

Chaotic dynamics in a monetary economy with habit persistence

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Abstract

This paper develops a cash-in-advance model for persistent habits of consumption. With a binding cash-in-advance constraint, the economic transition can be represented by a one-dimensional or two-dimensional dynamical system, depending on persistent habits of only cash-goods consumption or both cash-goods and credit-goods consumption, respectively. We verify the existence of entropic chaos and ergodic chaos for the former case and identify the presence of entropic chaos for the latter.

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1. Introduction

In recent years, economists have realized that rational expectation models can have more than one equilibrium; they have also encountered the complicated problem of analyzing qualitative behavior for equilibrium paths. The chaotic equilibrium has the most complicated dynamics among all kinds of equilibria.² In this case, a small difference in initial conditions or a slight perturbation of parameter values may result in an extremely different dynamical process for

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² There are bubbles, sunspots, cycles, and chaotic equilibria; refer to Barnett et al. (1989) and Benhabib (1992).

an economy over time. These phenomena have led researchers to investigate the possibility of endogenous fluctuations in economic models, while traditional macroeconomic models still use unexpected shocks to explain fluctuations in outputs and prices within the economy.

In the literature on monetary economics, money has been introduced in several models through a cash-in-advance constraint. Assuming that households can consume cash goods and credit goods, Michener and Ravikumar (1998) extended the work of Woodford (1994) and provided examples of chaotic motions in a cash-in-advance environment. The study by Auray et al. (2002) incorporated habit persistence in a monetary economic model. They showed that under a simple cash-in-advance economy, habit persistence can cause endogenous oscillations and chaotic equilibria.

In this paper, we develop a cash-in-advance model with consumptions of both cash goods and credit goods in the economy, and prove the existence of chaotic dynamics. Different from Michener and Ravikumar, our model allows that agents have persistent consumption behavior habits. We join the literature of “catching up with the Joneses” formation by assuming that each individual compares the current consumption with past average consumption.³ Moreover, households can consume both cash goods and credit goods in our model while households consume only cash goods in Auray et al.

For our model with the binding cash-in-advance constraint and households having only the persistent habit of cash-goods consumption, we derive a one-dimensional dynamical system, representing the economic transition from a first-order difference in real money balance. We then verify the existence of chaos in the sense of positive topological entropy (entropic chaos) and the presence of a unique ergodic invariant measure absolutely continuous with respect to the Lebesgue measure (ergodic chaos), while the previous studies analyzing the complicated dynamics of economic models focused on the chaos in the sense of Li and Yorke (1975). On the other hand, assuming that agents had persistent habits of both cash-goods and credit-goods consumption, we obtain a second-order difference in real money balance that induces a two-dimensional dynamical system for the economic model, and we identify the presence of entropic chaos.

The remainder of this paper is organized as follows. In the next section, we develop a cash-in-advance economy with the habit persistence of cash goods and show that this exhibits entropic chaos as well as ergodic chaos. In the subsequent section, we extend the model by allowing agents to have persistent habits of both cash-goods consumption and credit-goods consumption, and show the existence of entropic chaos. The final section presents the conclusion of our study and suggestions for future projects.

2. The model

We consider an economy with infinitely living identical agents. Following Lucas and Stokey (1987), we assume that agents consume cash goods (c_t) and credit goods (d_t). We use p_t to denote the common price of these two goods and M_t to denote the nominal money balance in period t . Households are composed of shoppers and workers. The details of the trading scenario and the sequence of events are described in Lucas and Stokey and in Michener and Ravikumar. In period t , individuals use the real money balance M_t/p_t brought forward from the previous period

³ For studies applying habit formation to growth models, see Alvarez-Cuadrado et al. (2004) and Ryder and Heal (1973). For papers exploiting habit persistence in asset pricing models to resolve equity premium puzzles and to explain several asset pricing phenomena, see Abel (1990) and Campbell and Cochrane (1999).

$t - 1$ to buy cash goods. Hence, agents face the following cash-in-advance constraint of the cash goods:

$$c_t \leq \frac{M_t}{p_t}. \quad (1)$$

Households are endowed with constant v in every period. Cash goods and credit goods are produced by the production function, $Y_t = f(v) = v$. Assuming that both cash goods and credit goods will perish after one period, we have

$$c_t + d_t \leq v. \quad (2)$$

In period t , the government injects money into the economy by giving a nominal lump-sum transfer T_t to households. We assume that the nominal money supply \bar{M} grows at the rate of θ , that is, $\bar{M}_{t+1} = (1 + \theta)\bar{M}_t$. Thus, $T_t = \bar{M}_{t+1} - \bar{M}_t$. Households allocate the transfer, the real money balance carried from the last period, and the value of output on the cash goods, the credit goods and the money balance they plan to carry to the next period. Therefore, the budget constraint for households can be written as

$$c_t + d_t + \frac{M_{t+1}}{p_t} \leq v + \frac{M_t}{p_t} + \frac{T_t}{p_t}. \quad (3)$$

Agents have the persistent habit of cash-goods consumption and will compare their current consumption of cash goods with the average cash-goods consumption of the previous period. This preference is separable in cash goods and credit goods, and is represented as

$$\sum_{t=0}^{\infty} \eta^t U(V(c_t, \bar{c}_{t-1}), G(d_t)), \quad (4)$$

where $\eta \in (0, 1)$ is the discount factor, and \bar{c}_{t-1} is the average consumption of cash goods in period $t - 1$. We make the following assumptions about the utility function:

Assumption 1. The utility function is strictly increasing in c_t and d_t . That is,

$$\frac{\partial U}{\partial c_t}(V(c_t, \bar{c}_{t-1}), G(d_t)) > 0$$

and

$$\frac{\partial U}{\partial d_t}(V(c_t, \bar{c}_{t-1}), G(d_t)) > 0.$$

Assumption 2. The utility function is twice differentiable and strictly concave in c_t and d_t . That is,

$$\frac{\partial^2 U}{\partial c_t^2}(V(c_t, \bar{c}_{t-1}), G(d_t)) < 0$$

and

$$\frac{\partial^2 U}{\partial d_t^2}(V(c_t, \bar{c}_{t-1}), G(d_t)) < 0.$$

Assumption 3. The utility function is decreasing in \bar{c}_{t-1} . That is,

$$\frac{\partial U}{\partial \bar{c}_{t-1}}(V(c_t, \bar{c}_{t-1}), G(d_t)) \leq 0.$$

Assumption 1 demonstrates that by holding past average cash-goods consumption constant, an increase in current consumption will not reduce the utility. **Assumption 2** indicates that the marginal utility of the cash/credit-goods consumption decreases with an increase in the cash/credit-goods consumption. **Assumption 3** states that by holding current consumptions unchanged, an increase in the past average consumption of cash goods will not enlarge the utility.

In order to study the dynamics of the economy over time, we set the real balance $m_t = M_t/p_t$. Money market clearing implies that $\bar{M}_t = M_t$. Given the growth rate of money supply, the perfect foresight equilibrium comprises the sequences of prices, transfers, and individual decisions $\{c_t, d_t, m_t\}$ such that (i) the household maximization problem will be solved for $\{c_t, d_t, m_t\}$ by maximizing the utility function subject to Eqs. (1) and (3) and the non-negativity constraints of c_t, d_t and m_t ; (ii) the equilibrium in the goods market implies that Eq. (2) holds with equality; and (iii) money market clears.

