A Classification of Graph Capacity Functions*

Lih-Hsing Hsu
DEPARTMENT OF COMPUTER SCIENCE AND INFORMATION SCIENCE
NATIONAL CHIAO TUNG UNIVERSITY
HSINCHU, TAIWAN 30050, REPUBLIC OF CHINA
e-mail: lhhsu@cc.nctu.edu.tw

ABSTRACT

Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, the sum of G and H, $G + H$, is the disjoint union of G and H. The product of G and H, $G \times H$, is the graph with the vertex set $V(G \times H)$ that is the Cartesian product of $V(G)$ and $V(H)$, and two vertices (g_1, h_1), (g_2, h_2) are adjacent if and only if $[g_1, g_2] \in E(G)$ and $[h_1, h_2] \in E(H)$. Let $\mathcal{G}$ denote the set of all graphs. Given a graph G, the G-matching function, γ_G, assigns any graph $H \in \mathcal{G}$ to the maximum integer k such that kG is a subgraph of H. The graph capacity function for $G, P_G : \mathcal{G} \to \mathfrak{R}$, is defined as $P_G(H) = \lim_{n\to\infty}[\gamma_G(H^n)]^{1/n}$, where H^n denotes the n-fold product of $H \times H \times \cdots \times H$. Different graphs G may have different graph capacity functions, all of which are increasing. In this paper, we classify all graphs whose capacity functions are additive, multiplicative, and increasing; all graphs whose capacity functions are pseudo-additive, pseudo-multiplicative, and increasing; and all graphs whose capacity functions fall under neither of the above cases. © 1996 John Wiley & Sons, Inc.

1. INTRODUCTION

Most of the graph definitions used in this paper are standard (see, e.g., [1]). A *graph* $G = (V, E)$ consists of a finite set V and a subset E of $\{[u, v] | u \neq v, [u, v]$ is an unordered pair of elements of $V\}$. We call $V = V(G)$ the *vertex set* of G and $E = E(G)$ the *edge set* of G. Let $\mathcal{G}$ be the set of all graphs. Graph H is a *subgraph* of graph G, denoted by $H \subseteq G$, if $V(G) \subseteq V(H)$ and $E(H) \subseteq E(G)$. For $S \subseteq V$, the *induced subgraph* $G|_S$ of G is the subgraph that has S as vertex

*This work was supported in part by the National Science Council of the Republic of China under contract NSC83-0208-M009-034.

Journal of Graph Theory, Vol. 21, No. 3, 251–265 (1996)

CCC 0364-9024/96/030251-15

set and contains all edges of G having two vertices in S. Graph G_1 is *isomorphic* to graph G_2, denoted by $G_1 \cong G_2$, if there exists a one-to-one mapping ϕ, called an *isomorphism*, from $V(G_1)$ onto $V(G_2)$ such that ϕ preserves adjacency and nonadjacency, i.e., $[u,v] \in E(G_1)$ if and only if $[\phi(u),\phi(v)] \in E(G_2)$. A graph G is *vertex transitive* if for any two different vertices u and v of G there exists an isomorphism $\phi : G \to G$ such that $\phi(u) = v$. We use K_n to denote the *complete graph* with n vertices and C_n to denote the *cycle graph* with n vertices.

Let $G = (X,E)$ and $H = (Y,F)$ be two graphs. A function ϕ from X into Y is a *homomorphism* from G into H if $[u,v] \in E$ implies $[\phi(u),\phi(v)] \in F$. The *sum* of G and H is defined as the graph $G + H = (W,B)$ with $W = X_1 \cup Y_1$, $B = E_1 \cup F_1$, where $G_1 = (X_1,E_1) \cong G$, $H_1 = (Y_1,F_1) \cong H$, and $X_1 \cap Y_1 = \varnothing$. The *product* of G and H is defined as the graph $G \times H = (Z,K)$, where vertex set $Z = X \times Y$, the Cartesian product of X and Y, and edge set $K = \{[(x_1,y_1),(x_2,y_2)] | [x_1,x_2] \in E \text{ and } [y_1,y_2] \in F\}$. The adjacency matrix of $G + H$ is actually the *direct sum* of the adjacency matrix of G and the adjacency matrix of H, and the adjacency matrix of $G \times H$ is actually the *tensor product* of the adjacency matrix of G and the adjacency matrix of H. Let kG denote $G + G + \cdots + G$ (k times) and G^k denote $G \times G \times \cdots \times G$ (k times). For example, $K_{1,2}^2 = K_{2,2} + K_{1,4}$, where $K_{m,n}$ denotes the *complete bipartite graph* having two partitioned sets V_1, V_2 with $|V_1| = m$ and $|V_2| = n$.

