Fuzzy Reliability Using a Discrete Stress-Strength Interference Model

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Summary & Conclusions — **This paper uses a fuzzy line-segment method to calculate the fuzzy unreliability of a system when only discrete-interval probabilities of stress & strength inside an interference region are available. The discrete-interval probabilities are treated as fuzzy numbers. A stress-strength interference model and extended operations of fuzzy numbers are used to calculate the fuzzy unreliability. Probability density functions are approximated by piecewise fuzzy line-segments that are expressed by linear fuzzy polynomials; the centroid of the membership function for the fuzzy unreliability is treated as the point estimate of unreliability. This method requires less computer time than previous methods in the literature, and is useful since it does not require information on distribution types of stress & strength. Numerical examples demonstrate the method.**

1. INTRODUCTION

Acronyms & Abbreviations

LR	left, right
TFN	triangular fuzzy number
fz-	fuzzy, or fuzzified
LSM	line-segment method.

Stress-strength interference models have been widely used for reliability analysis. Unreliability, in such models, is Pr{stress > strength}. The point estimate and bounds on the failure probability can be computed by analytic or numerical approaches [1] once the distributions of stress & strength are available. However, in the real world, it is often difficult to know the true distributions over the complete range of the r.v. of the stress & strength. Although there are methods [2] of fitting parametric models for the distributions, Kapur [3] devised an approach that requires only information regarding the interval probabilities within an interference region for determining the bounds on the exact unreliability. Ref [3] assumed that failure occurs whenever stress and strength fall into the same subinterval. However, the domination degree of stress, which in essence represents Pr{stress > strength} in a subinterval, can be any value between 0 and 1. As a result, the bounds in [3] are overestimated. Other approaches [4 - 6] also deal with the domination degree of stress.

All approaches in [1 - 6] use only the distinct values 0, 0.5, or 1 for the domination degree of stress. To determine the domination degree of stress, Wang & Liu [7] developed a multiple-LSM based on a geometrical viewpoint regarding relationships between the distributions of stress & strength. Their results demonstrated that the accuracy is satisfactory even if there are only 8 subintervals. However, the time consumed in solving a quadratic programming problem increases rapidly with the number of variables.

This paper regards the interval probabilities of stress & strength as TFN of LR-type, which are efficient in computation. In section 2, the line segments introduced in [7] for approximating the pdf become *fz*-line-segments. Section 3 presents two numerical examples to demonstrate the low computation time and high accuracy of *fz*-LSM.

Notation

$\hat{h}$	measure of fuzziness
$L(\cdot)$, $R(\cdot)$	[left, right] reference function of a *fz*-number
Φ	failure probability
L_Φ, U_Φ	[lower, upper] bound of failure probability
N	number of subintervals
S, C	[stress, strain]
$\hat{\Phi}$	point estimate of Φ
$\mu(\cdot)$	membership function
a_{i-1}, a_i	[left, right] ends of subinterval i
$\hat{}$	implies: *fz*-number
$\oplus$, $\ominus$	[addition, subtraction]
$\odot$	multiplication of a scalar and a *fz*-number
$\otimes$	multiplication of two *fz*-numbers.

Other, standard notation is given in "Information for Readers & Authors" at the rear of each issue.

2. *fz*-LINE-SEGMENT METHOD

Assumptions

0. (see appendix)
1a. Stress & strength are continuous *s*-independent r.v.
1b. S_{max} & C_{min} are specified.
2. Pr{stress > S_{max}} = 0.
3. Pr{strain < C_{min}} = 0.
4. The fuzziness of *fz*-pdf remains constant throughout each subinterval. ◄

Thus,

$$\Phi = \Pr\{S > C\} = \int_{-\infty}^{+\infty} f_s(s) \cdot F_c(s)\, ds \tag{1}$$

$$= \int_{C_{\min}}^{S_{\max}} f_s(s)\cdot[F_c(s) - F_c(C_{\min})]\ ds$$

$$\approx \sum_{i=1}^{n} p_i \cdot \sum_{k=1}^{i-1} q_k + D_i \tag{2}$$

$D_i \equiv$ domination degree of stress

$$= \int_{a_{i-1}}^{a_i} f_s(s)\cdot[F_c(s) - F_c(a_{i-1})]\ ds; \tag{3}$$

$$p_i \equiv \Pr\{a_{i-1} < S \le a_i\},$$

$$q_i \equiv \Pr\{a_{i-1} < C \le a_i\}, \tag{4}$$

for $i=1, 2, \ldots, n$.

