	−73.27	−52.72	−55.10
	20	−58.19	−63.48	−42.94	−44.64
	25	−49.55	−54.44	−33.94	−35.17
	30	−41.49	−46.12	−25.67	−26.65
	35	−33.98	−38.49	−18.11	−19.00

DPS denotes the double-jump stochastic volatility model of Duffie, Pan, and Singleton [2000]. Heston denotes the stochastic volatility model of Heston [1993]. HJM denotes the forward rate model of Heath, Jarrow, and Morton [1992]. HW denotes the short rate model of Hull and White [1990]. Hence, DPS-HJM denotes the hybrid of the DPS and HJM models. Similarly, DPS-HW, Heston-HJM, and Heston-HW denote other hybrid models. Capped equity swaps are calculated with the following parameters: $R = 20\%$, $m = 5$, $t_0 = 0$, $t_m = 5$, $S(t_0) = S(0) = 1$, $Y(0) = 0.04$, $\sigma_Y = 0.61$, $\overline{Y} = 0.012$, $\kappa_Y = 5.06$, $\delta_1 = 0.1$, $\delta_2 = -0.1$, $\gamma = 0.2$, $\kappa = 0.1$, and $F(0,t) = 0.15 + 0.005t - 0.0002t^2$. In addition, the values $\lambda_{x,y} = 1.64$, $\mu_0 = -0.03$, $\mu_{x,y} = -7.87$, $\sigma_{x,y} = 0.22$, and $\theta_y = 0.0036$ are set for the DPS model, $\lambda_{x,y} = 0$ is set for the Heston model, $\gamma(t, T) = \lambda e^{\kappa(T-t)}$ is set for the HJM model, and $\gamma(t, T) = \gamma e^{-\kappa(T-t)}$ is set for the HW model.

hybrid models. Directly setting the value of $\lambda_{x,y}$ to zero reduces the kurtosis of the equity return and the embedded option value, and increases the capped equity swap value on the fixed rate payer side.

Note that the value of $\gamma(t, T)$ in the HW model is smaller than in the HJM model, which causes $a(t, T)$ in the HW model to be greater than $a(t, T)$ in the HJM model. Finally, the HW model incurs lower forward-start option values than the HJM model. Because values of equity swaps with a fixed notional principle do not depend on the stochastic interest rate process, their values are equivalent to those in the HW and HJM models. Hence, the capped equity swap value of the HW model is larger than that of the HJM model in the fixed notional principle case. However, in the variable notional principle case, a larger value of $a(t, T)$ results in a larger value of $\zeta(\bullet)$ and thereby results in a much smaller capped equity swap value. Hence, the capped equity swap value of the HW model is smaller than that of the HJM model in the variable notional principle case.

Another general model in which return and volatility jumps are nonsimultaneous and independent is included for comparison. This model includes stochastic volatility and independent double jumps (SVDJI). Let λ_x denote the arrival rate of the return jump and λ_y denote the arrival rate of the volatility jump. The return-jump amplitude is assumed to be a normal distribution, but is independent of the volatility jump:

TABLE VIII
Capped Equity Swaps with Various Jumps

Notional Principle	*Cap Rate (%)*	*DPS (%)*	*SVDJI (%)*	*SVJ (%)*	*SVJV (%)*
Fixed	15	−58.44	−39.79	−65.39	−39.53
	20	−51.10	−32.14	−58.29	−31.91
	25	−44.73	−26.01	−51.96	−25.83
	30	−39.26	−21.24	−46.36	−21.13
	35	−34.61	−17.64	−41.44	−17.62
Variable	15	−76.80	−51.98	−86.22	−51.60
	20	−66.93	−41.64	−76.68	−41.29
	25	−58.34	−33.33	−68.18	−33.05
	30	−50.97	−26.87	−60.65	−26.69
	35	−44.71	−21.99	−54.04	−21.92

Model	Model parameters	δ_2	κ_Y	$\overline{Y}$	σ_Y
DPS	$\mu_0 = -0.03$, $\mu_{x,y} = -7.87$, $\sigma_{x,y} = 0.22$, $\lambda_{x,y} = 1.64$, $\theta_V = 0.0036$	−0.1	5.06	1.19%	0.61
SVDJI	$\mu_x = -0.014$, $\lambda_x = 0.87$, $\lambda_y = 2.43$, $\sigma_x = 0.04$, $\theta_V = 0.0036$	−0.31	3.02	1.99%	0.54
SVJ	$\mu_x = -0.03$, $\lambda_x = 3.05$, $\sigma_x = 0.19$	−0.16	3.92	1.28%	0.41
SVJV	$\lambda_y = 1.36$, $\theta_y = 0.0016$	−0.26	2.58	2.33%	0.53

DPS denotes the stochastic volatility with simultaneous double jump model. SVDJI denotes the stochastic volatility with independent return jump and volatility jump model. SVJ denotes the stochastic volatility with return jump. SVJV denotes the stochastic volatility with volatility jump. Capped equity swap values are calculated with the following parameter specifications: $R = 20\%$, $m = 5$, $t_0 = -1$, $t_m = 5$, $S(t_0) = S(0) = 1$, $\delta_1 = 0.1$, $Y(0) = 0.04$, $\gamma(t, T) = \gamma e^{-\kappa(T-t)}$, $\gamma = 0.01$, $\kappa = 0.05$, and $F(0, t) = 0.15 + 0.005t - 0.0002t^2$.

$$x(t) \sim N\left(\log(1 + \mu_x) - \frac{1}{2}\sigma_x^2, \sigma_x^2\right). \tag{27}$$

The volatility-jump amplitude follows an exponential distribution with mean θ_y. The tri-variant characteristic function defined in Equation (11) has the same presentation as in Equation (12), but with a somewhat different expression of $A(t, u; \phi_S, \phi_V, \phi_Y, T, \tau)$. The relative-payoff forward-start option can be derived similarly to the proof in Appendix B. Both relative- and absolute-payoff forward-start option formulae are given in Appendix E. With the appropriate specifications of jump-related parameters, three models are nested within the SVDJI model—the Heston model ($\lambda_x = 0$ and $\lambda_y = 0$), the stochastic volatility and jump in return (SVJ) model ($\lambda_y = 0$), and the stochastic volatility and jump in volatility (SVJV) model ($\lambda_x = 0$). Table VIII presents a numerical study of various jump specifications. Model parameters for equity return distribution are taken from Table III in Bakshi and Cao (2003).

