

Chaotic dynamics in an overlapping generations model with myopic and adaptive expectations

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Abstract

In this paper, we study dynamic behavior of an overlapping generations model under three different expectations: perfect foresight, myopic expectations and adaptive expectations. We show that economic transition under myopic or adaptive expectations is very different from that under perfect foresight. When agents are perfectly foresighted, dynamics is simple, with a unique steady state that is globally attracting. However, cycles and chaotic motion can appear under myopic and adaptive expectations. We study the possibility of Li–Yorke chaos under myopic expectations and entropic chaos under adaptive expectations.

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1. Introduction

One problem in an overlapping generations (OLG) model with capital accumulation is how agents form future expectations when making inter-temporal decisions. Traditional models usually assume that agents are perfectly foresighted. It is well-known that complex dynamics can easily arise in OLG models with perfect foresight. Galor and Ryder (1989) showed that in a standard OLG model with capital accumulation and perfect foresight, multiple equilibria may exist. Their study provided a sufficient condition for a unique, non-trivial steady state equilibrium by calculating the third order derivatives of the utility and production functions.²

In reality, however, the requirement of perfect expectation does not seem reasonable. Agents are more likely to construct their expectations based on current or past information, or simply nothing. We say that agents are myopic if their expectations are formed arbitrarily. That is, they neither take the future into consideration nor learn something from the past. Studies by Benhabib and Day (1982) and Michel and de la Croix (2000) exhibited that when agents are myopic,

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² Recently, Wendner (2003) derived separate conditions on utility and production functions for the uniqueness and stability of an equilibrium without using the third order derivatives of the utility and production function.

complex dynamics can arise in the OLG models. By introducing credit into a standard OLG model without capital accumulation, Benhabib and Day showed that a constant credit expansion rate can generate “erratic” dynamics in prices and credit real value. Michel and de la Croix provided an OLG model with capital accumulation and found that only when the myopic dynamics is monotonic can a study of the myopic dynamics characterize the perfect foresight dynamics. They gave a sufficient condition for the uniqueness of the non-trivial steady state by calculating the second order derivatives of the utility and production functions. If the myopic dynamics is not monotonic (for example, if the inter-temporal elasticity of substitution is high), then cyclical oscillations will emerge and chaotic dynamics can appear under myopic expectations. In this situation, the economic transitions under myopic and perfect foresight are very different.

Besides myopic expectations, some studies assume a learning process whereby agents develop their expectations (adaptive expectations); see Chiarella (1988), Hommes (1994, 2002), Brock and Hommes (1997), Grandmont (1985, 1998), and Chiarella and He (2003). Grandmont (1985) used an OLG model without capital accumulation, but with elastic labor supply, to show that chaotic motion can arise when the curvature of a utility function is large due to strong income effects. By considering adaptive expectations of the real interest rate in an OLG model with capital accumulation, Longo and Valori (2001) studied the local properties around the steady states. Besides providing a uniqueness condition of the non-trivial steady state, they also showed that under certain conditions, period-doubling (saddle-node, Neimark–Hopf) bifurcation will occur and the dynamics will become complicated.

Following the literature of expectation formation in OLG models, we consider three different types of real interest rate expectations: perfect foresight, myopic expectations and adaptive expectations, in an OLG model with capital accumulation. Differing from previous studies, we focus on the global stability of equilibria and the possibility of chaotic dynamics. Our results indicate that economic dynamics is very different under different expectation formations of the real interest rate. We find that economic dynamics is simple under perfect foresight and can be rather complex under the other two types of expectation formations. Furthermore, we show that the inter-temporal elasticity of substitution is an important determinant to the complexity of economic dynamics under myopic expectations. Under adaptive expectations, besides the inter-temporal elasticity of substitution, the complexity of dynamics also depends on the total factor productivity and the learning process.

In an environment with perfect and myopic expectations, the economy can be represented by a one-dimensional (1D) dynamical system (a first order difference equation in capital per capita). Under perfect foresight, the dynamics is simple with a unique, non-trivial equilibrium that is a global attractor. However, under myopic expectations, economic dynamics becomes quite complex when the inter-temporal elasticity of substitution is high. The bifurcation diagram shown by Michel and de la Croix indicated that cycles and chaotic dynamics might emerge when the inter-temporal elasticity of substitution is sufficiently high, but they only provided numerical results without any rigorous proof. To complement the analysis of Michel and de la Croix, we provide another simple way to prove the uniqueness and stability of the non-trivial equilibrium under perfect foresight and prove the presence of chaotic dynamics in the sense of Li and Yorke when the inter-temporal elasticity of substitution is sufficiently high under myopic expectations.