The interval probabilities are, in general, obtained by experiment; thus they suffer from uncertainty. To deal with this uncertainty, we treat interval probabilities as *fz*-numbers of LR-type denoted by:

$$\tilde{F}_i = (m_{F_i}, \alpha_{F_i}, \beta_{F_i})_{\rm LR} > 0,$$

$m_{F_i} \equiv$ point estimate of the interval probability

$$\alpha_{F_i}, \beta_{F_i} \equiv \lambda\cdot m_{F_i};\ \lambda \in [0,1]. \tag{5}$$

The value of λ depends on the fuzziness of m_{F_i}. The unknown pdf's of stress & strength in each subinterval are approximated by *fz*-line segments, which can in turn be represented by piecewise linear *fz*-polynomials:

$$\tilde{f}_i(x) = (\tilde{\tau}_i \odot x)\oplus\tilde{\eta}_i, \tag{6}$$

$$\tilde{\tau}_i \equiv (m_{\tau_i}, \alpha_{\tau_i}, \beta_{\tau_i})_{\rm LR}$$

$$\tilde{\eta}_i \equiv (m_{\eta_i}, \alpha_{\eta_i}, \beta_{\eta_i})_{\rm LR}$$

To determine $\tilde{\tau}_i$ & $\tilde{\eta}_i$, use assumptions 5 - 7.

Assumptions (for ease of computation)

5. $l = a_i - a_{i-1}$, independent of i.

6. The value of the pdf at the midpoint of subinterval i equals the average value of the pdf in the subinterval:

$$\tilde{f}_i(\hat{a}_i) = \tilde{F}_i/l, \tag{7}$$

$$\hat{a}_i = ½(a_i + a_{i-1}).$$

7. $\tilde{F}_i$ remains invariant under the approximation of the pdf curve in subinterval i; thus,

$$\tilde{F}_i = \int_{a_{i-1}}^{a_i} \tilde{f}_i(x)\ dx. \tag{8}$$

8. $\alpha_{\tau_i} = \beta_{\tau_i} = ½\epsilon_{\tau_i}$, $0 < \epsilon_{\tau_i} \to 0$, to satisfy assumption #4. ◄(9)

Eq (7) & (8) each comprise 3 equations, for a total of 6 dependent equations. There are 3 degrees of freedom to choose 6 unknown variables for defining $\tilde{\tau}_i$ & $\tilde{\eta}_i$ in (6).

$$m_{\tau_i} = \begin{cases} ½(m_{F_{i+1}} - m_{F_{i-1}})/l^2, & \text{for } 1<i<n, \\ (m_{F_{i+1}} - m_{F_i})/l^2, & \text{for } i=1, \\ (m_{F_i} - m_{F_{i-1}})/l^2, & \text{for } i=n. \end{cases} \tag{10}$$

$$\hat{h}_{c,1}(\tilde{f}_i) = ½[(\alpha_{\tau_i}+\beta_{\tau_i})\cdot x + (\alpha_{\eta_i} + \beta_{\eta_i})]. \tag{11}$$

The $\tilde{\tau}_i$ are now well defined; thus $\tilde{\eta}_i$ can be obtained by solving (7).

The $\tilde{f}_i(x)$ must be positive over its domain $[a_{i-1}, a_i]$. Because of the linearity of (6), examine the sign of $\tilde{f}_i(x)$ only at both endpoints of subinterval i.

A. $\tilde{f}_i(a_{i-1}) = (m_{a_{i-1}}, \alpha_{a_{i-1}}, \beta_{a_{i-1}})_{\rm LR} \le 0$

Redefine $\tilde{\tau}_i$ such that $\tilde{f}_i(a_{i-1})$ may become a very small positive *fz*-number.

$$m_{a_{i-1}} \equiv \alpha_{a_{i-1}} = \alpha_{\tau_i}\cdot a_{i-1} + \alpha_{\eta_i}.$$