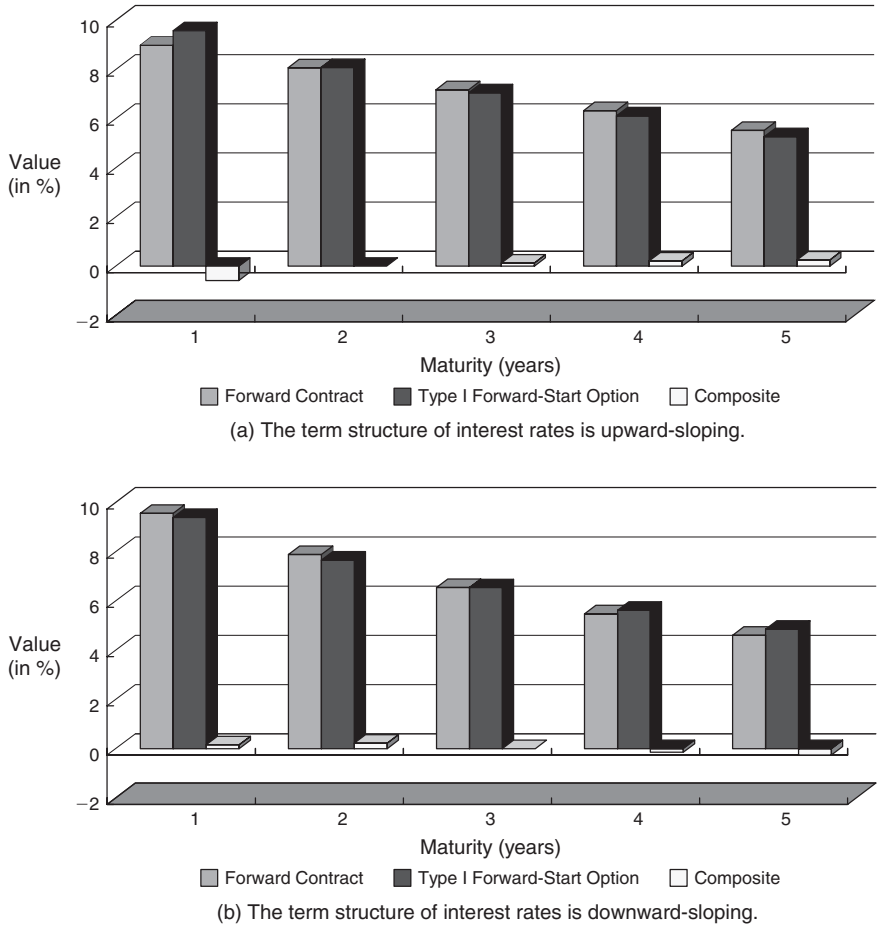


FIGURE 3
Capped equity swaps with a fixed notional principle. (a) The term structure of interest rates is upward-sloping. (b) The term structure of interest rates is downward-sloping.

COUNTERPARTY RISK AND FORWARD SMILE

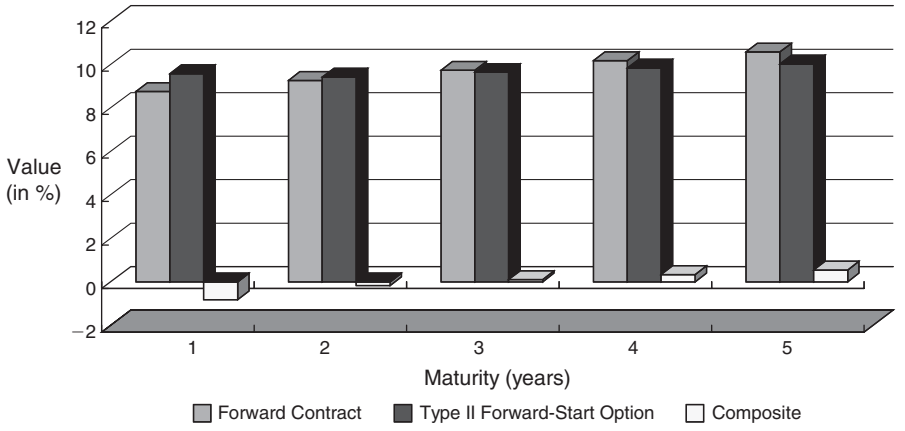
Because capped equity swaps are private arrangements between two parties and entail credit risks, it is important to identify which party has credit risk exposure as a result of the swap. The credit risk arises from the possibility of a default by the counterparty when the value of the contract is negative for the counterparty. The payoff of a capped equity swap in one period is a composite of a long forward contract and a short forward-start option. However, at the outset of a capped equity swap, the sum of values of composites is zero and some composites are positive, whereas others are negative.

Figure 3 illustrates forward contracts, forward-start options, and their composites underlying capped equity swaps with a fixed notional principle.

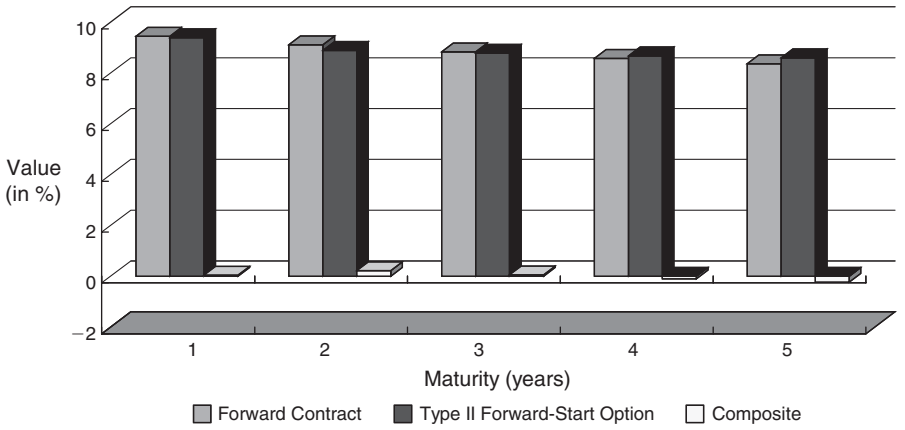
In Figure 3(a), the term structure of interest rates ($F(0, t) = 0.15 + 0.005t - 0.0002t^2$) is upward-sloping. Although values of forward contracts and forward-start options all decrease as maturity increases, their values are all positive. Because forward-start options cannot have negative values, forward contracts in capped swaps may be composed of all positive values. However, forward contract values decrease more slowly than relative-payoff forward-start option values and thus, values of composites (forward contract minus forward-start options) in the first two periods are negative, whereas those in the last three periods become positive. After paying the first constant payment, capped equity swaps with a fixed notional principle for the fixed rate payer have positive values during the remainder of the contract. Hence, the fixed rate payer bears the credit risk for almost the entire life of the contract. In Figure 3(b), the term structure of the interest rates ($F(0, t) = 0.15 - 0.005t + 0.0002t^2$) is downward-sloping. Although values of forward contracts and forward-start options are all positive and decrease as maturity increases, forward contract values decrease faster than relative-payoff forward-start option values. Thus, values of composites (forward contract minus forward-start options) in the first three periods are positive, whereas those in the last two periods become negative. After paying the first payment, capped equity swaps with a fixed notional principle for the capped equity-return payer have positive values during the remainder of the contract. Hence, the capped equity-return payer bears the credit risk for almost the entire life of the contract.