When agents have adaptive expectations of the real interest rate, we obtain a second order difference equation in capital per capita to represent a two-dimensional (2D) dynamical system of the economy. Because it is much more difficult to study a 2D dynamical system than a 1D dynamical system, we apply a recent method developed by Juang et al. (2005) and Li and Malkin (2006) to approximate a 2D dynamical system by using a 1D dynamical system and identify the existence of entropic chaos (positive topological entropy) when both the inter-temporal elasticity of substitution and the total factor productivity are sufficiently high and when agents rely heavily on current information to form their expectations.

The remainder of this paper is organized as follows. The next section develops an OLG model with capital accumulation and shows that dynamics under perfect foresight is always simple. Section 3 modifies the model by assuming that agents have myopic expectations and shows that chaotic dynamics can appear. An OLG model with adaptive expectations is analyzed in Section 4. The final section concludes our study. Proofs of a proposition and theorems are given in the Appendices. Note that appendices are available on JEBO website (will not appear in hard copy).

2. The model

We consider an infinite-horizon, discrete time OLG model. Agents live for two periods, each period covering approximately 30 years, corresponding to young agents and old agents. The population grows at the rate of n . Each agent is endowed with one unit of time in each period. Young agents use all the time for work to earn the real wage

rates (w_t) for consumptions (c_{1t}) and savings (s_t). When young agents become old, they spend all the time for leisure and consume (c_{2t+1}) their savings from the previous period. We consider a CES utility function:

$$U(c_{1t}, c_{2t+1}) = \begin{cases} \frac{c_{1t}^{1-(1/\sigma)} - 1}{1 - (1/\sigma)} + \beta \frac{c_{2t+1}^{1-(1/\sigma)} - 1}{1 - (1/\sigma)}, & \text{if } \sigma \neq 1, \\ \log c_{1t} + \beta \log c_{2t+1}, & \text{if } \sigma = 1, \end{cases} \quad (1)$$

where $\sigma > 0$ is the inter-temporal elasticity of substitution and $\beta \in (0, 1)$ is the discount factor. In this paper, we assume that $\sigma \neq 1$ so that the expectation of the real interest rate can affect the saving decisions of agents.

The budget constraints for young and old agents are

$$c_{1t} + s_t \leq w_t, \quad (2)$$

$$c_{2t+1} \leq R_{t+1}^e s_t, \quad (3)$$

where R_{t+1}^e represents the expected real interest rate. Hence, the young generation will maximize Eq. (1) subject to Eqs. (2) and (3), together with

$$c_{1t} \geq 0 \quad \text{and} \quad c_{2t+1} \geq 0.$$

The optimal saving decision given by the young agents is

$$s_t(w_t, R_{t+1}^e) = \frac{w_t}{1 + \beta^{-\sigma}(R_{t+1}^e)^{1-\sigma}}.$$

We use the Cobb–Douglas production function:

$$Y_t = AK_t^\alpha L_t^{1-\alpha}, \quad A > 0, \alpha \in (0, 1),$$

where K_t is capital and L_t is labor used for production in period t . The parameters A and α represent the total factor productivity and capital share, respectively.

The factor prices are

$$w_t = A(1 - \alpha)k_t^\alpha, \quad (4)$$

$$R_t = A\alpha k_t^{\alpha-1}, \quad (5)$$

where R_t represents the real interest rate and k_t is the per capita capital. The capital market clearing condition implies that $k_{t+1} = (1/(1+n))s_t$. Hence, given the initial condition, the equilibrium is composed by the sequence $\{k_t\}_{t \geq 0}$ that satisfies

$$k_{t+1} = \frac{1}{1+n} s_t(w_t, R_{t+1}^e) = \frac{w_t}{(1+n)[1 + \beta^{-\sigma}(R_{t+1}^e)^{1-\sigma}]}. \quad (6)$$

First, we study the economy where agents are perfectly foresighted. That is,

$$R_{t+1}^e = R_{t+1}.$$

Then Eq. (6) can be written as

$$k_{t+1} = \frac{1}{1+n} s_t(w_t, R_{t+1}) = \frac{w_t}{(1+n)(1 + \beta^{-\sigma}R_{t+1}^{1-\sigma})}. \quad (7)$$

Using Eqs. (4) and (5) to substitute factor prices in Eq. (7), the dynamics of capital per capita can be represented as the following difference equation:

$$A(1 - \alpha)k_t^\alpha - (1+n)k_{t+1}[1 + \beta^{-1}(\alpha\beta A)^{1-\sigma}k_{t+1}^{(\alpha-1)(1-\sigma)}] = 0. \quad (8)$$

In the following theorem, we show that the model under perfect foresight has simple dynamics. Although this result has been shown in de la Croix and Michel (2002, pp. 45 and 46), we give another simple and explicit proof. While they obtained the result with an implicit function, we prove it by explicit calculations.