Solve (7), then,

$$m_{\tau_i} = (2/l^2)\cdot(m_{F_i} - l\cdot m_{a_{i-1}}),$$

$$m_{\eta_i} = m_{a_{i-1}} - m_{\tau_i}\cdot a_{i-1}. \quad ◄$$

B. $\tilde{f}_i(a_i) \le 0$

Similarly,

$$m_{a_i} \equiv \alpha_{a_i} = \alpha_{\tau_i}\cdot a_i + \alpha_{\eta_i},$$

$$m_{\tau_i} = (2/l^2)\cdot(l\cdot m_{a_i} - m_{F_i}),$$

$$m_{\eta_i} = m_{a_i} - m_{\tau_i}\cdot a_i. \quad ◄$$

D_i, see (3), also becomes the *fz*-number,

$$\tilde{D}_i = (m_{D_i}, \alpha_{D_i}, \beta_{D_i})_{\rm LR}.$$

The r.h.s of (3) leads to:

$$\tilde{D}_i = \int_{a_{i-1}}^{a_i} \tilde{f}_{s_i}(s) \otimes \left[\int_{a_{i-1}}^{s} \tilde{f}_{c_i}(c)\ dc\right] ds.$$

By (A-2) and further manipulation:

$$m_{D_i} = m_{\tau_{s_i}}\cdot m_{\tau_{c_i}}\cdot B_{1,i} + m_{\tau_{s_i}}\cdot m_{\eta_{c_i}}\cdot A_{1,i} + m_{\eta_{s_i}}\cdot m_{\tau_{c_i}}\cdot A_{2,i}$$

$$+ m_{\eta_{s_i}}\cdot m_{\eta_{c_i}}\cdot B_2;$$

$$\gamma_{D_i} = \psi_i(\tau,\tau;\gamma)\cdot B_{1,i} + \psi_i(\tau,\eta;\gamma)\cdot A_{1,i} + \psi_i(\eta,\tau;\gamma)\cdot A_{2,i} + \psi_i(\eta,\eta;\gamma)\cdot B_2, \text{ for } \gamma=\alpha,\ \beta;$$

$$A_{1,i} \equiv \tfrac{1}{3}a_i^3 - \tfrac{1}{2}a_i^2\cdot a_{i-1} + \frac{1}{6}a_{i-1}^3,$$

$$A_{2,i} \equiv \frac{1}{6}a_i^3 - \tfrac{1}{2}a_i\cdot a_{i-1}^2 + \tfrac{1}{3}a_{i-1}^3;$$

$$B_{1,i} \equiv \tfrac{1}{2}\hat{a}_i^2\cdot l^2,\ B_2 \equiv \tfrac{1}{2}l^2;$$

$$\psi_i(y,z;\gamma) \equiv m_{y_{s_i}}\cdot\gamma_{z_{c_i}} + m_{y_{c_i}}\cdot\gamma_{z_{s_i}}$$

Finally, (2) becomes

$$\tilde{\Phi} \approx \sum_{i=1}^{n} \tilde{p}_i \otimes \left[\sum_{k=1}^{i-1} \tilde{q}_k\right] \oplus \sum_{i=1}^{n} \tilde{D}_i. \quad (12)$$

The centroid of $\mu_{\tilde{\Phi}}$ can be treated as a point estimate of the crisp unreliability, and

$$m_{\tilde{\Phi}} - \alpha_{\tilde{\Phi}},\ m_{\tilde{\Phi}} + \beta_{\tilde{\Phi}}$$

can be treated as the [lower, upper] bounds of the crisp unreliability.

Since *fz*-unreliability is obtained from (12), the *fz*-LSM is anticipated to be very efficient no matter how many subintervals the interference interval has. Furthermore, (6) is derived from a geometrical viewpoint regarding the distributions of stress & strength based on the point estimates of the interval probabilities. The number of subintervals to be discretized and the accuracy of the point estimates, on the other hand, depend on the amount of experimental data available. If the experimental data are sufficient, not only is the accuracy of the interval probabilities guaranteed with many subintervals, but the interference interval can be enlarged. Thus, the accuracy of the *fz*-LSM depends only on the amount of experimental data available and is not affected by the distributions of stress & strength.

3. NUMERICAL EXAMPLES

Two examples are computed and the results compared with exact unreliabilities to illustrate the low computation time and high accuracy of the *fz*-LSM. The examples involve 3 distributions (*s*-normal, exponential, Weibull) to investigate the performance of the *fz*-LSM *vs* distributions of stress & strength. One can estimate the interval probabilities by sampling and then statistically estimating the probabilities. However, for objective comparison in these examples, instead of the sampling method, the exact interval probabilities are used as the estimated interval probabilities, to avoid sampling errors when estimating the interval probabilities. Furthermore, uncertainties involved in the estimated interval probabilities are considered by letting $\lambda=2\%$ in (5). All computation are executed on a CONVEX 240 computer. The *fz*-LSM is examined in terms of the CPU time consumed compared with that consumed by methods in [6, 7].