Figure 4 shows forward contracts, forward-start options, and their composites underlying capped equity swaps with a variable notional principle. In Figure 4(a), the term structure of interest rates ($F(0, t) = 0.15 + 0.005t - 0.0002t^2$) is upward-sloping. Notice that values of forward contracts and forward-start options all increase as maturity increases and forward contract values increase faster than absolute-payoff forward-start option values. However, similar to Figure 3(a), values of composites in the first two periods are negative, whereas those in the last three periods are positive. Hence, the fixed rate payer still bears the credit risk for almost the entire life of the contract. In Figure 4(b), the term structure of interest rates ($F(0, t) = 0.15 - 0.005t + 0.0002t^2$) is downward-sloping. Like those in Figure 3(b), values of forward contracts and forward-start options all decrease as maturity increases. Forward contract values decrease faster than absolute-payoff forward-start option values and thus, values of composites in the first three periods are positive, whereas those in the last two periods are negative. Therefore, the capped equity-return payer bears the credit risk for almost the entire life of the contract.

Finally, we are interested in the Black–Scholes pricing formula for the forward-start option because we need to use it when expressing the forward volatility of our model. Forward volatility is important because it determines



(a) The term structure of interest rates is upward-sloping.



(b) The term structure of interest rates is downward-sloping.

FIGURE 4
Capped equity swaps with a variable notional principle. (a) The term structure of interest rates is upward-sloping. (b) The term structure of interest rates is downward-sloping.

the conditional behavior of the process. Assuming deterministic term structures for volatility and interest rates, the value of the relative-payoff forward-start option at time T_0 can be expressed as $c(T_0; T_0, T, K) = BS(1, T - T_0, K, r', \sigma')$ where[2] r' is the forward interest rate[3] and σ' is the forward volatility rate between T_0 and T. Hence, we have

$$BS(1, T - T_0, K, r', \sigma') = \frac{c(0; T_0, T, K)}{V(0, T_0)}. \tag{28}$$

[2]The Black–Scholes volatilities are best implied from the relative-payoff forward-start options because $c(T_0; T_0, T, K)$ does not depend on $S(T_0)$ but $\bar{c}(T_0; T_0, T, K)$ does.
[3]The forward interest rate $r'$is given by $r' = -\ln(V(0, T)/V(0, T_0))/(T - T_0)$.

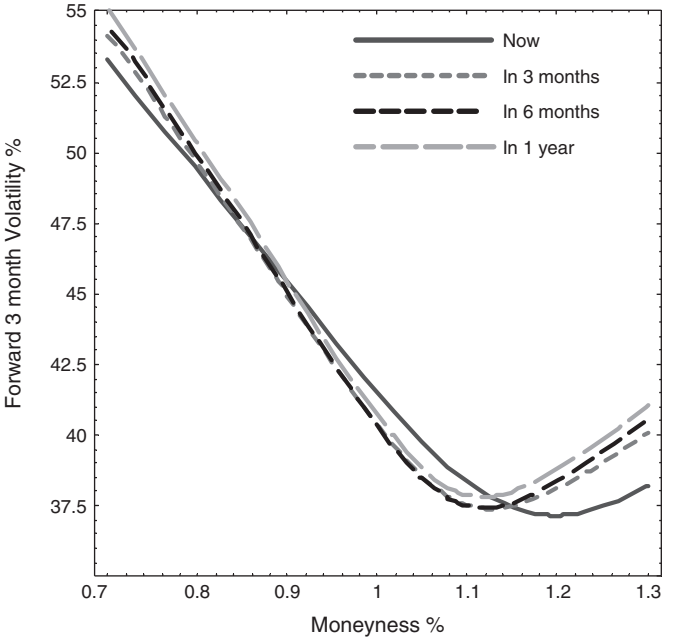


FIGURE 5
Forward volatility for three-month maturity. The forward smiles are calculated using the following parameter values: $Y(0) = 0.1$, $\sigma_Y = 1$, $\bar{Y} = 0.1$, $\kappa_Y = 2$, $\delta_1 = 0.1$, $\delta_2 = -0.1$, $\lambda_{x,y} = 1.64$, $\mu_0 = -0.03$, $\mu_{x,y} = -7.87$, $\sigma_{x,y} = 0.22$, $\theta_y = 0.0036$, $\gamma(t, T) = \gamma e^{-\kappa(T-t)}$, $\gamma = 0.01$, $\kappa = 0.05$, and $F(0, t) = 0.15 + 0.005t - 0.0002t^2$, for $T - T_0 = 0.25$ (three months). The current smile ($T_0 = 0$) is also plotted for reference.

The values of forward volatility σ', implied by Equation (28), are plotted in Figure 5 for a wide range of moneyness K and a variety of time horizons T_0, T. Figure 5 shows that forward smiles are more convex than the current smile because there is additional uncertainty associated with the value of future initial volatility to which this smile corresponds. The stochasticity of the initial volatility introduces excess kurtosis and results in a more convex forward smile.

Figure 6 describes the sensitivities of the forward smile to some model parameters. Setting $\delta_1 = 0$ implies that no correlation exists between forward rates and equity returns and thus incurs a lower volatility and forward smile. Note that setting $\delta_2 = 0$ and $\mu_{x,y} = 0$ affects the forward smile similarly because both $\delta_2 = -0.1$ and $\mu_{x,y} = -7.87$ imply a negative correlation between equity returns and variance rates. Setting $\delta_2 = 0$ or $\mu_{x,y} = 0$ increases the deep out-of-the-money ($K \gg 1$) option value and decreases the deep in-the-money ($K \ll 1$) option value. Hence, the forward volatility of the out-of-the-money option increases and in-the-money option decreases. Setting $\lambda_{x,y} = 0$ implies that there is