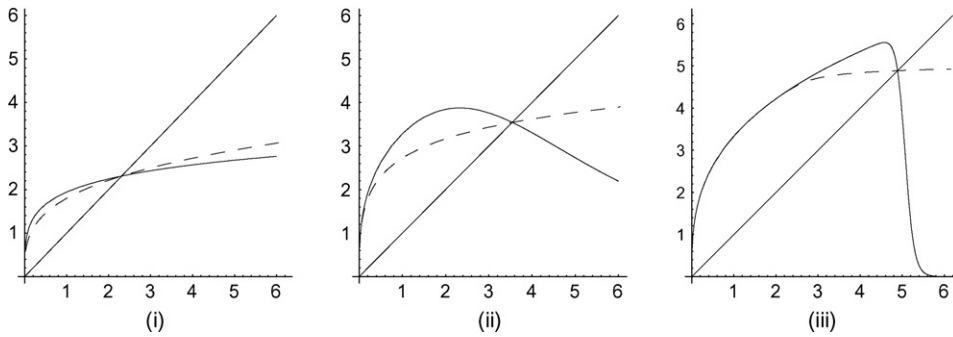


Fig. 1. The graphs of the diagonal line and the systems $k_t \mapsto k_{t+1}$ under perfect foresight (dashed) and under myopic expectations (solid) with (i) $\sigma = 1.45$, (ii) $\sigma = 5$ and (iii) $\sigma = 75$, and $A = 10$, $\alpha = 0.34$, $\beta = 0.995^{30}$ and $n = 1.023^{30} - 1$.

Proposition 1. *The dynamics $k_t \mapsto k_{t+1}$ in Eq. (8) has a unique steady state, and it is globally stable in $(0, \infty)$.*

Proof. See Appendix A in supplementary material. \square

From the proof of Proposition 1, we can express k_{t+1} as a function of k_t (that is, $k_{t+1} = g^{-1}(k_t)$, where g^{-1} represents the inverse of the function g). The graphs of g^{-1} with $\sigma = 1.45, 5$ and 75 are drawn by the dashed curves in Fig. 1(i)–(iii), respectively. Notice that the function g^{-1} is monotonic and the dynamics is simple. There is a unique and stable positive steady state as demonstrated by Proposition 1. The curve will be stretched upward with an increase in σ , and the value of the positive steady state will increase. However, changing σ will not affect the stability of the non-trivial steady state.

3. Myopic expectations

Following Michel and de la Croix, we assume that when agents are myopic, they expect that the real interest rate in the next period will be the same as the one in the current period. That is,

$$R_{t+1}^e = R_t. \tag{9}$$

Combining Eqs. (6) and (9), we get that given the initial value of k_0 , the myopic expectation equilibrium consists of the sequence $\{k_t\}_{t \geq 0}$ that satisfies

$$k_{t+1} = \frac{1}{1+n} s_t(w_t, R_t) = \frac{w_t}{(1+n)(1+\beta^{-\sigma} R_t^{1-\sigma})}.$$

Therefore, the dynamics of the economy can be expressed as the following equation:

$$k_{t+1} = f_{\sigma}(k_t), \tag{10}$$

where $f_{\sigma} : (0, \infty) \rightarrow (0, \infty)$ is a one-parameter family of functions defined by

$$f_{\sigma}(x) = \frac{A(1-\alpha)x^{\alpha}}{(1+n)[1+\beta^{-1}(\alpha\beta A)^{1-\sigma}x^{(1-\alpha)(\sigma-1)}]} \tag{11}$$

with parameter σ . The economic dynamics $k_t \mapsto k_{t+1}$ in Eq. (10) is equivalent to the dynamics $x \mapsto f_{\sigma}(x)$. For iterations, we denote the identity function by f_{σ}^0 and inductively define $f_{\sigma}^t = f_{\sigma} \circ f_{\sigma}^{t-1}$ for positive integer t . Thus every equilibrium path $\{k_t\}_{t \geq 0}$ of Eq. (10) corresponds to the trajectory $\{f_{\sigma}^t(x)\}_{t \geq 0}$ of Eq. (11) with $x = k_0$.