Using λ_t and μ_t to represent the associated Kuhn-Tucker multipliers of the constraints Eqs. (1) and (3), respectively, the first-order conditions are

$$\frac{\partial U}{\partial V}(V(c_t, \bar{c}_{t-1}), G(d_t)) \frac{\partial V}{\partial c_t}(c_t, \bar{c}_{t-1}) = (\lambda_t + \mu_t)p_t, \tag{5}$$

$$\frac{\partial U}{\partial G}(V(c_t, \bar{c}_{t-1}), G(d_t)) \frac{dG}{dd_t}(d_t) = \mu_t p_t, \tag{6}$$

$$\eta(\lambda_{t+1} + \mu_{t+1}) = \mu_t. \tag{7}$$

Notice that constraints bind at household’s optimal decisions due to the monotonicity of the utility function. Combining Eqs. (5)–(7) and the clearing condition of the goods market, we get

$$\begin{aligned} & \frac{\partial U}{\partial V}(V(c_{t+1}, \bar{c}_t), G(v - c_{t+1})) \frac{\partial V}{\partial c_{t+1}}(c_{t+1}, \bar{c}_t) \\ &= \frac{1 + \theta}{\eta} \frac{m_t}{m_{t+1}} \frac{\partial U}{\partial G}(V(c, \bar{c}_{t-1}), G(v - c_t)) \frac{dG}{dd_t}(v - c_t). \end{aligned} \tag{8}$$

Unlike Michener and Ravikumar who needed the unbinding cash-in-advance constraint to generate chaos, here we focus on the case when the cash-in-advance constraint always binds and will show the existence of chaos. We consider a separable utility function that satisfies **Assumptions 1–3**:

$$U(V(c_t, \bar{c}_{t-1}), G(d_t)) = V(c_t, \bar{c}_{t-1}) + \kappa G(d_t), \tag{9}$$

where

$$V(c_t, \bar{c}_{t-1}) = \begin{cases} \frac{1}{1 - \rho} \left[\left(\frac{c_t}{\bar{c}_{t-1}^\xi} \right)^{1-\rho} - 1 \right], & \text{if } \rho \neq 1, \\ \log \left(\frac{c_t}{\bar{c}_{t-1}^\xi} \right), & \text{if } \rho = 1, \end{cases} \tag{10}$$

and

$$G(d_t) = \begin{cases} \frac{d_t^{1-\gamma} - 1}{1 - \gamma}, & \text{if } \gamma \neq 1, \\ \log(d_t), & \text{if } \gamma = 1, \end{cases} \tag{11}$$

with $\rho > 0$, $\xi \geq 0$ and $\gamma > 0$. The parameter value $\kappa > 0$ measures, given the same utility, how many units of cash goods are needed to substitute for one unit of credit goods. The parameter ξ measures the degree of habit persistence of cash-goods consumption. As indicated by Abel, if $\xi = 0$ then households do not compare their current cash-goods consumptions with their past average cash-goods consumption within the economy and Eq. (4) is the time-separable utility function as usual.

Because the dynamics of the real money balance will disappear provided $\rho = 1$, we make the following assumption in order to avoid this situation.

Assumption 4. $\rho \neq 1$.

One-dimensional dynamical systems

Using the utility function form in Eq. (9), we derive a first-order difference equation for the real money balance from Eq. (8):

$$m_{t+1} = \left\{ \frac{\kappa(1 + \theta)}{\eta} m_t^{1+\xi(1-\rho)} (v - m_t)^{-\gamma} \right\}^{1/(1-\rho)} \tag{12}$$

For the rest of this section, we study the dynamical behavior of Eq. (12). Without loss of generality, we assume that $v = 1$. For simplicity, let $\omega = (\kappa(1 + \theta)/\eta)^{1/(1-\rho)}$, $\alpha = (1/1 - \rho) + \xi$, $\beta = -(\gamma/1 - \rho)$, and $x = m_t$. For searching chaotic behavior, we focus on the case when $\alpha > 0$ and $\beta > 0$. Then the dynamics of Eq. (12) with $m_t \mapsto m_{t+1}$ is equivalent to the dynamics of the family of functions $x \mapsto f_{\omega,\alpha,\beta}(x)$, where $f_{\omega,\alpha,\beta} : I \rightarrow \mathbb{R}^+$ is defined by

$$f_{\omega,\alpha,\beta}(x) = \omega x^\alpha (1 - x)^\beta, \tag{13}$$

where ω , α , and β are positive real parameters, and $I = \mathbb{R}^+$ if β is an even integer and $I = [0, 1]$ if β is not even. The restriction of the domain I depending on β is necessary in order to focus our interests on the points whose images are still positive. For simplicity, we write $f = f_{\omega,\alpha,\beta}$, denote the identity function by f^0 , and inductively define $f^n = f \circ f^{n-1}$ for positive integer n . The restriction of f on a subset J of I is denoted by $f|_J$.

For the case when β is not an even integer, Chen and Li (2006) proved that under certain conditions the family Eq. (13) has chaotic dynamics in the sense of Li and Yorke, Devaney (1989), and Smale (1965). In this section, we study the case when β is even and will prove that Eq. (13) has entropic chaos and ergodic chaos.

First, we recall the definition of topological entropy and entropic chaos; refer to Robinson (1999).

Definition 1. Let $g : X \rightarrow X$ be a continuous map on the space X with metric d . For $n \in \mathbb{N}$ and $\epsilon > 0$, a set $S \subset X$ is called an (n, ϵ) -separated set for g if for every pair of points $x, y \in S$ with $x \neq y$, there exists an integer k with $0 \leq k < n$ such that $d(g^k(x), g^k(y)) > \epsilon$. The topological entropy of g is defined to be

$$h_{\text{top}}(g|X) = \lim_{\epsilon \rightarrow 0, \epsilon > 0} \limsup_{n \rightarrow \infty} \frac{\log(\max\{\#(S) : S \subset X \text{ is an } (n, \epsilon)\text{-separated set for } g\})}{n},$$

where $\#(S)$ is the cardinality of elements of S .

We say that g has entropic chaos on X if $h_{\text{top}}(g|X) > 0$.

Topological entropy has played a fundamental role in the theory of chaos. Its concept is a mathematical formulation of exponential divergence of nearby initial conditions. It describes the

total exponential complexity of the orbit structure with a single number in a rough but expressive way. The topological entropy is positive for chaotic systems and is zero for non-chaotic systems.

It is also possible to define a measure theoretic entropy $h_\mu(g)$ for an invariant measure μ . Then the *Variational Principle* says that topological entropy is the supremum of metric theoretic entropies; more precisely, if $g : X \rightarrow X$ is a homeomorphism of a compact metric space (X, d) then $h_{\text{top}}(g) = \sup\{h_\mu(g) : \mu \text{ is an } f\text{-invariant Borel probability measures on } X\}$; refer to Theorem 4.5.3 of Katok and Hasselblatt (1995).

Back to the family Eq. (13); f has critical points at $0, \alpha/(\alpha + \beta)$ and 1 . Consider the case when $\alpha < 1$ or $\alpha = 1$ and $\omega > 1$. Then f has three fixed points, namely, $0, q, p$, with $0 < q < 1 < p$; see Fig. 1(a). We prove the existence of entropic chaos.