Let f be a real-valued function defined by $\mathcal{G}$. The function f is *additive* if $f(G + H) = f(G) + f(H)$ for any $G, H \in \mathcal{G}$, and f is *pseudo-additive* if $f(G + H) = f(G) + f(H)$ for any $G, H \in \mathcal{G}$ such that $f(G) \neq 0$ and $f(H) \neq 0$. The function f is *multiplicative* if $f(G \times H) = f(G) \cdot f(H)$ for any $G, H \in \mathcal{G}$, and f is *pseudo-multiplicative* if $f(G \times H) = f(G) \cdot f(H)$ for any $G, H \in \mathcal{G}$ such that $f(G) \neq 0$ and $f(H) \neq 0$. The function f is *increasing* if $f(G) \leq f(H)$ whenever G is a subgraph of H. A graph function f is *AMI* if it is additive, multiplicative, and increasing. A graph function f is *PAMI* if it is pseudo-additive, pseudo-multiplicative, and increasing. Obviously, if a graph function f is AMI then f is PAMI. It is interesting to attempt to classify all multiplicative increasing graph functions. However, this problem is still unsolved [3,4,5,10].

Given a graph G, we define the *G-matching function*, γ_G, that maps $\mathcal{G}$ into a nonnegative integer. To be specific, let G be a fixed graph. For any graph H, $\gamma_G(H)$ is defined as the maximum integer k such that kG is isomorphic to a subgraph of H. Note that $\gamma_{K_2}(H)$ is the edge independence number of the graph H. For example, $\gamma_{K_2}(K_{1,2}) = 1$ and $\gamma_{K_2}(K_{1,2}^2) = 3$. It was proved in [3] that $\lim_{n\to\infty}[\gamma_{K_2}(K_{1,2}^n)]^{1/n} = 2^{3/2}$. Generalizing this concept, we define the *capacity function* for G, $P_G : \mathcal{G} \to \Re$, as $P_G(H) = \lim_{n\to\infty}[\gamma_G(H^n)]^{1/n}$. Different graphs G will yield different graph capacity functions.

Given two graphs G_1 and G_2, let $\{H_{i,j} | 1 \leq j \leq \gamma_H(G_i), 1 \leq i \leq 2\}$ form a set of disjoint subgraphs of G_i such that $H_{i,j} \cong H$ for every i and j. Then the set $\{H_{1,k} \times H_{2,l} | 1 \leq k \leq \gamma_H(G_1), 1 \leq l \leq \gamma_H(G_2)\}$ forms a set of disjoint subgraphs of $G_1 \times G_2$ such that each $H_{1,k} \times H_{2,l}$ contains a subgraph isomorphic to H. Thus, $\gamma_H(G_1 \times G_2) \geq \gamma_H(G_1) \cdot \gamma_H(G_2)$.

Since we have $0 \leq \gamma_G(H^n) \leq |V(H)|^n$ for any graph G, it follows that $\sup_{n\to\infty}[\gamma_G(H^n)]^{1/n}$ exists. Fekete's Theorem states that if a sequence of numbers $\{a_i\}_{i=1}^{\infty}$ is sub-additive (i.e., $a_{m+n} \leq a_m + a_n$), then $\lim_{n\to\infty} a_n/n = \inf_{n\to\infty} a_n/n$. Let $a_n = -\log \gamma_G(H^n)$. Then $\{a_n\}_{i=1}^{\infty}$ is a sub-additive sequence, because $\gamma_G(H^{m+n}) \geq \gamma_G(H^m) \cdot \gamma_G(H^n)$ and $-\log \gamma_G(H^{m+n}) \leq (-\log \gamma_G(H^m)) + (-\log \gamma_G(H^n))$. It follows from Fekete's Theorem that $\lim_{n\to\infty} \log(\gamma_G(H^n))^{1/n} = \sup_{n\to\infty} \log(\gamma_G(H^n))^{1/n}$. Hence $P_G(H)$ always exists. It is easy to verify that every graph capacity function P_G is increasing.