The function $k_{t+1} = f_{\sigma}(k_t)$ with $\sigma = 1.45, 5$ and 75 are represented by the solid curves in Fig. 1(i)–(iii), respectively. Fig. 1(i) shows that the function f_{σ} is monotonic and the economy will converge to a positive steady state when σ is small ($\sigma = 1.45$).³ In this case, the myopic dynamics can be used to approximate the economic behavior under perfect

³ A sufficient condition for the monotonicity of the function f_{σ} is that $\alpha > (1-\alpha)(\sigma-1)$. Hence, given the value of α , the function f_{σ} is monotonic if $\sigma < (1 + (\alpha/(1-\alpha)))$ and vice versa.

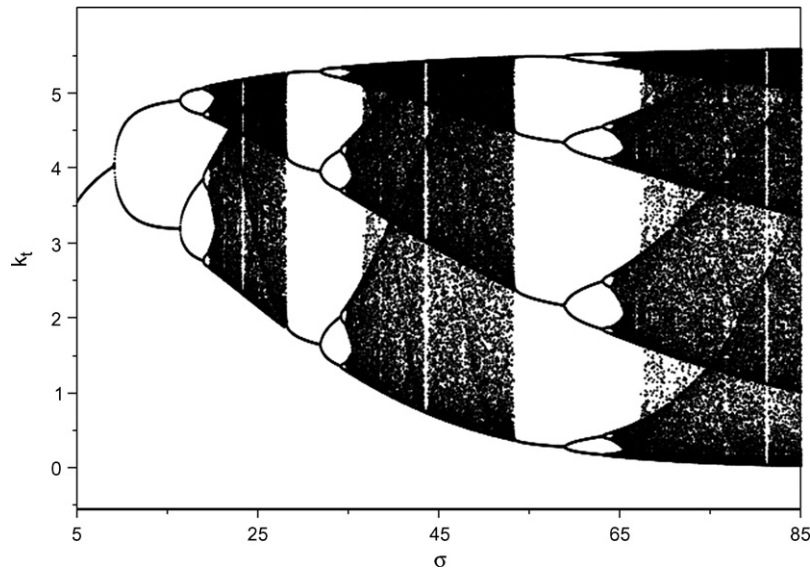


Fig. 2. The bifurcation diagram on σ for Eq. (10) with $A = 10$, $\alpha = 0.34$, $\beta = 0.995^{30}$, and $n = 1.023^{30} - 1$.

foresight. However, Fig. 1(ii) and (iii) shows that the function f_σ becomes non-monotonic when σ is large enough ($\sigma = 5$ and 75). Furthermore, as σ increases, the curve will be stretched upward, the positive steady state will become larger, and the dynamics of the function f_σ will become complicated. Hence, even if the positive steady states of the functions g^{-1} and f_σ are the same, the dynamics will be different because of the non-monotonicity of the function f_σ . The bifurcation diagram with varying σ shown in Fig. 2 indicates that an increase in σ will induce cycles and chaotic motion.

In the following theorem, we show that the dynamical system under myopic expectations exhibits chaotic motion according to Li and Yorke (1975). But first, we give the definition of Li–Yorke chaos.

Definition 1. Let $h : I \rightarrow I$ be a map, where I is an interval. We say that h has Li–Yorke chaos on I if

1. h has periodic points of all periods; here by a periodic point p of period t , we mean that $h^t(p) = p$ and $h^i(p) \neq p$ for $0 < i < t$;
2. there exists an uncountable set $S \subset I$ such that
 - (i) if $x, y \in S$ with $x \neq y$ then

$$\limsup_{t \rightarrow \infty} |h^t(x) - h^t(y)| > 0 \quad \text{and} \quad \liminf_{t \rightarrow \infty} |h^t(x) - h^t(y)| = 0;$$

- (ii) if $x \in S$ and $y \in I$ is periodic then

$$\limsup_{t \rightarrow \infty} |h^t(x) - h^t(y)| > 0.$$

The Li–Yorke Theorem (Theorem 1) says that for a continuous map h on an interval, if there is a point p such that

$$h^3(p) < p < h(p) < h^2(p),$$

then h exhibits Li–Yorke chaos. The existence of such an initial point p for the system f_σ in Eq. (11) is shown in Fig. 3(i).

Fig. 3(ii) indicates that for any parameter σ , the function f_σ attains the same value at a particular point. Here, we point out that, in fact, $x^* = (\alpha\beta A)^{1/1-\alpha}$; that is, for any $\sigma_1, \sigma_2 > 0$, we have that $f_{\sigma_1}(x^*) = f_{\sigma_2}(x^*)$. This fact will play a role in the proof of the following theorem, which gives a sufficient condition of the existence of Li–Yorke chaos under myopic expectations.