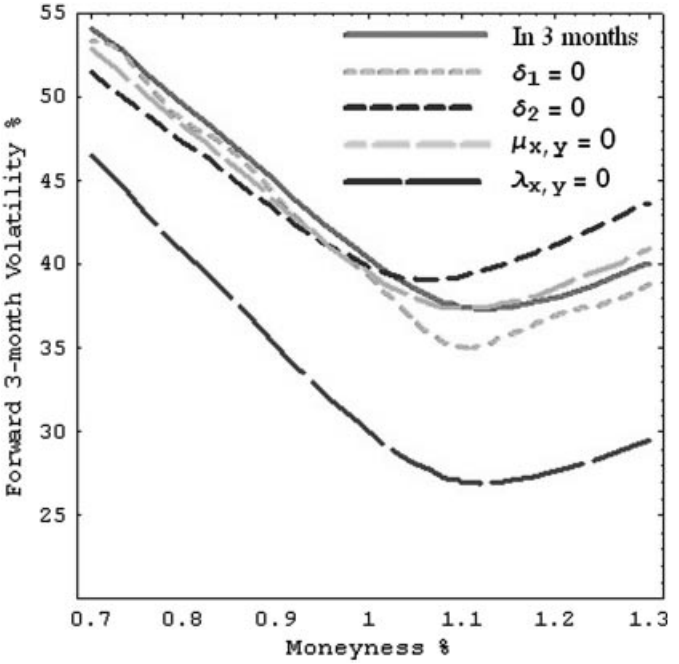


FIGURE 6
Sensitivities of forward three-month volatility in three months. The solid red line denotes the forward three-month volatility in three months calculated using the following parameter values: $Y(0) = 0.1$, $\sigma_Y = 1$, $\overline{Y} = 0.1$, $\kappa_Y = 2$, $\delta_1 = 0.1$, $\delta_2 = -0.1$, $\lambda_{x,y} = 1.64$, $\mu_0 = -0.03$, $\mu_{x,y} = -7.87$, $\sigma_{x,y} = 0.22$, $\theta_y = 0.0036$, $\gamma(t, T) = \gamma e^{-\kappa(T-t)}$, $\gamma = 0.01$, $\kappa = 0.05$, $F(0, t) = 0.15 + 0.005t - 0.0002t^2$, $T_0 = 0.25$, and $T = 0.5$. The green dotted line plots the forward three-month volatility in three months calculated with the same parameters, except that $\delta_1 = 0$. The other dotted lines are similarly obtained by omitting different parameters. The dotted lines are plotted for comparison.

no jump in the equity return and variance rate processes and thus, it incurs a lower volatility and forward smile.

CONCLUSIONS

A characteristic function-based approach is proposed to derive closed-form solutions of capped equity swaps under the double-jump stochastic volatility model with stochastic interest rates. Comparative statics using numerical examples demonstrate that stochastic volatility, double jumps, and stochastic interest rates all play a significant role in determining swap rates of capped equity swaps. In addition, studies of nested hybrid models and other special models show the generalizability of this method.

Moreover, analysis of the counterparty risk in capped equity swaps shows that the fixed rate payer bears the credit risk during almost the entire life of the contract when the term structure of interest rates is upward-sloping.

Conversely, the capped equity-return payer bears the credit risk when the term structure of interest rates is downward-sloping regardless of whether the notional principle is fixed or variable.

Finally, investigation of the forward smile shows that the stochasticity of future initial volatility incurs a more convex forward smile than the smile for the current day. Additionally, this investigation demonstrates that stochastic volatility, double jumps, and stochastic interest rates control the dynamics of the forward smile.

APPENDIX A

$$\Pi(\phi_S) \equiv \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left[\exp\left[\begin{array}{c} A(T_0, T - T_0; \phi_S, 0, 0, T, T) + iv \ln[V(T_0, T)] \\ + B(T - T_0; \phi_S; 0)Y(T_0) + iv \ln[K] \end{array}\right]\right]}{v} dv \tag{A1}$$

where

$$\begin{aligned} A(t, u; \phi_S, \phi_V, \phi_Y, T, \tau) \equiv{}& \frac{1}{2} i\phi_S(i\phi_S - 1)\int_0^u (\delta_1 - a(t + s, T))^2 ds \\ &+ \frac{1}{2} i\phi_V(i\phi_V - 1)\int_0^u (a(t + s, \tau) - a(t + s, T))^2 ds \\ &+ (i\phi_S)(i\phi_V)\int_0^u (\delta_1 - a(t + s, T))(a(t + s, \tau) - a(t + s, T)) ds \\ &+ \kappa_Y \overline{Y}\left[\begin{array}{l} \frac{-1}{\sigma_Y^2}(\varepsilon + i\phi_S\sigma_Y\delta_2 - \kappa_Y)u - \\ \frac{2}{\sigma_Y^2}\ln\left[1 - \frac{(\varepsilon + i\phi_S\sigma_Y\delta_2 - \kappa_Y + i\phi_Y\sigma_Y^2)(1 - \exp[-\varepsilon u])}{2\varepsilon}\right] \end{array}\right] \\ &+ \frac{\lambda_{x,y}(2\varepsilon - b)\exp[i\phi_S\mu_0 + \frac{1}{2}(i\phi_S)^2\sigma_{x,y}^2]}{p} u + \exp\left[i\phi_S\mu_0 + \frac{1}{2}(i\phi_S)^2\sigma_{x,y}^2\right] \\ &\times \frac{2\lambda_{x,y}\theta_y(i\phi_S(i\phi_S - 1) + (i\phi_Y\sigma_Y)^2 + 2(i\phi_Y)(i\phi_S\sigma_Y\delta_2 - \kappa_Y))}{pq} \ln\left[\frac{p + q\exp[-\varepsilon u]}{p + q}\right] \\ &- (i\phi_S)\lambda_{x,y}\left[\frac{\exp[\mu_0 + \frac{1}{2}\sigma_{x,y}^2]}{1 - \theta_y\mu_{x,y}} - 1\right]u - \lambda_{x,y}u, \end{aligned} \tag{A2}$$

and

$$B(u;\phi_S,\phi_Y) \equiv \frac{(i\phi_S(i\phi_S-1)+(i\phi_Y\sigma_Y)^2+2(i\phi_Y)(i\phi_S\sigma_Y\delta_2-\kappa_Y))(1-\exp[-\varepsilon u])}{2\varepsilon-(\varepsilon+i\phi_S\sigma_Y\delta_2-\kappa_Y+i\phi_Y\sigma_Y^2)(1-\exp[-\varepsilon u])} \tag{A3}$$

$$\text{with } \varepsilon \equiv \sqrt{(i\phi_S\sigma_Y\delta_2-\kappa_Y)^2-i\phi_S(i\phi_S-1)\sigma_Y^2},\ b \equiv \varepsilon+i\phi_Y\sigma_Y^2+i\phi_S\sigma_Y\delta_2-\kappa_Y,$$
$$p \equiv 2\varepsilon(1-(i\phi_Y+i\phi_S\mu_{x,y})\theta_y)-q, \text{ and}$$
$$q \equiv b(1-(i\phi_Y+i\phi_S\mu_{x,y})\theta_y)+\theta_y(i\phi_S(i\phi_S-1))+\theta_y((i\phi_Y\sigma_Y)^2 + 2(i\phi_Y)(i\phi_S\sigma_Y\delta_2-\kappa_Y)).$$