Theorem 1. *Let $f = f_{\omega, \alpha, \beta}$ be the family Eq. (13) with $0 < \alpha \leq 1$ and $\beta > 0$ even. Then we have the following properties:*

1. *if $f(\alpha/(\alpha + \beta)) = 1$, then $h_{\text{top}}(f|[0, 1]) \geq \log(2)$ and hence f has entropic chaos on $[0, 1]$;*
2. *if $f(\alpha/(\alpha + \beta)) > 1$, then $h_{\text{top}}(f|\Lambda) \geq \log(2)$ and hence f has entropic chaos on Λ , where $\Lambda = \{x : f^n(x) \in [0, 1] \text{ for all } n \geq 0\}$; and*
3. *if $f(\alpha/(\alpha + \beta)) \geq p$, then $h_{\text{top}}(f|\Lambda) \geq \log(3)$ and hence f has entropic chaos on Λ , where $\Lambda = \{x : f^n(x) \in [0, p] \text{ for all } n \geq 0\}$.*

Proof. We prove item 1 by adapting the method of Robinson for the first part and the method of Katok and Hasselblatt for the second part. Let $I_1 = [0, \frac{\alpha}{\alpha + \beta}]$ and $I_2 = [\frac{\alpha}{\alpha + \beta}, 1]$. Then $f^{-1}([0, 1]) \cap [0, 1] = [0, 1] = I_1 \cup I_2$. For $n \geq 1$ and for $i_0, i_1, \dots, i_{n-1} \in \{1, 2\}$, let

$$I_{i_0, i_1, \dots, i_{n-1}} = \bigcap_{k=0}^{n-1} f^{-k}(I_{i_k}) = \{x \in [0, 1] : f^k(x) \in I_{i_k} \text{ for } 0 \leq k \leq n-1\}.$$

Let $S_0 = [0, 1]$ and for $n \geq 1$, let

$$S_n = \bigcap_{k=0}^n f^{-k}([0, 1]) = \bigcap_{k=0}^{n-1} f^{-k}(I_1 \cup I_2) = \bigcup_{i_0, i_1, \dots, i_{n-1} \in \{1, 2\}} I_{i_0, i_1, \dots, i_{n-1}}.$$

Then $S_n = [0, 1]$ for all $n \geq 0$. Moreover, we claim for all $n \geq 1$, the following properties:

- (a) *if $i_0, i_1, \dots, i_{n-2} \in \{1, 2\}$, then $I_{i_0, i_1, \dots, i_{n-2}} = I_{i_0, i_1, \dots, i_{n-2}, 1} \cup I_{i_0, i_1, \dots, i_{n-2}, 2}$ is the union of two nonempty closed intervals with disjoint interiors;*
- (b) *if $i_0, i_1, \dots, i_{n-1}, i'_0, i'_1, \dots, i'_{n-1} \in \{1, 2\}$ with $(i_0, i_1, \dots, i_{n-1}) \neq (i'_0, i'_1, \dots, i'_{n-1})$, then $\text{int}(I_{i_0, i_1, \dots, i_{n-1}}) \cap \text{int}(I_{i'_0, i'_1, \dots, i'_{n-1}}) = \emptyset$ and so S_n is the union of 2^n closed intervals with pairwise disjoint interiors; and*
- (c) *the map f takes the component $I_{i_0, i_1, \dots, i_{n-1}}$ of S_n homeomorphically onto the component $I_{i_1, \dots, i_{n-1}}$ of S_{n-1} .*

The claim is true by induction on n . For $n = 1$, then $S_1 = [0, 1] \cap f^{-1}([0, 1]) = I_1 \cup I_2$ is the union of two nonempty closed intervals with disjoint interiors. The map f is monotonically increasing on I_1 and hence f maps I_1 homeomorphically onto $[0, 1] = S_0$. Similarly, f is monotonically decreasing on I_2 and hence f also maps I_2 homeomorphically onto $[0, 1] = S_0$. The set $S_1 = I_1 \cup I_2 = [0, 1]$. Assume the claim is true for n and we verify it for $n + 1$. Let

$I_{i_0, \dots, i_{n-1}}$ be a component of S_n . Then $f(I_{i_0, \dots, i_{n-1}}) = I_{i_1, \dots, i_{n-1}}$ is a component of S_{n-1} , and $I_{i_1, \dots, i_{n-1}} = I_{i_1, \dots, i_{n-1}, 1} \cup I_{i_1, \dots, i_{n-1}, 2}$. Therefore,

$$\begin{aligned} I_{i_0, \dots, i_{n-1}} &= f^{-1}(S_n) \cap I_{i_0, \dots, i_{n-1}} = f^{-1}(S_n \cap I_{i_1, \dots, i_{n-1}}) \cap I_{i_0} \\ &= [f^{-1}(I_{i_1, \dots, i_{n-1}, 1}) \cup f^{-1}(I_{i_1, \dots, i_{n-1}, 2})] \cap I_{i_0} \end{aligned}$$

is the union of two nonempty closed intervals with pairwise disjoint interiors, so (a) holds. Since there are 2^n choices of the index for $I_{i_0, \dots, i_{n-1}}$, the set S_{n+1} is the union of 2^{n+1} intervals, so (b) holds. The map f is monotone on the component $I_{i_0, \dots, i_{n-1}, j}$ of S_{n+1} , so it maps homeomorphically onto $I_{i_1, \dots, i_{n-1}, j}$ of S_n , so (c) holds. This completes the verification of the claim.

For $n \geq 2$, let Σ_n^1 be the *one-sided sequence space* $\{\mathbf{i} = (i_0, i_1, \dots) : i_k \in \{1, 2, \dots, n\} \text{ for all } k \geq 0\}$ with the metric $d(\mathbf{i}, \mathbf{j}) = \sum_{k=0}^{\infty} \delta(i_k, j_k)/3^k$, where $\delta(s, t)$ is zero if $s = t$ and is one if $s \neq t$. The *shift map* σ on Σ_n^1 is defined by $\sigma(\mathbf{i}) = \mathbf{j}$ where $j_k = i_{k+1}$ for all $k \geq 0$. Let $\bar{\Sigma}_2^1$ be the space obtained from Σ_2^1 by identifying (i_0, i_1, \dots) and (j_0, j_1, \dots) if there exists $n \geq 0$ such that $i_k = j_k$ for $0 \leq k \leq n - 1$, $i_n = 1$, $j_n = 2$ and $i_k = 2$ and $j_k = 1$ for all $k > n$. Define $g : [0, 1] \rightarrow \bar{\Sigma}_2^1$ by $g(x) = (i_0, i_1, \dots)$ where $f^k(x) \in I_{i_k}$ for all $k \geq 0$.

We prove that g is a semi-conjugacy from f on $[0, 1]$ to σ on $\bar{\Sigma}_2^1$. First we check the condition that $g \circ f = \sigma \circ g$. Let $x \in [0, 1]$, $(i_0, i_1, \dots) = g(x)$ and $(j_0, j_1, \dots) = g(f(x))$. Then $f^{k+1}(x) \in I_{i_{k+1}}$ and $f^{k+1}(x) = f^k(f(x)) \in I_{j_k}$. Hence, $i_{k+1} = j_k$ so $g(f(x)) = \sigma(g(x))$. Next we check that g is surjective. Let $(i_0, i_1, \dots) \in \bar{\Sigma}_2^1$. By the above claim, $\{I_{i_0, \dots, i_n}\}_{n=0}^{\infty}$ is a nested sequence of nonempty closed intervals. Thus, there exists $x_0 \in \bigcap_{n=0}^{\infty} I_{i_0, \dots, i_n} = \bigcap_{k=0}^{\infty} f^{-k}(I_{i_k})$. Therefore, $f^k(x_0) \in I_{i_k}$ for all $k \geq 0$ and hence $g(x_0) = (i_0, i_1, \dots)$. Last we must check that g is continuous. Let $x \in [0, 1]$ and $(i_0, i_1, \dots) = g(x)$. Let $\epsilon > 0$. Pick an integer n such that $3^{-n} < \epsilon$. Consider the interval I_{i_0, \dots, i_n} . Take $\delta > 0$ so small that if $y \in [0, 1]$ with $|y - x| < \delta$, then $y \in I_{i_0, \dots, i_n}$. For $y \in [0, 1]$ with $|y - x| < \delta$, let $(j_0, j_1, \dots) = g(y)$. Then $j_k = i_k$ for $0 \leq k \leq n$. Thus, $d(g(x), g(y)) \leq \sum_{k=n+1}^{\infty} 3^{-k} = 2^{-1}3^{-n} < \epsilon$. This proves the continuity of g . We conclude that g is a semi-conjugacy from f to σ .