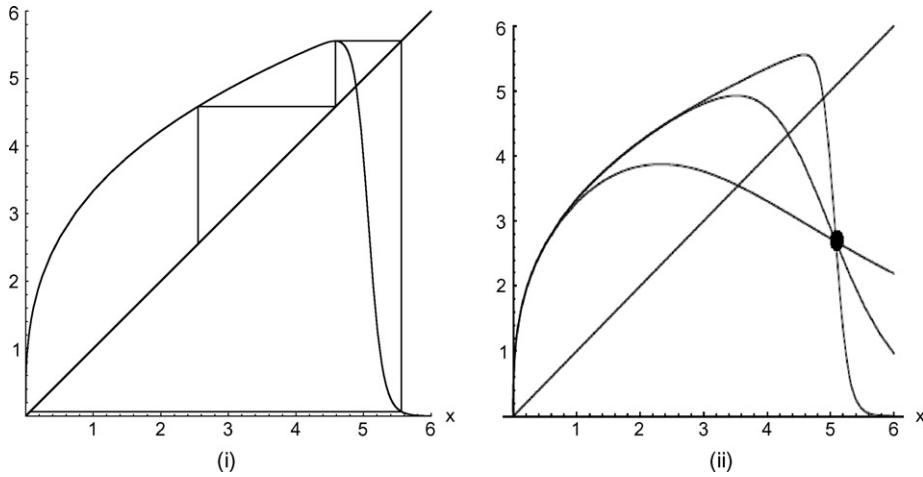


Fig. 3. (i) The graphs of $y = f_{75}(x)$ and $y = x$, and the first three iterates of an initial point under f_{75} ; (ii) the graphs of $f_{\sigma}(x)$ with $\sigma = 5, 15, 75$ and $A = 10, \alpha = 0.34, \beta = 0.995^{30}$ and $n = 1.023^{30} - 1$.

Theorem 1. Let $A > 0, 0 < \alpha < 1, \beta > 0$ and $n > ((1 - \alpha)/\alpha) - 1$ satisfy $\beta^{1+\alpha} < ((1 - \alpha)/\alpha(1 + n))^{1+\alpha} < \beta(1 + \beta)^{\alpha}$. If σ is sufficiently large, then Eq. (10) exhibits Li–Yorke chaos.

Proof. See Appendix B in supplementary material. □

If the function f_{σ} is monotonic, then we can use myopic dynamics to approximate the dynamic transition under perfect foresight. However, if the function f_{σ} of the myopic dynamics is non-monotonic, cycles will start to emerge as σ increases as shown in Fig. 2. Theorem 1 provides a sufficient condition for the presence of chaos in the sense of Li and Yorke, and it indicates that the inter-temporal elasticity of substitution plays an important role in determining the existence of complex dynamics. When σ is large enough, the system gets into the region of Li–Yorke chaos. The numerical example given by de la Croix and Michel (p. 50) with parameter values $A = 20, \beta = 0.3, \alpha = \frac{1}{3}$ and $n = 1.025^{30} - 1$ satisfies the condition in Theorem 1 and will display Li–Yorke chaos.

We give a simple criterion for the existence of Li–Yorke chaos in the 1D dynamical system in this section. In the next section, we study an OLG model with adaptive expectations and find that economic transition can be represented by a 2D dynamical system. However, the existence of periodic points of period three implies that Li–Yorke chaos works only in 1D dynamical systems but not for higher-dimensional dynamical systems. Hence, we consider entropic chaos in the 2D dynamical system in the next section. Note that in 1D dynamical systems, the existence of Li–Yorke chaos also implies the presence of entropic chaos. We consider Li–Yorke chaos instead of entropic chaos in the 1D dynamical system because it is more convenient to verify and easier for us to use an approximation method to identify the existence of entropic chaos in the 2D dynamical system in the next section. It is well-known that for 1D dynamical systems, the topological entropy is positive if and only if there exists a periodic point of period that is not the powers of 2. One may also use the criterion for entropic chaos in 1D dynamical systems, but this will cause the analysis to be more complex than our current analysis when we approximate the 2D dynamical system by the 1D dynamical system. For this reason, we study Li–Yorke chaos and entropic chaos under myopic and adaptive expectations, respectively.