The study of these capacity functions is interesting for several reasons. First, observe that graph capacity functions are similar to the Shannon capacity function [11], but defined on a different product. Second, the author has studied a conjecture posed by Lovász [9] on the classification of multiplicative increasing graph functions. It was observed in [3] that some graph capacity functions can be viewed as lower bounds for multiplicative increasing graph functions. Later, it was proved in [5] that some capacity functions are not only multiplicative increasing, but also additive. However, not all graph capacity functions are necessarily so well-behaved. From a mathematical point of view, it is interesting to classify capacity functions for graphs. In this paper, we classify all graphs whose capacity functions are AMI, PAMI, and non-PAMI, respectively. For convenience, we say graph G is *AMI* (*PAMI*, *non-PAMI*, respectively) if and only if P_G is AMI (PAMI, non-PAMI, respectively).

Graph G is an (m,n)*-graph* if clique number $\omega(G) = m$ and chromatic number $\chi(G) = n$. Mycielski [10] proved that the possible $(\omega(G), \chi(G))$ are (1,1) and those (m,n) with $2 \leq m \leq n$. The following theorems from [5,6,7] will be useful in the discussion below.

Theorem 1.1. If $P_G = P_H$, then $(\omega(G), \chi(G)) = (\omega(H), \chi(H))$.

Theorem 1.2. (1) If K is a subgraph of H then $P_H \leq P_K$.

(2) $P_{H^k} = P_H$ for any positive integer k.
(3) $P_G = P_H$ if and only if $H^n \subseteq G^t \subseteq H^m$ for some n, t, $m \in N$.

Theorem 1.3. The following statements are equivalent:

(1) $P_G \geq P_H$.
(2) There exists some integer t such that $G \subseteq H^t$.
(3) For any two distinct vertices u and v of G, there exists a homomorphism $\phi : G \rightarrow H$ such that $\phi(u) \neq \phi(v)$.

2. SOME PROPERTIES OF GRAPH CAPACITY FUNCTIONS

In [5], Hsu et al. defined a class of so-called uniform graphs. The definition of uniform graphs is rather abstract. However, these graphs include all vertex transitive graphs. It was proved in [5] that every uniform graph is PAMI. A proper subset of uniform graphs called primary uniform graphs, which include complete graphs, odd cycles, and the Petersen graph, was also defined in [5], and it was proved that every primary uniform graph is AMI. The formal definition of uniform graphs is stated below.

Let G and H be two graphs with $V(G) = \{x_1, x_2, \ldots, x_u\}$ and $V(H) = \{y_1, y_2, \ldots, y_v\}$. For any positive integer m let $\vec{z} = (z_1, z_2, \ldots, z_m)$ be a vertex in H^m. Then $\vec{a} = (a_1, a_2, \ldots, a_v)$, where $a_i = |\{z_j | z_j = y_i, 1 \leq j \leq m\}|/m$, is called the *distribution* of $\vec{z}$. Let $D(H) = \{(a_1, a_2, \ldots, a_v) | a_i \geq 0, \sum_{i=1}^{v} a_i = 1\}$. We define a u-ary relation $R_G(H)$ on $D(H)$ as follows. We say $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u) \in R_G(H)$, with $\vec{a}_i \in D(H)$, if and only if $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u)$ satisfies one of the following two conditions.

(a) There exists a positive integer m such that in H^m we can find $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_u \in V(H^m)$ with the distribution of $\vec{x}_i$ to be $\vec{a}_i$ for every i and the induced subgraph of $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_u\}$ in H^m contains a subgraph isomorphic to G with $\vec{x}_i$ corresponding to x_i for every i.
(b) There exists a sequence $\{(\vec{a}_{i,1}, \vec{a}_{i,2}, \ldots, \vec{a}_{i,u})\}_{i=1}^{\infty}$ in $R_G(H)$ that satisfies the above condition such that $\lim(\vec{a}_{i,1}, \vec{a}_{i,2}, \ldots, \vec{a}_{i,u}) = (\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u)$.

A graph G is *uniform* if it satisfies the following condition: For any graph H, if $(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u)$ is in $R_G(H)$ and satisfy the condition (a) then $(\sum_{i=1}^{u} \vec{a}_i/u, \sum_{i=1}^{u} \vec{a}_i/u, \ldots, \sum_{i=1}^{u} \vec{a}_i/u,)$ (u times) is also in $R_G(H)$ and satisfy the condition (a).