The space $\bar{\Sigma}_2^1$ is obtained from Σ_2^1 by identifying two sequences if they are itineraries of the same point in S . Thus, the shift map σ on Σ_2^1 and $f|_{[0, 1]}$ both naturally project to the shift map σ on $\bar{\Sigma}_2^1$. Notice that the semi-conjugacy $\bar{g} : \Sigma_2^1 \rightarrow \bar{\Sigma}_2^1$ is injective outside a countable set, namely, the itineraries of points in the backward orbits of turning points. Notice further that by the Variational Principle it suffices to consider non-atomic measures, since purely atomic measures have zero entropy. Consider a non-atomic σ -invariant measure ζ on Σ_2^1 and pull it back via the semi-conjugacy \bar{g} to a measure $\bar{g}_*\zeta$ on $\bar{\Sigma}_2^1$. Thus, \bar{g} establishes a bijective corresponding between ζ and $\bar{g}_*\zeta$ so the measure theoretic entropies coincide. By the Variational Principle, we have $h_{\text{top}}(\sigma|_{\bar{\Sigma}_2^1}) = \sup_{\zeta} h_{\zeta}(\sigma|_{\bar{\Sigma}_2^1}) \geq \sup_{\zeta} h_{\bar{g}_*\zeta}(\sigma|_{\bar{\Sigma}_2^1}) = \sup_{\zeta} h_{\zeta}(\sigma|_{\Sigma_2^1}) = h_{\text{top}}(\sigma|_{\Sigma_2^1})$. Since g is a semi-conjugacy from $f|_{[0, 1]}$ to $\sigma|_{\bar{\Sigma}_2^1}$, $h_{\text{top}}(f|_{[0, 1]}) \geq h_{\text{top}}(\sigma|_{\bar{\Sigma}_2^1}) = h_{\text{top}}(\sigma|_{\Sigma_2^1}) = \log(2)$. We have finished the proof of item 1.

We prove item 2 by making a slight modification from the proof of item 1. Since $f(\alpha/(\alpha + \beta)) > 1$, there exists p_1, p_2 such that $0 < p_1 < p_2 < 1$ and $f(p_1) = f(p_2) = 1$. Let $I_1 = [0, p_1]$ and $I_2 = [p_2, 1]$. Then $f^{-1}([0, 1]) \cap [0, 1] = I_1 \cup I_2$. For $n \geq 1$ and $i_0, \dots, i_{n-1} \in \{1, 2\}$, let $I_{i_0, \dots, i_{n-1}} = \bigcap_{k=0}^{n-1} f^{-k}(I_{i_k})$. Let $S_0 = [0, 1]$ and for $n \geq 1$, let $S_n = \bigcup_{i_0, \dots, i_{n-1} \in \{1, 2\}} I_{i_0, \dots, i_{n-1}}$. Then $\Lambda = \bigcap_{n=0}^{\infty} S_n$. Moreover, for all $n \geq 1$, we have

- (a) if $i_0, \dots, i_{n-2} \in \{1, 2\}$, then $I_{i_0, \dots, i_{n-2}} \cap S_n = \bigcup_{j \in \{1, 2\}} I_{i_0, \dots, i_{n-2}, j}$ is the union of three nonempty closed intervals with pairwise disjoint interiors, which are subsets of $I_{i_0, \dots, i_{n-2}}$;

- (b) the set S_n is the union of 2^n closed intervals; and
- (c) the map f takes the component $I_{i_0, \dots, i_{n-1}}$ of S_n homeomorphically onto the component $I_{i_1, \dots, i_{n-1}}$ of S_{n-1} .

Define $g : \Lambda \rightarrow \Sigma_2^1$ by $g(x) = (i_0, i_1, \dots)$ where $f^k(x) \in I_{i_k}$ for all $k \geq 0$. Then g is a semi-conjugacy from f on Λ to σ on Σ_2^1 . Therefore, we have $h_{\text{top}}(f|\Lambda) \geq h_{\text{top}}(\sigma|\Sigma_2^1) \geq \log(2)$.

We prove item 3 by using the same argument as above. Since $f(\alpha/(\alpha + \beta)) \geq p$, there exists p_1, p_2 such that $0 < p_1 \leq p_2 < 1$ and $f(p_1) = f(p_2) = p$. Note that $p_1 = p_2$ if and only if $f(\alpha/(\alpha + \beta)) = p$. Let $I_1 = [0, p_1], I_2 = [p_2, 1]$ and $I_3 = [1, p]$. Then $f^{-1}([0, p]) \cap [0, p] = I_1 \cup I_2 \cup I_3$. For $n \geq 1$ and $i_0, \dots, i_{n-1} \in \{1, 2, 3\}$, let $I_{i_0, \dots, i_{n-1}} = \bigcap_{k=0}^{n-1} f^{-k}(I_{i_k})$. Let $S_0 = [0, p]$ and for $n \geq 1$, let $S_n = \bigcup_{i_0, \dots, i_{n-1} \in \{1, 2, 3\}} I_{i_0, \dots, i_{n-1}}$. Then $\Lambda = \bigcap_{n=0}^{\infty} S_n$. Moreover, for all $n \geq 1$, we have

- (a) if $i_0, \dots, i_{n-2} \in \{1, 2, 3\}$, then $I_{i_0, \dots, i_{n-2}} \cap S_n = \bigcup_{j \in \{1, 2, 3\}} I_{i_0, \dots, i_{n-2}, j}$ is the union of three nonempty closed interval with pairwise disjoint interiors, which are subsets of $I_{i_0, \dots, i_{n-2}}$;
- (b) the set S_n is the union of 3^n closed intervals; and
- (c) the map f takes the component $I_{i_0, \dots, i_{n-1}}$ of S_n homeomorphically onto the component $I_{i_1, \dots, i_{n-1}}$ of S_{n-1} .

Let $\bar{\Sigma}_3^1$ be the space obtained from Σ_3^1 by identifying (i_0, i_1, \dots) and (j_0, j_1, \dots) if there exists $n \geq 0$ such that $i_k = j_k$ for $0 \leq k \leq n - 1, i_n = 2, j_n = 3$ and $i_k = 3$ and $j_k = 2$ for all $k > n$ or if $p = q$ and there exists $n \geq 0$ such that $i_k = j_k$ for $0 \leq k \leq n - 1, i_n = 1, j_n = 2$ and $i_k = 2$ and $j_k = 1$ for all $k > n$.

Define $g : \Lambda \rightarrow \bar{\Sigma}_3^1$ by $g(x) = (i_0, i_1, \dots)$ where $f^k(x) \in I_{i_k}$ for all $k \geq 0$. Then g is a semi-conjugacy from f on Λ to σ on $\bar{\Sigma}_3^1$. By using the Variational Principle as in the proof of item 1, we have $h_{\text{top}}(f|\Lambda) \geq h_{\text{top}}(\sigma|\bar{\Sigma}_3^1) = h_{\text{top}}(\sigma|\Sigma_3^1) \geq \log(3)$. \square

Next, we consider the family Eq. (13) with $\alpha > 1$; see Fig. 1(b). Suppose that $f(\alpha/(\alpha + \beta)) \geq \alpha/(\alpha + \beta)$. Then f has four fixed points, namely, $0, r, q, p$ with $0 < r < \alpha/(\alpha + \beta) < q < 1 < p$. Moreover, r has a preimage in $(q, 1)$, namely, r_+ with $f(r_+) = r$. We prove that the topological entropy is positive.

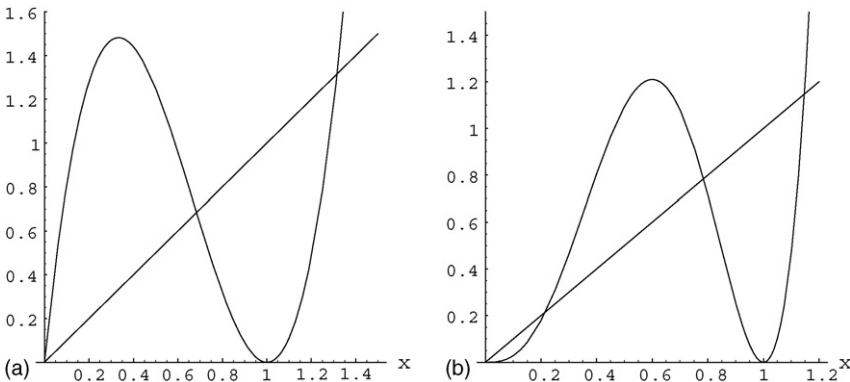


Fig. 1. The graphs of (a) $f_{10,1,2}$ and (b) $f_{35,3,2}$.