4. Adaptive expectations

Following Longo and Valori, we consider a first order autoregressive adaptive expectation of the real interest rate. That is, R_{t+1}^e is constructed based on current and past information with decreasing weights. With $\lambda \in (0, 1), R_{t+1}^e$ is represented by

$$R_{t+1}^e = \sum_{i=0}^{\infty} \lambda(1 - \lambda)^i R_{t-i} = (1 - \lambda)R_t^e + \lambda R_t. \tag{12}$$

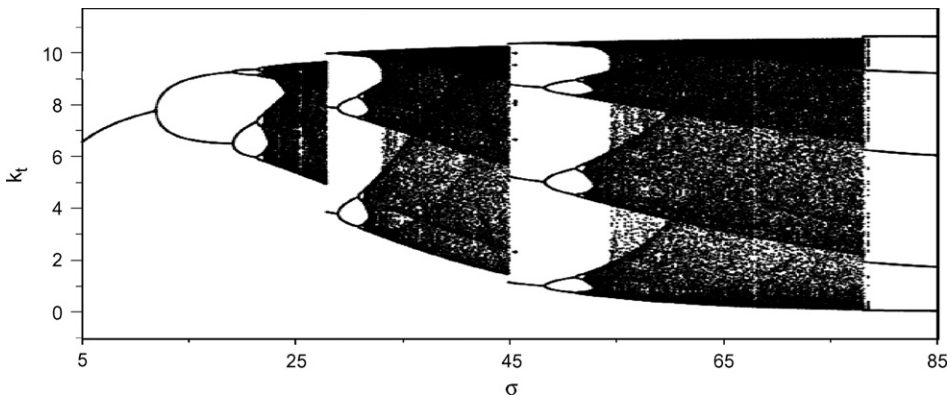


Fig. 4. The bifurcation diagram on σ for Eq. (13) with $A = 15, \lambda = 0.9, \alpha = 0.34, \beta = 0.995^{30}$ and $n = 1.023^{30} - 1$.

Eq. (12) shows that R_{t+1}^e can be represented by R_t^e and R_t with weights $(1 - \lambda)$ and λ , respectively. An increase in λ indicates that agents rely more and more heavily on the current information (R_t) when forming their expectations. Notice that when $\lambda = 1$, agents have myopic expectations. Substituting Eq. (12) into Eq. (6), the dynamics of the economy can be expressed as the sequence $\{k_t\}_{t \geq 0}$ that satisfies of the following difference equation for all $t \geq 0$:

$$A(1 - \alpha)k_{t+1}^\alpha = k_{t+2}(1 + n) \left\{ 1 + \beta^{-\sigma} \left[\lambda \alpha A k_{t+1}^{\alpha-1} + (1 - \lambda) \beta^{\sigma/1-\sigma} \left(\frac{A(1 - \alpha)k_t^\alpha}{(1 + n)k_{t+1}} - 1 \right)^{1/1-\sigma} \right]^{1-\sigma} \right\}. \quad (13)$$

Eq. (13) implies that under adaptive expectations, the economic transition can be represented by the following 2D dynamical system:

$$F_{\sigma,\lambda}(k_t, k_{t+1}) = (k_{t+1}, g_{\sigma,\lambda}(k_t, k_{t+1})), \quad (14)$$

where $g_{\sigma,\lambda}(k_t, k_{t+1}) = (A(1 - \alpha)k_{t+1}^\alpha / ((1 + n)\{1 + \beta^{-\sigma}\}[\lambda \alpha A k_{t+1}^{\alpha-1} + (1 - \lambda) \beta^{\sigma/1-\sigma} ((A(1 - \alpha)k_t^\alpha / (1 + n)k_{t+1}) - 1)^{1/1-\sigma}]^{1-\sigma}))$, and σ, λ are considered as parameters. Thus the Jacobian matrix of the two-dimensional system in Eq. (14) is

$$DF_{\sigma,\lambda}(k_t, k_{t+1}) = \begin{bmatrix} 0 & 1 \\ \frac{\partial g_{\sigma,\lambda}}{\partial k_t}(k_t, k_{t+1}) & \frac{\partial g_{\sigma,\lambda}}{\partial k_{t+1}}(k_t, k_{t+1}) \end{bmatrix}.$$

First, we give a local stability analysis of a steady state for the 2D dynamical system (Eq. (14)) by using the 1D dynamical system (Eq. (10)) to approximate this system. Considering $\lambda = 1$, we have $g_{\sigma,1}(k_t, k_{t+1}) = f_\sigma(k_{t+1})$; hence the matrix $DF_{\sigma,1}(k_t, k_{t+1})$ has two eigenvalues: 0 and $f'_\sigma(k_{t+1})$. In the 1D dynamical system $k_{t+1} \mapsto f_\sigma(k_{t+1})$, the value of f'_σ at a steady state (\bar{x}) passes through -1 as σ increases. Hence, f_σ has a period-doubling bifurcation at some $\sigma_0 > 0$; see Fig. 1. As σ increases, the map f_σ undergoes a successive period-doubling bifurcation; see Fig. 2. For the dynamics of the 2D dynamical system $F_{\sigma,\lambda}$, the continuity of the system in λ implies that there is a steady state near (\bar{x}, \bar{x}) on the diagonal line for λ close to 1. As the parameter σ increases, the continuity of the derivative gives us that $DF_{\sigma,\lambda}$ at the steady state has one eigenvalue passing through -1 and the other one passing through 0 while both of them remain real valued. This indicates that the steady state gives up its stability to an asymptotically stable periodic orbit of period 2. As the parameter σ increases further, the system undergoes a successive period-doubling bifurcation. Fig. 4 presents the bifurcation diagram of Eq. (13) with varying σ . Moreover, because eigenvalues will pass through -1 and 0 instead of 1, the scenario of Hopf bifurcation is not observed in our numerical computation.