In [6], it is proved that every vertex transitive graph is uniform. We are not able to prove that every uniform graph is vertex transitive. However, we know by the following theorem that the capacity function for any uniform graph is equal to the capacity function for some vertex transitive graph.

Theorem 2.1. There exists a vertex transitive graph T such that $P_G = P_T$ if G is uniform.

Proof. Assume that $|V(G)| = v$. Let $\vec{e}_1 = (1,0,0,\ldots,0,0)$, $\vec{e}_2 = (0,1,0,\ldots,0,0),\ldots,$ $\vec{e}_v = (0,0,0,\ldots,0,1)$ be elements in $D(G)$. Let $\vec{a} = (1/v, 1/v, \ldots, 1/v)$ (v times). Obviously, $\vec{a} = (\sum_{i=1}^{v} \vec{e}_i/v, \sum_{i=1}^{v} \vec{e}_i/v, \ldots, \sum_{i=1}^{v} \vec{e}_i/v)$. Since G is a subgraph of $G^1(= G)$, we have $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_v)$ is in $R_G(G)$ and satisfies the condition (a). Because G is uniform, $(\vec{a}, \vec{a}, \ldots, \vec{a})$ is in $R_G(G)$ and satisfies the condition (a). Thus, there exists a positive integer m such that in G^m we can find a copy of G, say G', such that the distribution of each vertex of G' is $\vec{a}$. Let T denote the subgraph of G^m induced by the set $\{\vec{x} \in V(G^m)|$ the distribution of $\vec{x}$ is $\vec{a}\}$. Obviously T is vertex transitive. Since $G \cong G' \subseteq T \subseteq G^m$, it follows from Theorem 1.2 that $P_G = P_T$. ∎

In [6], it was proved that P_G is PAMI if G is uniform. The above theorem shows that $\{P_G | G \text{ is uniform}\} = \{P_T | T \text{ is vertex transitive}\}$. One might ask whether for any PAMI graph G there exists a vertex transitive graph T such that $P_G = P_T$. The answer is No, and a counterexample will be given later in this section.

Let us define a real-valued function ϵ on $D(H)$ by assigning $\epsilon(\vec{a}) = \prod_{i=1}^{v} a_i^{-a_i}$, where $\vec{a} = (a_1, a_2, \ldots, a_v)$. Let $I_G(H) = \{\vec{a} \in D(H) | (\vec{a}, \vec{a}, \ldots, \vec{a})\,(u \text{ times}) \text{ is in } R_G(H)\}$. Since $I_G(H)$ is compact and ϵ is continuous, there exists $\vec{c}$ in $I_G(H)$ such that $\epsilon(\vec{c}) = \max\{\epsilon(\vec{a}) | \vec{a} \in I_G(H)\}$. Hsu et al. [5] proved the following theorem.

Theorem 2.2. If G is uniform, then

$$P_G(H) = \max_{(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_u) \in R_G(H)} \min\{\epsilon(\vec{a}_1), \epsilon(\vec{a}_2), \ldots, \epsilon(\vec{a}_u)\} = \max_{\vec{a}_1 \in I_G(H)} \epsilon(\vec{a}).$$

Definition 2.1. Let G be a graph with $V(G) = \{x_1, x_2, \ldots, x_v\}$ and let $r_1, r_2, \ldots, r_v$ be positive integers. Let $\vec{r} = (r_1, r_2, \ldots, r_v)$. We use $G^{\vec{r}}$ to denote the graph with $V(G^{\vec{r}}) = \{x_{i,j} | 1 \leq i \leq v, 1 \leq j \leq r_i\}$ and $(x_{i,j}, x_{k,l}) \in E(G^{\vec{r}})$ if and only if $(x_i, x_k) \in E(G)$. Assume $c_1, c_2, \ldots, c_n$ are positive integers. We use $G^{\vec{c}_1}$ to denote the graph $G^{\vec{r}}$ with $r_i = c_1$ for every i. Moreover, if $V(G) = C_1 \cup C_2$, we use $G^{\vec{c}_1\vec{c}_2}$ to denote the graph $G^{\vec{r}}$ with c_1 corresponding to every vertex u in C_1 and c_2 corresponding to every vertex v in C_2. Similarly, we can define $G^{\vec{c}_1\vec{c}_2\ldots\vec{c}_n}$ analogously.