Theorem 2. Let $f = f_{\omega, \alpha, \beta}$ be the family Eq. (13) with $\alpha > 1$ and $\beta > 0$ even. Then we have the following properties:

1. if $f(\alpha/(\alpha + \beta)) \geq r_+$, then $h_{\text{top}}(f|\Lambda) \geq \log(2)$ and hence f has entropic chaos on Λ , where $\Lambda = \{x : f^n(x) \in [r, r_+] \text{ for all } n \geq 0\}$; and
2. if $f(\alpha/(\alpha + \beta)) \geq p$, then $h_{\text{top}}(f|\Lambda) \geq \log(3)$ and hence f has entropic chaos on Λ , where $\Lambda = \{x : f^n(x) \in [r, p] \text{ for all } n \geq 0\}$.

Proof. First we prove item 1. Since f is monotonically increasing on $[r, \frac{\alpha}{\alpha+\beta}]$ and is monotonically decreasing on $[\frac{\alpha}{\alpha+\beta}, r_+]$, the assumption $f(\alpha/(\alpha + \beta)) \geq r_+$ implies that $[r, r_+] \cap f([r, r_+])$ consists of two nonempty closed intervals, say I_1 and I_2 , such that $f(I_1) = f(I_2) = [r, r_+]$. By using the same argument as in the proof of Theorem 1, the desired result follows.

Next, we prove item 2. Since f is monotone on $[r, \frac{\alpha+\beta}{\beta}]$, $[\frac{\alpha}{\alpha+\beta}, 1]$ and $[1, p]$, respectively, the assumption $f(\alpha/(\alpha + \beta)) \geq p$ implies that $[r, p] \cap f([r, p])$ consists of three empty closed intervals, say I_1, I_2 and I_3 , such that $f(I_1) = f(I_2) = f(I_3) = [r, p]$. Again, by using the same argument as in the proof of Theorem 1, we have the desired result. \square

Next, we give the definitions of ergodic invariant measure and ergodic chaos; refer to Robinson.

Definition 2. Let I be a bounded closed interval and $g : I \rightarrow I$ be a continuous map. A measure ζ is said to be g -invariant if $\zeta(g^{-1}(A)) = \zeta(A)$ for all measurable set A . A g -invariant measure ζ is said to be ergodic if $\zeta(I \setminus A) = 0$ for any measurable set A with $g(A) = A$ and $\zeta(A) > 0$. We say such a map g has ergodic chaos on I if there exists a unique ergodic g -invariant measure that is absolutely continuous with respect to the Lebesgue measure.

The concept of ergodicity indicates that almost all orbits are dense in the support of the measure because for any set of positive measure A , the orbit of A has full measure. Alternatively, the Birkhoff Ergodic Theorem [refer to Theorem 4.1.2 of Katok and Hasselblatt] implies that if g has an ergodic invariant measure ζ on I , then for any ζ -integrable real-valued function φ on I , $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(g^i(x)) = \int_I \varphi(x) d\zeta(x)$ holds for ζ -almost all $x \in I$. That is, ergodicity reveals that for any integrable function, the time average along almost all orbits is equal to the space average of the function. A dynamical system may have many ergodic invariant measures. The one that is absolutely continuous with respect to the Lebesgue measure is the most interesting and important.

In Boldrin et al. (2001), it is shown that the family Eq. (13) with $\alpha = 1$ exhibits ergodic chaos under certain conditions. Here, we prove the existence of ergodic chaos for Eq. (13) with integers $\alpha > 1$ and $\beta \geq 1$.

Theorem 3. Let $f = f_{\omega, \alpha, \beta}$ be the family Eq. (13) with $\alpha > 1$ and $\beta \geq 1$ both integers, and $f(\alpha/(\alpha + \beta)) > \alpha/(\alpha + \beta)$. Let $0 < r < q < r_+ < 1$ be as mentioned above. If $f(\alpha/(\alpha + \beta)) \leq r_+$, $f'(q) < -1$ and $f^k(\alpha/(\alpha + \beta)) = q$ for some $k \geq 2$, then f has ergodic chaos on $[r, r_+]$.

Proof. By Proposition 1 of Boldrin et al. [originally due to Misiurewicz (1980)], it is sufficient to show that the Schwarzian derivative of f is negative, that is, $S_f(x) \equiv \frac{f'''(x)}{f'(x)} - \frac{3}{2} (\frac{f''(x)}{f'(x)})^2 < 0$ for all $x \in [r, r_+] \setminus \{0, 1, \frac{\alpha}{\alpha+\beta}\}$. For convenience, let $A = (-1)^\beta \omega(\alpha + \beta)$ and $a_1 = \alpha/(\alpha + \beta)$, and let a_i be 0 for $2 \leq i \leq \alpha$ and be 1 for $\alpha + 1 \leq i \leq \alpha + \beta - 1$. Then $f'(x) = A \prod_i (x - a_i)$, $f''(x) = A \sum_j \prod_{i \neq j} (x - a_i)$, and $f'''(x) = A \sum_j \sum_{k \neq j} \prod_{i \neq j, k} (x - a_i)$. Here both the product

and the summation are taken over all integers between 1 and $\alpha + \beta - 1$. Thus,

$$S_f(x) = \sum_j \sum_{k \neq j} \frac{1}{(x-a_j)(x-a_k)} - \frac{3}{2} (\sum_j \frac{1}{x-a_j})^2$$

$$= - \sum_j (\frac{1}{x-a_j})^2 - \frac{1}{2} (\sum_j \frac{1}{x-a_j})^2 < 0 \quad \text{for } x \neq a_j.$$

This completes the proof of the theorem. \square

At the end of this section, we conclude the result of simple dynamics.

Theorem 4. *Let $f = f_{\omega, \alpha, \beta}$ be the family Eq. (13) with $\alpha > 0, \beta > 0$ and $0 < \omega < ((\alpha + \beta)^{\alpha+\beta-1} / (\alpha^{\alpha-1} \beta^\beta))$. Let J be the interval $[0, p]$ if β is an even integer and be $[0, 1]$ if β is not even. Then for every $x \in J, f^n(x)$ converges to zero as n tends to infinity.*

Proof. Since $f(\alpha/(\alpha + \beta))$ is the unique local maximum of f on J , the assumption implies that $f(x) < x$ for all $x \neq 0$ in J . Fix $x \in J$ not equal to 0. Then $\{f^n(x)\}_{n=0}^\infty$ is a strictly decreasing sequence bounded below by 0, and hence there exists $\bar{x} \geq 0$ in J such that $\lim_{n \rightarrow \infty} f^n(x) = \bar{x}$. By the continuity of f , we have $f(\bar{x}) = f(\lim_{n \rightarrow \infty} f^n(x)) = \lim_{n \rightarrow \infty} f^{n+1}(x) = \bar{x}$, that is, \bar{x} is a fixed point of f . Since 0 is the unique fixed point of f on $J, \bar{x} = 0$, so the desired result follows. \square

Note that since $\omega = ((1 + \theta)/\eta)^{1/1-\rho}$, the above condition $0 < \omega < ((\alpha + \beta)^{\alpha+\beta-1} / (\alpha^{\alpha-1} \beta^\beta))$ is equivalent to $\theta > -1 + \eta((\alpha + \beta)^{\alpha+\beta-1} / (\alpha^{\alpha-1} \beta^\beta))^{1-\rho} > 0$ (respectively $0 < \theta < -1 + \eta((\alpha + \beta)^{\alpha+\beta-1} / (\alpha^{\alpha-1} \beta^\beta))^{1-\rho}$) if $\rho > 0$ (respectively $0 < \rho < 1$). Therefore, with the persistence of the cash goods habit, the dynamics of our model will eliminate chaotic behavior and become simple convergence when the money supply growth rate is sufficiently large (respectively small) provided $\rho > 1$ (respectively $0 < \rho < 1$). This result of bifurcating at the critical value $\rho = 1$ is different from the earlier ones. Woodford argued that an increase of money supply growth can eliminate multiple equilibria. Matsuyama (1991) demonstrated that a high money supply growth rate can cause instability in the price levels and showed the result is also robust with respect to different settings of a money-in-the-utility model. The numerical examination by Michener and Ravikumar demonstrated that the multiplicity of equilibria cannot be eliminated, regardless of whether the money supply growth rate is high or low in a cash-in-advance economy.