APPENDIX B

$$c(0;T_0,T,K) = E_0^Q\left[\frac{1}{B(0,T)}\left(\max\left\{\frac{S(T)-KS(T_0)}{S(T_0)},0\right\}\right)\right]$$

$$= E_0^Q\left[\frac{1}{B(0,T_0)}E_{T_0}^Q\left[\frac{1}{B(T_0,T)}\max\left\{\frac{S(T)-KS(T_0)}{S(T_0)},0\right\}\right]\right]$$

$$= E_0^Q\left[\frac{1}{B(0,T_0)}c(T_0;T_0,T,K)\right] = V(0,T_0)E_0^{T_0}[c(T_0;T_0,T,K)]$$

$$= V(0,T_0)E_0^{T_0}[\bar{c}(T_0;T_0,T,K)/S(T_0)]$$

$$= V(0,T_0)E_0^{T_0}[\Pi(-i-v)-V(T_0,T)K\Pi(-v)]$$

$$\frac{1}{2}V(0,T_0)\left(1-K\frac{V(0,T)}{V(0,T_0)}\right)-\frac{1}{\pi}V(0,T_0)$$

$$\times\left(\int_0^\infty \frac{\mathrm{Im}\left[\begin{array}{l}\exp A(T_0,T-T_0;-i-v,0,0,T,T)+iv\ln[K]]\times\\ E_0^{T_0}[\exp[B(T-T_0;-i-v,0)Y(T_0)+iv\ln[V(T_0,T)]]]\end{array}\right]}{v}dv\right)$$

$$+\frac{1}{\pi}V(0,T_0)K$$

$$\times\left(\int_0^\infty \frac{\mathrm{Im}\left[\begin{array}{l}\exp[A(T_0,T-T_0;-v,0,0,T,T)+iv\ln[K]]\times\\ E_0^{T_0}[\exp[B(T-T_0;-v,0)Y(T_0)+(1+iv)\ln[V(T_0,T)]]]\end{array}\right]}{v}dv\right)$$

$$= \frac{1}{2}\left(V(0,T_0) - KV(0,T)\right) - \frac{1}{\pi}V(0,T_0)$$
$$\times\left(\int_0^{\infty}\frac{\mathrm{Im}\begin{bmatrix}\exp[A(T_0,T-T_0;-i-v,0,0,T,T)+iv\ln[K]]\times\\ J(0,T_0;0,v,-iB(T-T_0;-i-v,0),T_0,T)\end{bmatrix}}{v}dv\right)+$$
$$\frac{1}{\pi}V(0,T_0)K\left(\int_0^{\infty}\frac{\mathrm{Im}\begin{bmatrix}\exp[A(T_0,T-T_0;-v,0,0,T,T)+iv\ln[K]]\times\\ J(0,T_0;0,-i+v,\ -iB(T-T_0;-v,0),T_0,T)\end{bmatrix}}{v}dv\right) \tag{B1}$$

where[4,5]

$$J(t,u;\phi_S,\phi_V,\phi_Y,T,\tau) = \exp\begin{bmatrix}A(t,u;\phi_S,\phi_V,\phi_Y,T,\tau)+B(u;\phi_S,\phi_Y)Y(t)+\\ i\phi_S\log[S^T(t)]+i\phi_V\log[V^T(t,\tau)]+i\phi_Y Y(t)\end{bmatrix}$$

Define $\underline{A}$ and $\underline{B}$ as follows:

$$\underline{A}(T_0,T;\phi_S,\phi_V',v,K) \equiv A(T_0,T-T_0;\phi_S,0,0,T,T) + iv\ln[K]$$
$$+\frac{1}{2}i\phi_V'(i\phi_V'-1)\int_0^{T_0}(a(s,T)-a(s,T_0))^2ds + \left(\frac{2\underline{\varepsilon}-\underline{b}}{\underline{p}}-1\right)\lambda_{x,y}T_0, \tag{B2}$$

$$\underline{B}(T_0,T;\phi_S,Y(0)) \equiv$$
$$\kappa_Y\overline{Y}\begin{bmatrix}\frac{-1}{\sigma_Y^2}(\underline{\varepsilon}+\kappa_Y)T_0+\\ \frac{-2}{\sigma_Y^2}\ln\left[1-\frac{(\underline{\varepsilon}-\kappa_Y+B(T-T_0;\phi_S,0)\sigma_Y^2)(1-\exp[-\underline{\varepsilon}T_0])}{2\underline{\varepsilon}}\right]\end{bmatrix}$$
$$+\frac{2\lambda_{x,y}\theta_y((B(T-T_0;\phi_S,0)\sigma_Y)^2+2B(T-T_0;\phi_S,0)(-\kappa_Y))}{\underline{pq}}\ln\left[\frac{\underline{p}+\underline{q}\exp[-\underline{\varepsilon}T_0]}{\underline{p}+\underline{q}}\right]$$
$$+\left(\frac{(B(T-T_0;\phi_S,0)^2\sigma_Y^2+2B(T-T_0;\phi_S,0)(-\kappa_Y))(1-\exp[-|\underline{\varepsilon}T_0])}{2\underline{\varepsilon}-(\underline{\varepsilon}-\kappa_Y+B(T-T_0;\phi_S,0)\sigma_Y^2)(1-\exp[-\underline{\varepsilon}T_0])}\right)Y(0)$$
$$+B(T-T_0;\phi_S,0)Y(0), \tag{B3}$$

[4]See Theorem I in Guo and Hung (2008).
[5]The iteration of improper integrals, namely Fubini's theorem, can be justified by Loya (2005).

with $\underline{\varepsilon} \equiv |\kappa_Y|$, $\underline{b} \equiv \underline{\varepsilon} + B(T - T_0; \phi_S, 0)\sigma_Y^2 - \kappa_Y$,

$\underline{q} \equiv \underline{b}(1 - B(T - T_0; \phi_S, 0)\theta_y) + \theta_y((B(T - T_0; \phi_S, 0)\sigma_Y)^2 +$

$2B(T - T_0; \phi_S, 0)(-\kappa_Y))$, and

$\underline{p} \equiv 2\underline{\varepsilon}(1 - B(T - T_0; \phi_S, 0)\theta_y) - \underline{q}$.

Given (B2) and (B3), (B1) can be written as

$$c(0; T_0, T, K) = -V(0, T_0)$$

$$\times \left(\frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\mathrm{lm}\left[\exp\left[\begin{matrix} A'(T_0, T; -v, -i + v, v, K) + B'(T_0, T; -v, Y(0)) + \\ iv(\ln[V(0, T)/V(0, T_0)]) \end{matrix} \right] \right]}{v} dv \right)$$

$$- V(0, T)K$$

$$\times \left(\frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\mathrm{lm}\left[\exp\left[\begin{matrix} A'(T_0, T; -v, -i + v, v, K) + B'(T_0, T; -v, Y(0)) + \\ iv(\ln[V(0, T)/V(0, T_0)]) \end{matrix} \right] \right]}{v} dv \right)$$

$$= V(0, T_0)\Phi(-i - v, v) - V(0, T)K\Phi(-v, -i + v), \tag{B4}$$

where

$$\Phi(\phi_S, \phi_V') \equiv \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\mathrm{lm}\left[\exp\left[\begin{matrix} \underline{A}(T_0, T; \phi_S, \phi_V', v, K) + \\ \underline{B}(T_0, T; \phi_S, Y(0)) + iv(\ln[V(0, T)/V(0, T_0)]) \end{matrix} \right] \right]}{v} dv$$