3.1 Example 1 (Adapted from [4])

Given

1. Strength is a 3-parameter Weibull distributed r.v. with location parameter $C_{\min}=30$, shape parameter $\beta=2$, and scale parameter $\theta - C_{\min} = 30$.

2. Stress is *s*-normally distributed with mean $\mu_s=30$ and standard deviation $\sigma_s = 3$. ◄

$\Phi_{\text{exact}} = 0.0049$. To calculate unreliability bounds, let the interference region be [30, 50]. Figure 1 shows the exact & approximate pdf's of stress & strength. Table 1 compares results from [6], [7], [0] ([0] = current study); the *fz*-LSM results in a small bound width and the most accurate point estimate of unreliability. Even if the number of subintervals is only 6, the relative error resulting from the *fz*-LSM is only 2.9% and the bounds contain the exact unreliability. The *fz*-LSM requires only 0.03 sec of CPU time to calculate the unreliability with 10 subintervals, while the method in [7] requires 0.18 sec. The time consumed by the *fz*-LSM is almost invariant with respect to N. By contrast, the time consumed by methods [3-7] that resort to solving a quadratic programming problem increases appreciably with N, *eg*, when N=100, the *fz*-LSM consumes only 0.04 sec, whereas that of [7] consumes 26 sec.

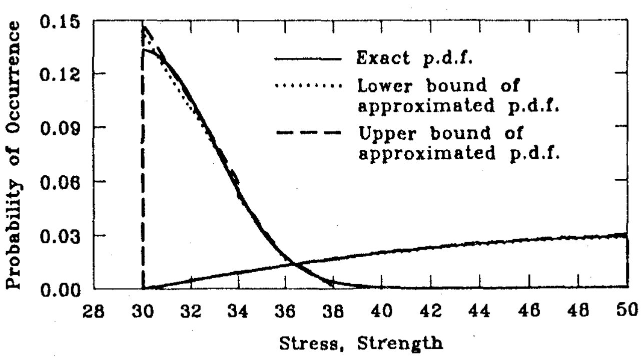

Figure 1. Approximate & Exact pdf Curves

TABLE 1
Comparison of Results

Method	N	$10^3 L_\Phi$	$10^3 U_\Phi$	U_Φ/L_Φ	$10^3\tilde{\Phi}$	Error (%)
[7]	6	4.96	5.13	1.03	5.05	-2.99
[6]	6	2.43	9.14	3.77	5.78	-18
[0]	6	4.81	5.28	1.10	5.04	-2.9
[7]	8	4.84	5.05	1.04	4.94	-0.88
[6]	8	2.65	8.86	3.35	5.76	-18
[0]	8	4.72	5.16	1.09	4.94	-0.86
[7]	10	4.81	5.05	1.05	4.93	-0.60
[6]	10	3.23	7.19	2.22	5.21	-6.30
[0]	10	4.72	5.14	1.09	4.93	-0.58

3.2 Example 2 (Adapted from [1])

Given

1. Component strength is s-normally distributed with $\mu_c = 100$ MPa and $\sigma_c = 10$ MPa.
2. Component stress is exponentially distributed with mean $= 50$ MPa. ◄

$\Phi_{\text{exact}} = 0.13806$. Let the interference region be $[\mu_c - 5\sigma_c, \mu_s + 5\sigma_s] = [50, 300]$ MPa.

Table 2 compares the results of 3 methods and figure 2 depicts the exact and approximate pdf's of stress & strength. The *fz*-LSM again outperforms the others, and the relative error of the point estimate is only 1.5% when $N=8$. The method takes only 0.15 sec when $N = 10$ or 100. By contrast, the method in [7] requires 0.3 and 78 sec to compute the unreliability when $N = 10$ and 100 subintervals, respectively.