3. Habit persistence in both cash-goods and credit-goods

In this section, we assume that agents have persistent habits for both cash-goods and credit-goods consumptions. With a slight modification of Eq. (4), the preference is now represented as

$$\sum_{t=0}^\infty \eta^t U(V(c_t, \bar{c}_{t-1}), G(d_t, \bar{d}_{t-1})),$$

where \bar{d}_{t-1} is the average consumption of credit goods in period $t - 1$. Note that here G is a function of two variables instead of one variable, as in Eq. (4). In an analogy to Assumption 3, we assume that an increase in the past average consumption of credit goods will magnify the utility while keeping current consumptions fixed.

Assumption 5. The utility function is decreasing in \bar{d}_{t-1} . That is,

$$\frac{\partial U}{\partial \bar{d}_{t-1}}(V(c_t, \bar{c}_{t-1}), G(d_t, \bar{d}_{t-1})) \leq 0.$$

The specific function form of $G(d_t, \bar{d}_{t-1})$ is given as

$$G(d_t, \bar{d}_{t-1}) = \begin{cases} \frac{(d_t - \tau \bar{d}_{t-1})^{1-\gamma} - 1}{1 - \gamma}, & \text{if } \gamma \neq 1, \\ \log(d_t - \tau \bar{d}_{t-1}), & \text{if } \gamma = 1. \end{cases} \tag{14}$$

The parameter $\tau \in (0, 1)$ indicates the degree of habit persistence of average credit-goods consumption. We assume $\gamma \neq 1$.

Here the setting of the persistent habit of credit-goods consumption in Eq. (14) is different from that of cash-goods consumption in Eq. (11) in order to simplify the analysis of the economic dynamics with habit persistence for both goods. Both forms have been used to describe the persistence of consumption habit in the previous studies. The former function form was adopted by de la Croix (1996) and Auray et al. while the latter one was utilized by Abel.

Two-dimensional dynamical systems

Given the utility function of $G(d_t, \bar{d}_{t-1})$ as in Eq. (14), we find that Eq. (8) reduces to

$$m_{t+2} = \left\{ \frac{\kappa(1 + \theta)}{\eta} m_{t+1}^{1+\xi(1-\rho)} [(1 - \tau)v - m_{t+1} + \tau m_t]^{-\gamma} \right\}^{(1/1-\rho)}. \tag{15}$$

Hence, the economic dynamics is represented by a second-order difference equation.

For the dynamics of Eq. (15), without loss of generality, we assume that $v = 1$. For describing chaotic behaviors, we assume $1/(1 - \rho) + \xi = -(\gamma/(1 - \rho)) = 1$. Let $\omega = (\kappa(1 + \theta)/\eta)^{(1/1-\rho)}$, $x = m_t$, and $y = m_{t+1}$. Then the dynamics of Eq. (15) with $(m_t, m_{t+1}) \mapsto (m_{t+1}, m_{t+2})$ is equivalent to the dynamics of the family of maps $(x, y) \mapsto F_{\omega, \tau}(x, y)$, where

$$F_{\omega, \tau}(x, y) = (y, \omega y(1 - y) + \tau \omega y(x - 1)). \tag{16}$$

Numerical simulations indicate that the dynamics of $F_{\omega, \tau}$ vary from simple to chaotic as parameters change; see Fig. 2.

First, following the pioneer article of Smale in the theory of chaotic dynamical systems, we show that $F_{\omega, \tau}$ has entropic chaos due to the existence of a so-called *Smale horseshoe*; see Fig. 3.

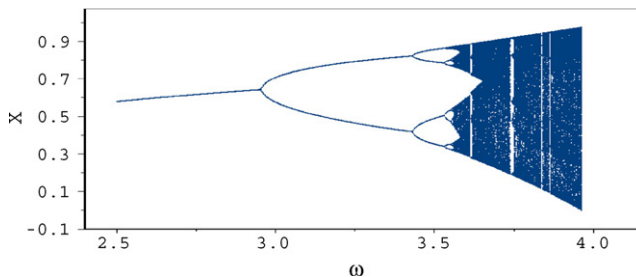


Fig. 2. The x -coordinate orbit diagram of $F_{\omega, 0.05}(x, y)$ in ω .

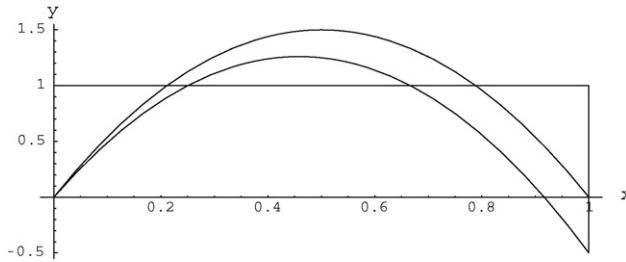


Fig. 3. The set $[0, 1] \times [0, 1]$ and its image $F_{\omega, \tau}([0, 1] \times [0, 1])$ form a “horseshoe”.

Theorem 5. Let $F_{\omega, \tau}$ be the family Eq. (16) with $\omega > 4$ and $0 < \tau < 1 - (2/\sqrt{\omega})$. Then $h_{\text{top}}(F_{\omega, \tau}|_{\Lambda}) \geq \log(2)$ and hence $F_{\omega, \tau}$ has entropic chaos on Λ , where

$$\Lambda = \{(x, y) : F_{\omega, \tau}^n(x, y) \in [0, 1] \times [0, 1] \text{ for all } n \in \mathbb{Z}\}.$$

Proof. Let $S = [0, 1] \times [0, 1]$ be the unit square, and let $g(y) = \omega y(1 - y) - \tau \omega y$ for $y \in [0, 1]$. Then $g([0, 1])$ is the bottom boundary of the image $F_{\omega, \tau}(S)$; refer to Fig. 3. The maximum of g on $[0, 1]$ is $g((1 - \tau)/2) = \omega((1 - \tau)/2)^2$, which is greater than 1 since $0 < \tau < 1 - (2/\sqrt{\omega})$. Thus, $F_{\omega, \tau}(S) \cap S$ has two vertical strips, namely, V_1 on the left and V_2 on the right. Similarly, $F_{\omega, \tau}^{-1}(S) \cap S$ has two horizontal strips, namely, H_1 on the bottom and H_2 on the top. We have $F_{\omega, \tau}(H_k) = V_k$ for $k = 1, 2$.