□

APPENDIX C

$$\zeta(0, T_0, T) = \exp\left[\int_0^{T_0} (\delta_1 - a(s, T_0))(a(s, T) - a(s, T_0))ds \right] \tag{C1}$$

$$\overline{\Pi}(\phi_S, \phi_V) \equiv \frac{1}{2} -$$

$$\frac{1}{\pi} \int_0^\infty \frac{\mathrm{Im}[\exp[\overline{A}(T_0, T; \phi_S, \phi_V) + \overline{B}(T_0, T; \phi_S) + iv(\ln[V(0, T)/V(0, T_0)])]]}{v} dv \tag{C2}$$

$$\begin{aligned}\overline{A}(T_0, T; \phi_S, \phi_V) \equiv\ & A(T_0, T - T_0; \phi_S, 0, 0, T, T) + iv \ln[K] \\ & + iv \int_0^{T_0} (\delta_1 - a(s, T_0))(a(s, T) - a(s, T_0))ds \\ & + \frac{1}{2} i\phi_V(i\phi_V - 1) \int_0^{T_0} (a(s, T) - a(s, T_0))^2 ds \\ & - \lambda_{x,y}\left[\frac{\exp[\mu_0 + \frac{1}{2}\sigma_{x,y}^2]}{1 - \theta_y \mu_{x,y}}\right] T_0\end{aligned} \tag{C3}$$

$$\begin{aligned}\overline{B}(T_0, T; \phi_S) \equiv\ & \kappa_Y \overline{Y} \left[\begin{array}{l} \frac{-1}{\sigma_Y^2}(\overline{\varepsilon} + \sigma_Y \delta_2 - \kappa_Y) T_0 + \\ \frac{-2}{\sigma_Y^2} \ln\left[1 - \frac{(\overline{\varepsilon} + \sigma_Y \delta_2 - \kappa_Y + B(T - T_0; \phi_S, 0)\sigma_Y^2)(1 - \exp[-\overline{\varepsilon} T_0])}{2\overline{\varepsilon}}\right] \end{array}\right] \\ & + \frac{\lambda_{x,y}(2\overline{\varepsilon} - \overline{b})\exp[\mu_0 + \frac{1}{2}\sigma_{x,y}^2]}{\overline{p}} T_0 + \exp\left[\mu_0 + \frac{1}{2}\sigma_{x,y}^2\right] \\ & \times \frac{2\lambda_{x,y}\theta_y((B(T - T_0; \phi_S, 0)\sigma_Y)^2 + 2B(T - T_0; \phi_S, 0)(\sigma_Y \delta_2 - \kappa_Y))}{\overline{pq}} \ln\left[\frac{\overline{p} + \overline{q}\exp[-\overline{\varepsilon} T_0]}{\overline{p} + \overline{q}}\right] \\ & + \left(\frac{(B(T - T_0; \phi_S, 0)^2 \sigma_Y^2 + 2B(T - T_0; \phi_S, 0)(\sigma_Y \delta_2 - \kappa_Y))(1 - \exp[-|\overline{\varepsilon} T_0|])}{2\overline{\varepsilon} - (\overline{\varepsilon} + \sigma_Y \delta_2 - \kappa_Y + B(T - T_0; \phi_S, 0)\sigma_Y^2)(1 - \exp[-\overline{\varepsilon} T_0])}\right) Y(0) \\ & + B(T - T_0; \phi_S, 0) Y(0)\end{aligned} \tag{C4}$$

with $\overline{\varepsilon} \equiv |\sigma_Y \delta_2 - \kappa_Y|$, $\overline{b} \equiv \overline{\varepsilon} + B(T - T_0; \phi_S, 0)\sigma_Y^2 + \sigma_Y \delta_2 - \kappa_Y$,
$\overline{q} \equiv \overline{b}(1 - (B(T - T_0; \phi_S, 0) + \mu_{x,y})\theta_y) +$
$\theta_y((B(T - T_0; \phi_S, 0)\sigma_Y)^2 + 2B(T - T_0; \phi_S, 0)(\sigma_Y \delta_2 - \kappa_Y))$, and
$\overline{p} \equiv 2\overline{\varepsilon}(1 - (B(T - T_0; \phi_S, 0) + \mu_{x,y})\theta_y) - \overline{q}$.

APPENDIX D

Lemma I: If the price of $S(t)$ relative to the money market account, $B(0, t)$, is a martingale under the risk-neutral measure, Q, and the prices of $S(t)$ and $V(t, \tau)$ relative to the discount bond with maturity T, $V(t, T)$, the forward-neutral measure, yields

$$E_0^Q\left[\frac{S(T)V(T, \tau)}{B(0, T)}\right] = V(0, T)E_0^T[S(T)V(T, \tau)],\ T \leq \tau. \tag{D1}$$

Proof: Consider a self-financing portfolio that replicates the payoff, $S(T)V(T, \tau)$, at time T, and consists of the risky asset, $S(t)$, and the money market account, $B(0, t)$. Let $\eta(t)$ be the number of units of the risky asset held in the portfolio, and let $\alpha(t)$ denote the amount in the money market account. Therefore, the value of the portfolio is given by

$$C(T; \tau) = C(0; \tau) + \int_0^T \alpha(s)dB(0, s) + \int_0^T \eta(s)dS(s), \; T \leq \tau. \tag{D2}$$

Let $S^*(t)$ denote the relative price of $S(t)$ with respect to the money market account $B(0, t)$:

$$S^*(t) = \frac{S(t)}{B(0, t)}. \tag{D3}$$

Other relative prices are defined similarly. Because $B^*(0, t) = 1$ and $dB^*(0, t) = 0$ for all t, it follows that $C^*(T; \tau) = C^*(0; \tau) + \int_0^\tau \eta(s)dS^*(s)$. Hence, if $S^*(t)$ is a martingale under the risk-neutral measure Q, the relative price process $C^*(t; \tau)$ is also a martingale. Let E^Q denote the expectation operator under Q. Because $C^*(T; \tau) = C(T; \tau) / B(0, T)$ and $B(0, 0) = 1$, we obtain

$$E_0^Q\left[\frac{C(T; \tau)}{B(0, T)}\right] = E_0^Q[C^*(T; \tau)] = C^*(0; \tau) = C(0; \tau). \tag{D4}$$

The same European derivative security also can be priced under the forward-neutral measure. Hence, there is also a self-financing portfolio that replicates the payoff, $S(T)V(T, \tau)$, at time T, and consists of the risky asset, $S(t)$, and the discount bond, $V(t, \tau)$. Let $\eta(t)$ be the number of units of the risky asset, and let $\beta(t)$ denote the number of discount bonds held in the portfolio. Therefore, the value of the portfolio is given by

$$C(T; \tau) = C(0; \tau) + \int_0^T \beta(s)dV(s, \tau) + \int_0^T \eta(s)dS(s). \tag{D5}$$