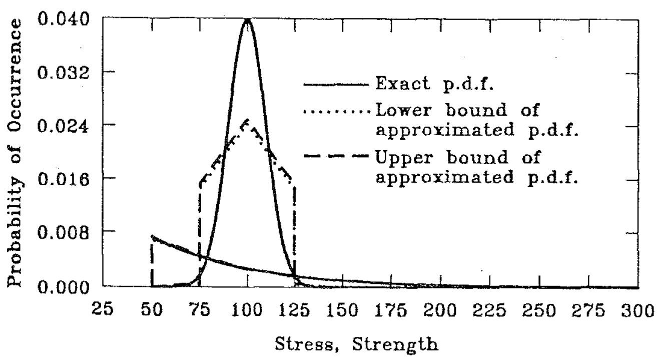

Figure 2. Approximate & Exact pdf Curves

TABLE 2
Comparison of Results

Method	N	$10^3 L^{\Phi}$	$10^3 U_{\Phi}$	U^{Φ}/L^{Φ}	' ∘ ψ $\hat{\Phi}$	Error (%)
[7]	6	1.25	1.32	1.05	1.29	6.8
[6]	6	1.51	1.11	0.73	1.31	5.2
[0]	6	1.23	1.34	1.09	1.29	6.8
[7]	8	1.36	1.44	1.06	1.40	-1.5
[6]	8	1.40	1.07	0.77	1.23	11
[0]	8	1.34	1.46	1.08	1.40	-1.5
[7]	10	1.34	1.42	1.06	1.38	-0.04
[6]	10	1.48	1.11	0.75	1.30	5.8
[0]	10	1.33	1.44	1.08	1.38	0.01

Now let the interference interval be further enlarged from [50, 300] MPa to [0.001, 450] MPa to investigate the effect of truncation of the interference interval on the accuracy of the *fz*-LSM. For any value of λ in (5), the relative error obtained is less than 0.1% when the $N > 34$. Consequently, the error resulting from the approximation of the pdf curves approaches zero if N is large enough.

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APPENDIX

A.1 Assumption 0

If $\tilde{M}$ is neither all-positive nor all-negative, one can decompose it into two positive *fz*-numbers:

$$\tilde{M} = \tilde{M}_1 \ominus \tilde{M}_2,$$

$$\tilde{M}_j \equiv (m_{M_j}, \alpha_{M_j}, \beta_{M_j})_{\text{LR}} > 0, \; j=1,2.$$

To make sure that $\tilde{M}_1$ & $\tilde{M}_2$ are positive, let

$$\alpha_{M_1} = \beta_{M_2} = \tfrac{1}{2}\alpha_M,$$

$$\beta_{M_1} = \alpha_{M_2} = \tfrac{1}{2}\beta_M,$$

$$m_{M_2} = \max(\tfrac{1}{2}\alpha_M - m_M, \tfrac{1}{2}\beta_M).$$

A.2 Definition 1 [8]

A *fz*-number $\tilde{M}$ is of LR-type iff there exist reference functions L & R, and scalars $\alpha_M > 0$, $\beta_M > 0$, such that,

$$\mu_{\tilde{M}}(x) = \begin{cases} L((m_M - x)/\alpha_M), & \text{for } x \le m_M, \\ R((x - m_M)/\beta_M), & \text{for } x \ge m_M; \end{cases}$$

m_M is a real number;

$$\tilde{M} = (m_M, \alpha_M, \beta_M)_{\text{LR}}.$$

Λ ($\Lambda = L, R$) which maps $\mathbb{R}^+ \to [0,1]$ and is decreasing, must satisfy:

$$\Lambda(0) = 1, \; \Lambda(1) = 0,$$

$$\Lambda(x) < 1 \text{ for all } x > 0,$$

$$\Lambda(x) > 0 \text{ for all } x < 1.$$

In this paper, a TFN is used as the reference function:

$$L(z) = R(z) = \max(1 - |z|, 0). \qquad \text{(A-1)}$$

A.3 Definition 2 [9]

A *fz*-number $\tilde{M}$ is [positive, negative] iff $\mu_{\tilde{M}}(x) = 0$, for all $[x < 0, x > 0]$.

A.4 Definition 3

Let,

1. $\tilde{f}(x) = (m_f(x),\ \alpha_f(x),\ \beta_f(x))_{LR}$ be a *fz*-number for all $x \in$ [a, b].

2. $m_f(x),\ \alpha_f(x),\ \beta_f(x)$ are positive integrable functions on $[a, b]$. ◄

From [10]:

$$\int_a^b \tilde{f}(x)\ dx = \left(\int_a^b m_f(x)\ dx,\ \int_a^b \alpha_f(x)\ dx,\ \int_a^b \beta_f(x)\ dx\right)_{LR}. \quad \text{(A-2)}$$

A.5 Definition 4 [11]

For *fz*-number $\tilde{M}$ in an infinite universal set $X = [a,b]$,

$$\hat{h}_{c,w}(\tilde{M}) = 1 - \left[\left(\int_a^b |2\mu_{\tilde{M}}(x) - 1|^w\ dx\right)\Big/(b-a)\right]^{1/w}. \quad \text{(A-3)}$$

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