For simplicity, we write F for $F_{\omega, \tau}$. For integers $m \leq 0$ and $n \geq 0$, let $S_m^n = \bigcap_{i=m}^n F^i(S)$. Then $S_0^1 = V_1 \cup V_2$ is a union of two vertical strips in S . As in the one dimensional case, for $n \geq 1$,

$$S_0^n = F(S_0^{n-1}) \cap S = [F(S_0^{n-1}) \cap V_1] \cup [F(S_0^{n-1}) \cap V_2] = F(S_0^{n-1} \cap H_1) \cup F(S_0^{n-1} \cap H_2).$$

In particular, for $n = 2$, $S_0^2 = F(S_0^1 \cap H_1) \cup F(S_0^1 \cap H_2) = F([V_1 \cup V_2] \cap H_1) \cup F([V_1 \cup V_2] \cap H_2)$ is the union of 2^2 vertical strips in S_0^1 . By induction, S_0^n is the union of 2^n vertical strips. Taking $n \rightarrow \infty$, we have $S_0^\infty = \bigcap_{n=1}^\infty S_0^n$ is the union of infinitely many vertical strips or segments (occurring while the widths of strips converges to zero as $n \rightarrow \infty$). If $z \in S_0^\infty$, then $z \in F^i(S)$ and $F^{-i}(z) \in S$ for all $i \geq 0$. Thus, S_0^∞ is the set of points whose backward iterates stay in S .

Considering the sets S_m^0 , we have $S_{-1}^0 = H_1 \cup H_2$ is the union of two horizontal strips in S . Then S_{-2}^0 is the union of four horizontal strips in S_{-1}^0 . Continuing by induction, we have that for $m \leq 0$, S_m^0 is the union of 2^{-m} horizontal strips and $S_{-\infty}^0 = \bigcap_{m=-\infty}^0 S_m^0$ is the union of infinitely many horizontal strips or segments (occurring while the heights of strips converges to zero as $m \rightarrow -\infty$). If $z \in S_{-\infty}^0$, then $z \in F^{-i}(S)$ and $F^i(z) \in S$ for all $i \geq 0$. Thus, $S_{-\infty}^0$ is the set of points whose forward iterates stay in S .

By the definition of Λ , we have that $\Lambda = S_0^\infty \cap S_{-\infty}^0$ is the intersection of infinitely many vertical strips (or segments) and infinitely many horizontal strips (or segments), and Λ is the set of points such that both the forward and backward iterates stay in S .

Let $\Sigma_2^2 = \{\mathbf{i} = (\dots, i_{-1}, i_0, i_1, \dots) : i_k \in \{1, 2\} \text{ for all } k \in \mathbb{Z}\}$ be the two-sided sequence space with the metric $d(\mathbf{i}, \mathbf{j}) = \sum_{k=-\infty}^\infty \frac{\delta(i_k, j_k)}{4^{|k|}}$, where $\delta(s, t)$ is 0 if $s = t$ and is 1 if $s \neq t$. The shift map σ on Σ_2^2 is defined by $\sigma(\mathbf{i}) = \mathbf{j}$ where $j_k = i_{k+1}$ for all $k \in \mathbb{Z}$. Let $\bar{\Sigma}_2^2$ be the space obtained from Σ_2^2 by identifying $(\dots, i_{-1}, i_0, i_1, \dots)$ and $(\dots, j_{-1}, j_0, j_1, \dots)$ if $i_k = j_m = 1$ for all $k \in \mathbb{Z}$ and all $m \leq 1$, that is, by identifying two sequences if they are itineraries of the same point in S . Define $h : \Lambda \rightarrow \bar{\Sigma}_2^2$ by $h(z) = (\dots, i_{-1}, i_0, i_1, \dots)$ where $F^k(z) \in H_{i_k}$ for all $k \in \mathbb{Z}$. We prove that h is a semi-conjugacy from $F|_{\Lambda}$ to $\sigma|_{\bar{\Sigma}_2^2}$.

First we prove that $\sigma \circ h = h \circ F$ on Λ . Let $h(z) = (\dots, i_{-1}, i_0, i_1, \dots)$ and $h(F(z)) = (\dots, j_{-1}, j_0, j_1, \dots)$. Then $F^{k+1}(z) \in H_{i_{k+1}}$ but also $F^{k+1}(z) = F^k(F(z)) \in H_{j_k}$. Thus, $i_{k+1} = j_k$ and $\sigma(h(z)) = h(F(z))$.

Next we prove the continuity of h . Let $h(z) = (\dots, i_{-1}, i_0, i_1, \dots)$. A neighborhood of $(\dots, i_{-1}, i_0, i_1, \dots)$ is given by $U = \{(\dots, j_{-1}, j_0, j_1, \dots) : j_k = i_k \text{ for } -k_0 \leq k \leq k_0\}$. With k_0 fixed, the continuity of F insures that there is a $\delta > 0$ such that if $w \in \Lambda$ with $|w - z| \leq \delta$, then $F^k(w) \in H_{i_k}$ for $-k_0 \leq k \leq k_0$. Thus, if $w \in \Lambda$ with $|w - z| \leq \delta$ then $h(w) \in U$.

Last we check that h is surjective. We apply induction on n to show that $\bigcap_{k=1}^n F^k(H_{i_{-k}})$ is a vertical strip for all strings of symbols $(\dots, i_{-1}, i_0, i_1, \dots) \in \bar{\Sigma}_2^n$. Let $(\dots, i_{-1}, i_0, i_1, \dots) \in \bar{\Sigma}_2^n$. For $n = 1$, this set is just $F(H_{i_{-1}}) = V_{i_{-1}}$, which is a vertical strip. Then

$$\bigcap_{k=1}^n F^k(H_{i_{-k}}) = F \left(\bigcap_{k=2}^n F^{k-1}(H_{i_{-k}}) \right) \cap F(H_{i_{-1}})$$

is a vertical strip. Letting n go to infinity, $\bigcap_{k=1}^\infty F^k(H_{i_{-k}})$ is a vertical strip or segment. Similarly, $\bigcap_{k=-\infty}^0 F^k(H_{i_{-k}})$ is a horizontal strip or segment. Thus, $\bigcap_{k=-\infty}^\infty F^k(H_{i_{-k}})$ is nonempty; say z is in this intersection. Therefore, $h(z) = (\dots, i_{-1}, i_0, i_1, \dots)$ and h is surjective. This completes the proof that h is a semi-conjugacy from $F|\Lambda$ to $\sigma|\bar{\Sigma}_2^n$.

By using the Variational Principle as in the proof of Theorem 1, we have that $h_{\text{top}}(F|\Lambda) \geq h_{\text{top}}(\sigma|\bar{\Sigma}_2^n) = h_{\text{top}}(\sigma|\Sigma_2^2) = \log(2)$. The proof of the theorem is complete. \square

Using the method of Yokoo (2000), we show the existence of entropic chaos for the family Eq. (16) with ω slightly less than 4.

Theorem 6. *Let $F_{\omega,\tau}$ be the family Eq. (16) with $3.7 \leq \omega < 4$ and $\tau > 0$ sufficiently small. Then there exists a set $\Lambda \subset (0, 1) \times (0, 1)$ such that $F_{\omega,\tau}(\Lambda) = \Lambda$ and $F_{\omega,\tau}$ has entropic chaos on Λ .*

Proof. Let $g(y) = \omega y(1 - y)$, $a = g^2(\frac{1}{2})$ and $b = g(1/2)$. Since $3.7 \leq \omega < 4$, we have $0 < a < (\omega - 1)/\omega < b < 1$, $g((\omega - 1)/\omega) = (\omega - 1)/\omega$, and $g([a, b]) = [a, b]$. Let $p = (\frac{\omega-1}{\omega}, \frac{\omega-1}{\omega})$, then $F_{\omega,0}(p) = p$.

We first claim that there exist a compact region $M \subset (0, 1) \times (0, 1)$ and a number $\tau_0 > 0$ such that for every $\tau \in (0, \tau_0)$, the following assertions hold:

- (a) $F_{\omega,\tau}(M) \subset \text{int}(M)$ and $p \in \text{int}(M)$;
- (b) $F_{\omega,\tau}|M : M \rightarrow M$ is a C^1 -diffeomorphism onto its image; and
- (c) the Jacobian matrix evaluated at p , $D_p F_{\omega,\tau}$, has two real eigenvalues $\nu_1(\tau)$ and $\nu_2(\tau)$ with $0 < |\nu_1(\tau)| < 1 < |\nu_2(\tau)|$ and $|\nu_1(\tau)\nu_2(\tau)| < 1$.