Lemma 2.1. Assume that G is a vertex transitive graph with v vertices. Let $r_1, r_2, \ldots, r_v$ be positive integers and let $\vec{r} = (r_1, r_2, \ldots, r_v)$. Then $P_G(G^{\vec{r}}) = v \cdot \prod_{i=1}^{v} r_i^{1/v}$.

Proof. Let $\vec{c} = (c_{1,1}, c_{1,2}, \ldots, c_{1,r_1}, c_{v,2}, \ldots, c_{v,r_v})$ be the vector in $I_G(G^{\vec{r}})$ with each $c_{i,j}$ corresponds to the vertex $x_{i,j}$ such that $P_G(G^{\vec{r}}) = \epsilon(\vec{c})$. Since the graph G is vertex transitive and the function ϵ is concave, we have $\sum_{j=1}^{r_i} c_{i,j} = 1/v$ for every i. By the symmetry among $x_{i,1}, x_{i,2}, \ldots, x_{i,r_i}$, we have $c_{i,j} = (vr_i)^{-1}$ for every i and j. Thus, $P_G(G^{\vec{r}}) = v \cdot \prod_{i=1}^{v} r_i^{1/v}$. ∎

A graph $\hat{G}$ is a *homomorphic image* of another graph G if there exists a homomorphism $\Phi : G \to \hat{G}$ which is onto and if for every $(\hat{u}_1, \hat{u}_2) \in E(\hat{G})$ there exists $(u_1, u_2) \in E(G)$ such that $\Phi(u_i) = \hat{u}_i, i = 1, 2$. A graph G is *primary* if for any homomorphic image $\hat{G}$ of G there exists a positive integer k such that G is a subgraph of $\hat{G}^k$. A graph G is *primary uniform* if it is primary and uniform.

It was proved in [5] that complete graphs, odd cycles, and the Petersen graph are primary uniform. However, a uniform graph is not necessarily primary. It is easy to check that C_4 has K_2 as a homomorphic image but C_4 is not a subgraph of K_2^n for every $n \in N$. Thus, C_4 is uniform but not primary.

Lemma 2.2. $P_G \geq P_{\hat{G}^{\vec{2}}}$ for any homomorphic image $\hat{G}$ of G.

Proof. Obviously, for any two distinct vertices u and v in G there exists a homomorphism $\phi : G \rightarrow \hat{G}^{\vec{2}}$ such that $\phi(u) \neq \phi(v)$. By Theorem 1.3, $P_G \geq P_{\hat{G}^{\vec{2}}}$. ∎

Theorem 2.3. Let $\hat{G}$ be a homomorphic image of G such that $P_{\hat{G}} \geq P_G$. If H is any graph such that $P_G(H) \neq 0$, then $P_G(H) = P_{\hat{G}}(H)$. Furthermore, G is PAMI if $\hat{G}$ is PAMI.

Proof. Let $V(G) = \{x_1, x_2, \ldots, x_u\}$, $V(\hat{G}) = \{y_1, y_2, \ldots, y_v\}$, and ϕ be a homomorphism from G onto $\hat{G}$. Since $P_G(H) \neq 0$, there exists a positive integer t such that a subgraph G' of H^t is isomorphic to G. Let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_u$ be vertices of G' with $\vec{x}_j$ corresponding to x_j. There are $\gamma_{\hat{G}}(H^m)$ disjoint $\hat{G}$'s in H^m for every m. Let $\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_{\gamma_{\hat{G}}(H^m)}$ be such disjoint $\hat{G}$'s in H^m and let $V(\hat{G}_i) = \{\vec{y}_{i,y_1}, \vec{y}_{i,y_2}, \ldots, \vec{y}_{i,y_v}\}$ with $\vec{y}_{i,y_j}$ corresponding to y_j. For every $1 \leq i \leq \gamma_{\hat{G}}(H^m)$, $(\vec{x}_1, \vec{y}_{i,\phi(x_1)})$, $(\vec{x}_2, \vec{y}_{i,\phi(x_2)}), \ldots, (\vec{x}_u, \vec{y}_{i,\phi(x_u)})$ induce a subgraph G_i in H^{m+t} isomorphic to G. Then $G_1, G_2, \ldots, G_{\gamma_{\hat{G}}(H^m)}$ are mutually disjoint, because $\hat{G}_1, \hat{G}_2, \ldots, \hat{G}_{\gamma_{\hat{G}}(H^m)}$ are mutually disjoint. We have $\gamma_G(H^{m+t}) \geq \gamma_{\hat{G}}(H^m)$. Thus

$$P_{\hat{G}}(H) = \lim_{m\to\infty}[\gamma_{\hat{G}}(H^m)]^{1/m} \leq \lim_{m\to\infty}[\gamma_G(H^{m+t})]^{1/m} = \lim_{m\to\infty}[\gamma_G(H^{m+t})]^{1/(m+t)} = P_G(H).$$