Let $S^T(t)$ be the forward price of $S(t)$ defined by $V(t, T)$:

$$S^T(t) = \frac{S(t)}{V(t, T)}, \; t \leq T. \tag{D6}$$

Other forward prices are defined similarly. Subsequently, we have

$$C^T(T; \tau) = C^T(0; \tau) + \int_0^T \beta dV^T(s, \tau) + \int_0^T \eta(s)dS^T(s). \tag{D7}$$

Hence, if $V^T(t, \tau)$ and $S^T(t)$ are both martingales under the forward-neutral measure Q^T, the forward price process $C^T(t; \tau)$ is also a martingale. Denoting the expectation operator under Q^T by E^T, it follows that $E_0^T[C^T(T; \tau)] = C^T(0; \tau)$. Because $V(T, T) = 1$ and $C^T(T; \tau) = C(T; \tau)/V(T, T)$, we obtain

$$C(0; \tau) = V(0, T)E_0^T[C^T(T; \tau)] = V(0, T)E_0^T[C(T; \tau)]. \tag{D8}$$

From (D4) and (D8), we have

$$E_0^Q\left[\frac{S(T)V(T,\tau)}{B(0,T)}\right] = E_0^Q\left[\frac{C(T;\tau)}{B(0,T)}\right] = V(0,T)E_0^T[C(T;\tau)] = V(0,T)E_0^T[S(T)V(T,\tau)]. \tag{D9}$$

Lemma I. □

Next, we complete the derivation of the valuation formula of the equity swap with a variable notional principle.

$$\begin{aligned}
PV^{/}(t_0, t_m) &= \sum_{i=1}^{m} E_0^Q\left[\frac{1}{B(0,t_i)}\left(\left(\frac{S(t_i)}{S(t_{i-1})} - 1\right) - R\right)\frac{S(t_{i-1})}{S(t_0)}\right] \\
&= \frac{1}{S(t_0)}\sum_{i=1}^{m} E_0^Q\left[\frac{S(t_i)}{B(0,t_i)}\right] - \frac{1+R}{S(t_0)}E_0^Q\left[\frac{S(t_0)}{B(0,t_1)}\right] - \frac{1+R}{S(t_0)}\sum_{i=2}^{m} E_0^Q\left[\frac{S(t_{i-1})}{B(0,t_{i-1})B(t_{i-1},t_i)}\right] \\
&= \frac{1}{S(t_0)}\sum_{i=1}^{m} S(0) - (1+R)E_0^Q\left[\frac{1}{B(0,t_1)}\right] - \frac{1+R}{S(t_0)}\sum_{i=2}^{m} E_0^Q\left[\frac{S(t_{i-1})}{B(0,t_{i-1})}E_{t_{i-1}}^Q\left[\frac{1}{B(t_{i-1},t_i)}\right]\right] \\
&= m\frac{S(0)}{S(t_0)} - (1+R)V(0,t_1) - \frac{1+R}{S(t_0)}\sum_{i=2}^{m} E_0^Q\left[\frac{S(t_{i-1})}{B(0,t_{i-1})}V(t_{i-1},t_i)\right] \\
&= m\frac{S(0)}{S(t_0)} - (1+R)V(0,t_1) - \frac{1+R}{S(t_0)}V(0,t_{i-1})E_0^{t_{i-1}}[S^{t_{i-1}}(t_{i-1})V^{t_{i-1}}(t_{i-1},t_i)] \\
&= m\frac{S(0)}{S(t_0)} - (1+R)V(0,t_1) - \frac{1+R}{S(t_0)}\sum_{i=2}^{m} V(0,t_{i-1})J(0,t_{i-1};-i,-i,0,t_{i-1},t_i) \\
&= m\frac{S(0)}{S(t_0)} - (1+R)V(0,t_1) \\
&\quad - \frac{1+R}{S(t_0)}\sum_{i=2}^{m} V(0,t_{i-1})S^{t_{i-1}}(0)V^{t_{i-1}}(0,t_i)\exp\left[\int_0^{t_{i-1}}(\delta_1 - a(s,t_{i-1}))(a(s,t_i) - a(s,t_{i-1}))ds\right] \\
&= \frac{S(0)}{S(t_0)}\left[m - (1+R)\sum_{i=2}^{m}\frac{V(0,t_i)}{V(0,t_{i-1})} \times \zeta(0,t_{i-1},t_i)\right] - (1+R)V(0,t_1). \qquad \text{(D10)}
\end{aligned}$$

APPENDIX E
The SVDJI-HJM Model

$$
\begin{aligned}
A(t, u; \phi_S, \phi_V, \phi_Y, T, \tau) \equiv{}& \frac{1}{2} i\phi_S(i\phi_S - 1) \int_0^u (\delta_1 - a(t+s, T))^2 ds \\
&+ \frac{1}{2} i\phi_V(i\phi_V - 1) \int_0^u (a(t+s, \tau) - a(t+s, T))^2 ds \\
&+ (i\phi_S)(i\phi_V) \int_0^u (\delta_1 - a(t+s, T))(a(t+s, \tau) - a(t+s, T)) ds \\
&+ \kappa_Y \bar{Y} \left[\begin{array}{l} \dfrac{-1}{\sigma_Y^2} (\varepsilon + i\phi_S \sigma_Y \delta_2 - \kappa_Y) u - \\ \dfrac{1}{\sigma_Y^2} \ln\left[1 - \dfrac{(\varepsilon + i\phi_S \sigma_Y \delta_2 - \kappa_Y + i\phi_Y \sigma_Y^2)(1 - \exp[-\varepsilon u])}{2\varepsilon} \right] \end{array} \right] \\
&- \lambda_y u + \frac{\lambda_y (2\varepsilon - b)}{\bar{\bar{p}}} u \\
&+ \frac{2\lambda_y \theta_y (i\phi_S(i\phi_S - 1) + (i\phi_Y \sigma_Y)^2 + 2(i\phi_Y)(i\phi_S \sigma_Y \delta_2 - \kappa_Y))}{\bar{\bar{p}}\bar{\bar{q}}} \ln\left[\frac{\bar{\bar{p}} + \bar{\bar{q}} \exp[-\varepsilon u]}{\bar{\bar{p}} + \bar{\bar{q}}} \right] \\
&- (i\phi_S)\lambda_x \mu_x u + \lambda_x \left[(1 + \mu_x)^{i\phi_S} \exp\left[\frac{1}{2} i\phi_S (i\phi_S - 1) \sigma_x^2 \right] - 1 \right] u
\end{aligned}
\tag{E1}
$$

where

$$
\bar{\bar{q}} \equiv b(1 - (i\phi_Y)\theta_y) + \theta_y(i\phi_S(i\phi_S - 1) + (i\phi_Y \sigma_Y)^2 + 2(i\phi_Y)(i\phi_S \sigma_Y \delta_2 - \kappa_Y)) \tag{E2}
$$

$$
\bar{\bar{p}} \equiv 2\varepsilon(1 - (i\phi_Y)\theta_y) - \bar{\bar{q}}. \tag{E3}
$$