The proof of the claim is as follows. Given $a_1 \in (0, a)$, there exists $b_1 \in (b, 1)$ such that $g(b_1) > a_1$. Then $g([a_1, b_1]) \subset (a_1, b_1)$. Similarly, given $a_2 \in (0, a_1)$, there exists $b_2 \in (b_1, 1)$ such that $g(b_2) > a_2$. Then $[a_1, b_1] \subset (a_2, b_2)$ and $g([a_2, b_2]) \subset (a_2, b_2)$. Let $M = [a_2, b_2] \times [a_1, b_1]$, then $M \subset (0, 1) \times (0, 1)$ and $F_{\omega,0}(M) = [a_1, b_1] \times g([a_1, b_1]) \subset (a_2, b_2) \times (a_1, b_1) = \text{int}(M)$. Since the map $(x, y) \mapsto y(x - 1)$ is continuous on the compact set M , we have that for any sufficiently small $\tau > 0$, $F_{\omega,\tau}(M) \subset \text{int}(M)$, so item 1 follows. It is easy to see that $F_{\omega,\tau}$ is one to one on M . On the other hand, the Jacobian matrix of $F_{\omega,\tau}$ at point $z = (x, y)$ is given by

$$D_z F_{\omega,\tau} = \begin{bmatrix} 0 & 1 \\ \tau\omega y & \omega - 2\omega y + \tau(x - 1) \end{bmatrix}. \tag{17}$$

Hence, we have $\det(D_z F_{\omega,\tau}) = -\tau\omega y < 0$ for all $z = (x, y) \in M$ and $\tau > 0$. Therefore, item 2 follows. Let $v_1(\tau)$ and $v_2(\tau)$ with $|v_1(\tau)| \leq |v_2(\tau)|$ be the two eigenvalues of Eq. (17) evaluated at $p = (\frac{\omega-1}{\omega}, \frac{\omega-1}{\omega})$. Then $\lim_{\tau \rightarrow 0} v_1(\tau) = 0$ and $\lim_{\tau \rightarrow 0} v_2(\tau) = 2 - \omega \geq -1.7$. By the continuity of $v_i(\tau)$ with $i = 1, 2$ with respect to τ and by $|v_1(\tau)v_2(\tau)| = \det |D_p F_{\omega,\tau}| > 0$ for $\tau > 0$, item 3 follows.

The rest of the proof involves basic terminologies in the theory of dynamical systems; refer to Katok and Hasselblatt and Robinson for definitions. Since $3.7 \leq \omega < 4$, $g^2(\frac{1}{2}) < (1/\omega)$, and hence there exists $y_0 \in (\frac{1}{2}, \frac{\omega-1}{\omega})$ such that $g^2(y_0) = 1/\omega$ and $g^3(y_0) = (\omega - 1)/\omega$. Moreover, there exists a sequence $\{y_{-i}\}_{i=1}^{\infty}$ such that $g(y_{-i}) = y_{-i+1}$ in the interval $(\frac{1}{2}, g^2(\frac{1}{2}))$ for all $i \geq 1$ and $\lim_{i \rightarrow \infty} y_{-i} = (\omega - 1)/\omega$. Therefore, y_0 is a transverse homoclinic point with respect to the fixed point $(\omega - 1)/\omega$ for g . Let $q = (y_0, g(y_0))$, then $q \in M$ is a transverse homoclinic point with respect to the fixed point p for $F_{\omega,0}$. Let ℓ^s be the horizontal line segment in M passing through p , that is, $\ell^s = \{(x, y) \in M : y = (\omega - 1)/\omega\}$. Since every point in ℓ^s is mapped onto p by $F_{\omega,0}$, ℓ^s is a part of the stable manifold of p for $F_{\omega,0}$. Let $\ell^u = \{(x, y) \in M : y = g(x)\}$, an arc on the graph of g . Since each point on ℓ^u has a backward orbit converging to p , ℓ^u is a part of the unstable manifold of p for $F_{\omega,0}$. Clearly, ℓ^s and ℓ^u have a transverse intersection at $F_{\omega,0}^2(q) \in M$.

By the perturbation argument of invariant manifolds [see Appendices 1 and 4 of Palis and Takens (1993)], for $\tau > 0$ small, the C^1 diffeomorphism $F_{\omega,\tau}|M$ has a saddle point $p_\tau \in M$ near p . The stable and unstable manifolds of p for $F_{\omega,\tau}$, $W^s(p_\tau, F_{\omega,\tau})$ and $W^u(p_\tau, F_{\omega,\tau})$, contain arcs ℓ_τ^s and ℓ_τ^u which are C^1 close to ℓ^s and ℓ^u , respectively. Since transverse intersections are persistent in the C^1 sense, ℓ_τ^s and ℓ_τ^u have a transverse intersection $q_\tau \in M$ near q . Thus, for all sufficiently small $\tau > 0$, the diffeomorphism $F_{\omega,\tau}|M$ has a transverse homoclinic point for the hyperbolic fixed point p_τ .

By the Transverse Homoclinic Point Theorem [see Theorem VIII.4.5 of Robinson], there exist $k \in \mathbb{N}$ and a set $A_1 \subset M$ such that $f^k(A_1) = A_1$ and $F_{\omega,\tau}^k|A_1$ is topologically conjugate to the shift map $\sigma|\Sigma_2^2$. Let $A = \bigcup_{i=-\infty}^{\infty} F_{\omega,\tau}^i(A_1)$; then $A \subset M \subset (0, 1) \times (0, 1)$, $F_{\omega,\tau}(A) = A$ and $h_{\text{top}}(F_{\omega,\tau}|A) = \frac{1}{k} h_{\text{top}}(F_{\omega,\tau}^k|A_1) = \frac{1}{k} h_{\text{top}}(\sigma|\Sigma_2^2) = \frac{1}{k} \log(2) > 0$. The proof of the theorem is complete. \square

Theorems 5 and 6 demonstrate that under certain conditions, chaos will emerge when households have persistent habits of both cash-goods and credit-goods. Similar to the case when agents have the persistent habit of only the cash goods, the money supply growth rate also plays a significant role in determining the possibility of chaotic motion.

4. Conclusions

In this paper, we have shown the existence of chaotic behavior in a monetary economy with habit persistence in consumption of cash goods and credit goods, and focusing on entropic and ergodic chaos. The economic transition can be represented by a one-dimensional or two-dimensional dynamical system depending on the persistent habit of only cash-goods consumption or both cash-goods and credit-goods consumptions, respectively. We investigated the presence of chaotic motion under these two situations when the cash-in-advance constraint was always binding. We also showed that the money supply growth rate is an important determinant of the possibility of chaotic motion.

The dynamics of a cash-in-advance model become complicated when agents have persistent habits of both goods; thus future studies of the model are warranted. Some of the parameter values were restricted to simplify the two-dimensional dynamical system in the model. It would

be interesting to examine the possibility of chaotic dynamics when these restricted parameters are relaxed. Our two-dynamical system can be regarded as a perturbation of the one-dimensional dynamical system $y \mapsto \omega y(1 - y)$. Following the monograph of Palis and Takens, the striking phenomena caused by unfolding a homoclinic tangency may be further investigated; also refer to Li (2003).

Our results show that endogenous fluctuations of the real money balance are easily generated in a cash-in-advance economy with habit persistence. Because the economic performance over time will be very different, depending on whether the dynamical system is stochastic or chaotic, econometric studies to detect what the time series data stands for is an important task for the future. Our model here is a step forward in seeking the most appropriate dynamical systems.

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