We give more remarks in Figs. 4 and 5. Fig. 4 shows that deterministic cycles and complex motion will occur with an increase of σ . The parameter λ in Eq. (12) is the weight for adaptive expectations. Fig. 5 presents the bifurcation diagram with varying values of λ . If λ is small enough, there will be a unique and stable positive steady state. Hence, the adaptive expectations have a stabilizing effect on the local stability of the steady state. As λ increases, agents will

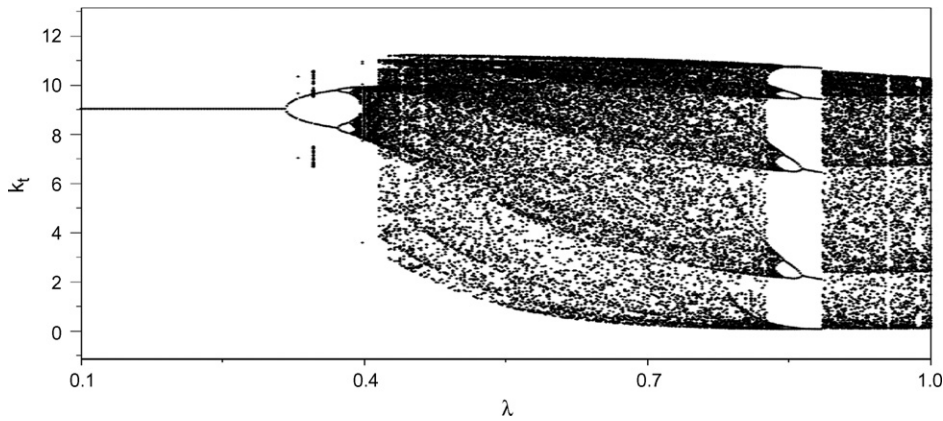


Fig. 5. The bifurcation diagram on λ for Eq. (13) with $A = 15, \sigma = 75, \alpha = 0.23, \beta = 0.995^{30}$, and $n = 1.023^{30} - 1$.

put a higher weight on the current real interest rate when forming their expectations, and complex dynamics starts to emerge when λ is large. When $\lambda = 1$, the system reduces to the one-dimensional model with myopic dynamics.

We give a rigorous proof for the existence of such chaotic dynamics by using the 1D dynamical system (Eq. (10)) under myopic expectations to approximate the 2D dynamical system (Eq. (13)) under adaptive expectations. First, we recall the definitions of topological entropy and entropic chaos; refer to Robinson (1999).

Definition 2. Let $g : X \rightarrow X$ be a continuous map on a space X with metric d . For $n \in \mathbb{N}$ and $\epsilon > 0$, a set $S \subset X$ is called an (n, ϵ) -separated set for g if for every pair of points $x, y \in S$ with $x \neq y$, there exists an integer k with $0 \leq k < n$ such that $d(g^k(x), g^k(y)) > \epsilon$. The topological entropy of g is defined to be

$$h_{\text{top}}(g|X) = \lim_{\epsilon \rightarrow 0, \epsilon > 0} \limsup_{n \rightarrow \infty} \frac{\log(\max\{\#(S) : S \subset X \text{ is an } (n, \epsilon)\text{-separated set for } g\})}{n},$$

where $\#(S)$ is the cardinality of elements of S .

We say that g has entropic chaos on X if $h_{\text{top}}(g|X) > 0$.

Topological entropy has played a fundamental role in the theory of chaos. Its concept is a mathematical formulation of exponential divergence of nearby initial conditions. It describes the total exponential complexity of the orbit structure with a single number in a rough but expressive way. The topological entropy is positive for chaotic systems and is zero for non-chaotic systems.

In the following theorem, we give a sufficient condition for the existence of entropic chaos under adaptive expectations.