Type I Forward-Start Call Option

The present value of Type I forward-start call option is given by

$$
c(0; T_0, T, K) = V(0, T_0) \underline{\Phi}(-i - v, v) - V(0, T) K \underline{\Phi}(-v, -i + v) \tag{E4}
$$

where

$$
\underline{\Phi}(\phi_S, \phi_V') = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\mathrm{Im}\left[\exp\left[\begin{array}{l} \underline{\underline{A}}\ (T_0, T; \phi_S, \phi_V', v, K) + \\ \underline{\underline{B}}\ (T_0, T; \phi_S, Y(0)) + iv(\ln[V(0, T)/V(0, T_0)]) \end{array} \right] \right]}{v} dv \tag{E5}
$$

$$\underline{\underline{A}}\ (T_0, T; \phi_S, \phi'_V, v, K) = A(T_0, T - T_0; \phi_S, 0, 0, T, T) + iv \ln[K] + \frac{1}{2} i\phi'_V(i\phi'_V - 1) \int_0^{T_0} (a(s, T) - a(s, T_0))^2 ds + \left(\frac{2\underline{\varepsilon} - \underline{b}}{\underline{p}} - 1\right)\lambda_y T_0 \tag{E6}$$

$$\underline{\underline{B}}(T_0, T; \phi_S, Y(0)) \equiv \kappa_Y \overline{Y} \left[\begin{array}{l} \dfrac{-1}{\sigma_Y^2}(\underline{\varepsilon} - \kappa_Y) T_0 + \\ \dfrac{-2}{\sigma_Y^2} \ln\left[1 - \dfrac{(\underline{\varepsilon} - \kappa_Y + B(T - T_0; \phi_S, 0)\sigma_Y^2)(1 - \exp[-\underline{\varepsilon} T_0])}{2\underline{\varepsilon}}\right] \end{array} \right] + \frac{2\lambda_y \theta_y ((B(T - T_0; \phi_S, 0)\sigma_Y)^2 + 2B(T - T_0; \phi_S, 0)(-\kappa_Y))}{\underline{p}\underline{q}} \ln\left[\frac{\underline{p} + \underline{q} \exp[-\underline{\varepsilon} T_0]}{\underline{p} + \underline{q}}\right] + \left(\frac{(B(T - T_0; \phi_S, 0)^2 \sigma_Y^2 + 2B(T - T_0; \phi_S, 0)(-\kappa_Y))(1 - \exp[-|\underline{\varepsilon} T_0|])}{2\underline{\varepsilon} - (\underline{\varepsilon} - \kappa_Y + B(T - T_0; \phi_S, 0)\sigma_Y^2)(1 - \exp[-\underline{\varepsilon} T_0])}\right) Y(0) + B(T - T_0; \phi_S, 0) Y(0). \tag{E7}$$

Type II Forward-Start Call Option

The present value of Type II forward-start call option is given by

$$\bar{c}(0; T_0, T, K) = S(0)\overline{\overline{\Pi}}(-i - v, v) - S(0)K \frac{V(0, T)}{V(0, T_0)} \mathfrak{s}(0, T_0, T)\overline{\overline{\Pi}}(-v, -i + v) \tag{E8}$$

where[6]

$$\overline{\overline{\Pi}}(\phi_S, \phi_V) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\mathrm{Im}\left[\exp\left[\begin{array}{l}\overline{\overline{A}}\ (T_0, T; \phi_S, \phi_V) + \\ \overline{\overline{B}}\ (T_0, T; \phi_S) + iv(\ln[V(0, T)/V(0, T_0)])\end{array}\right]\right]}{v} dv \tag{E9}$$

$$\overline{\overline{A}}(T_0, T; \phi_S, \phi_V) = A(T_0, T - T_0; \phi_S, 0, 0, T, T) + iv \ln[K] + iv \int_0^{T_0} (\delta_1 - a(s, T_0))(a(s, T) - a(s, T_0)) ds + \frac{1}{2} i\phi_V(i\phi_V - 1) \int_0^{T_0} (a(s, T) - a(s, T_0))^2 ds - \lambda_y T_0 \tag{E10}$$

[6]See Appendix E in Guo and Hung (2008).

$$\overline{\overline{B}}(T_0,T;\phi_S)\equiv\kappa_Y\overline{Y}\left[\begin{array}{l}\frac{-1}{\sigma_Y^2}(\bar{\varepsilon}+\sigma_Y\delta_2-\kappa_Y)T_0+\\ \frac{-2}{\sigma_Y^2}\ln\left[1-\frac{(\bar{\varepsilon}+\sigma_Y\delta_2-\kappa_Y+B(T-T_0;\phi_S,0)\sigma_Y^2)(1-\exp[-\bar{\varepsilon}T_0])}{2\bar{\varepsilon}}\right]\end{array}\right]$$

$$+\frac{\lambda_y(2\bar{\varepsilon}-\bar{b})}{\hat{p}}T_0$$

$$+\frac{2\lambda_y\theta_y((B(T-T_0;\phi_S,0)\sigma_Y)^2+2B(T-T_0;\phi_S,0)(\sigma_Y\delta_2-\kappa_Y))}{\hat{p}\hat{q}}$$

$$\ln\left[\frac{\hat{p}+\hat{q}\exp[-\bar{\varepsilon}T_0]}{\hat{p}+\hat{q}}\right]$$

$$+\left(\frac{(B(T-T_0;\phi_S,0)^2\sigma_Y^2+2B(T-T_0;\phi_S,0)(\sigma_Y\delta_2-\kappa_Y))(1-\exp[-|\bar{\varepsilon}T_0])}{2\bar{\varepsilon}-(\bar{\varepsilon}+\sigma_Y\delta_2-\kappa_Y+B(T-T_0;\phi_S,0)\sigma_Y^2)(1-\exp[-\bar{\varepsilon}T_0])}\right)Y(0)$$

$$+B(T-T_0;\phi_S,0)Y(0), \tag{E11}$$

$$\hat{q}=\bar{b}(1-B(T-T_0;\phi_S,0)\theta_y)+\theta_y((B(T-T_0;\phi_S,0)\sigma_Y)^2 + 2B(T-T_0;\phi_S,0)(\sigma_Y\delta_2-\kappa_Y)) \tag{E12}$$

and

$$\hat{p}\equiv 2\bar{\varepsilon}(1-B(T-T_0;\phi_S,0)\theta_y)-\hat{q}. \tag{E13}$$

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