Theorem 2. Let $0 < \alpha < 1, n > ((1 - \alpha)/\alpha) - 1$, and $\beta > 0$ with $\beta^{1+\alpha} < ((1 - \alpha)/\alpha(1 + n))^{1+\alpha} < \beta(1 + \beta)^\alpha$. If σ and A are sufficiently large, then for all $\lambda < 1$ near 1, Eq. (13) exhibits entropic chaos.

Proof. See Appendix C in supplementary material. \square

Theorem 2 provides a sufficient condition for the presence of entropic chaos under adaptive expectations. In fact, it shows that chaotic motion of the dynamical system under adaptive expectations is inherited from the chaotic behavior of the dynamical system under myopic expectations. This result differs from findings in the cobweb models where adaptive expectations may generate chaos while myopic expectations only cause simple dynamics (see Chiarella, 1988; Hommes, 1994). Furthermore, our study shows that the presence of entropic chaos in an OLG model under adaptive expectations requires that the inter-temporal elasticity of substitution and the total factor productivity are sufficiently high and the expectations heavily depend on the current information.

5. Conclusion

In this paper, we study economic transition in an OLG model with capital accumulation under three different types of expectations: perfect foresight, myopic expectations and adaptive expectations. We show that only simple dynamics

occurs under perfect foresight. However, the dynamics may become complex under myopic and adaptive expectations. Under myopic expectations, our results indicate that cycles will emerge as the inter-temporal elasticity of substitution increases and chaotic dynamics can occur when the inter-temporal elasticity of substitution is large enough. Under adaptive expectations, the inter-temporal elasticity of substitution and the weight of past information when forming expectations are important to determine the complexity of economic dynamics. By analyzing dynamics of different expectations in an OLG model with capital accumulation, our work highlights the importance of the expectation formation when constructing inter-temporal macroeconomic models. A study of different expectations in an optimal growth model would be an important future work.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at [doi:10.1016/j.jebo.2007.03.005](https://doi.org/10.1016/j.jebo.2007.03.005).

References

- Benhabib, J., Day, R.H., 1982. A characterization of erratic dynamics in the overlapping generations model. *Journal of Economic Dynamics and Control* 4, 37–55.
- Brock, W.A., Hommes, C.H., 1997. A rational route to randomness. *Econometrica* 65, 1059–1095.
- Chiarella, C., 1988. The cobweb model: its instability and the onset of chaos. *Economic Modeling* 5, 377–384.
- Chiarella, C., He, X.-Z., 2003. Dynamics of beliefs and learning under a_L -processes—the heterogeneous case. *Journal of Economic Dynamics and Control* 27, 503–531.
- de la Croix, D., Michel, P., 2002. *A Theory of Economic Growth: Dynamics and Policy in Overlapping Generations*. Cambridge University Press, Cambridge.
- Galor, O., Ryder, H.E., 1989. Existence, uniqueness, and stability of equilibrium in an overlapping generations model with productive capital. *Journal of Economic Theory* 49, 360–375.
- Grandmont, J.-M., 1985. On endogenous competitive business cycles. *Econometrica* 53, 995–1045.
- Grandmont, J.-M., 1998. Expectations formation and stability of large socioeconomic systems. *Econometrica* 66, 741–781.
- Hommes, C.H., 1994. Dynamics of the cobweb model with adaptive expectations and nonlinear supply and demand. *Journal of Economic Behavior and Organization* 24, 315–335.
- Hommes, C.H., 2002. Modeling the stylized facts in finance through simple nonlinear adaptive systems. *Proceedings of the National Academy of Sciences of the United States of America* 99, 7221–7228.
- Juang, J., Li, M.-C., Malkin, M., 2005. Multidimensional perturbations of chaotic one-dimensional maps and their applications. Research Report MIAS 2005-8, Institute of Mathematics, Academic Sinica, Taipei, Taiwan.
- Li, M.-C., Malkin, M., 2006. Topological horseshoes for perturbations of singular difference equations. *Nonlinearity* 19, 795–811.
- Li, T., Yorke, J., 1975. Period three implies chaos. *American Mathematical Monthly* 82, 985–992.
- Longo, M., Valori, V., 2001. Existence and stability of equilibria in OLG models under adaptive expectations. DiMaD Working Paper (Quaderni del Dipartimento di Matematica per le Decisioni), Università degli Studi di Firenze.
- Michel, P., de la Croix, D., 2000. Myopic and perfect foresight in the OLG model. *Economics Letters* 67, 53–60.
- Robinson, C., 1999. *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, 2nd ed. CRC Press, Boca Raton.
- Wendner, R., 2003. Existence, uniqueness, and stability of equilibrium in an OLG economy. *Economic Theory* 23, 165–174.