Since $P_{\hat{G}}(H) \geq P_G(H)$, we have $P_G(H) = P_{\hat{G}}(H)$. ∎

Corollary 2.1. Let G be an (n,n)-graph and H be a graph such that $P_G(H) \neq 0$. We have $P_G(H) = P_{K_n}(H)$. Therefore, G is PAMI for any (n,n)-graph.

Proof. Let $\hat{G} = K_n$. Since $\chi(G) = n$, $\hat{G}$ is a homomorphic image of G. Since $\omega(G) = n$, $\hat{G}$ is a subgraph of G. By Theorem 1.2, $P_{\hat{G}} \geq P_G$. By Theorem 2.3, the corollary holds. ∎

Example 1. C_4 is PAMI but not AMI.

Proof. Since C_4 is a vertex transitive graph, C_4 is PAMI and $P_{C_4}(C_4) = 4$. By Corollary 2.1, $P_{K_2}(C_4) = P_{C_4}(C_4) = 4$ follows. Since C_4 is not a subgraph of K_2^n for every $n \in N$, $P_{C_4}(K_2) = 0$. By the increasing property of P_{C_4}, $P_{C_4}(K_2 + C_4) > 0$ and $P_{C_4}(K_2 \times C_4) = P_{C_4}(2C_4) > 0$. By Corollary 2.1 and the AMI property of P_{K_2}, we have $P_{C_4}(K_2 + C_4) = P_{K_2}(K_2 + C_4) = 6$ and $P_{C_4}(K_2 \times C_4) = P_{K_2}(2C_4) = 8$. Thus, C_4 is not AMI. ∎

Example 2. By Corollary 2.1, the graph H in Figure 1 is PAMI. But there is no vertex transitive graph T such that $P_H = P_T$.

Proof. Suppose that there is a vertex transitive graph T such that $P_H = P_T$. It follows from Theorem 1.1 that $\omega(T) = 3$. Let $|V(T)| = n$. By Theorem 1.3, $T \subseteq H^m$ for some integer m. Thus, there exist $\vec{t}_1 = (g_{1,1}, g_{1,2}, \ldots, g_{1,m})$, $\vec{t}_2 = (g_{2,1}, g_{2,2}, \ldots, g_{2,m}), \ldots, \vec{t}_n = (g_{n,1}, g_{n,2}, \ldots, g_{n,m}) \in V(H^m)$ that induce a T. Since T is vertex transitive, the size of the maximum clique containing $\vec{t}_i$ in T is 3 for every i. Thus, the size of the maximum clique containing $g_{i,j}$ is at least 3 for every i and j. Candidates for all possible $g_{i,j}$ are those vertices in H such that the size of the maximum clique containing them is at least 3. Such vertices in H are $\{x_1, x_2, x_3\}$. These vertices generate a K_3. Therefore, we have $K_3 \subseteq T \subseteq K_3^m$. This

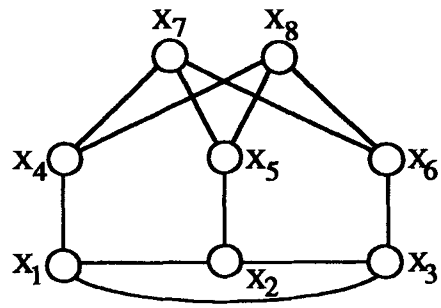

FIGURE 1. Graph H is PAMI, but there is no vertex transitive graph T such that $P_H = P_T$.

implies $P_{K_3} = P_T = P_H$. It follows from Theorem 1.3 (3) that there exists a homomorphism ϕ from H into K_3 such that $\phi(x_7) \neq \phi(x_8)$.

On the other hand, let $V(K_3) = \{y_1, y_2, y_3\}$ and let ϕ be any homomorphism from H into K_3. Since $\{x_1, x_2, x_3\}$ generates a K_3 in G, without loss of generality we assume that $\phi(x_i) = y_i$ for $i = 1, 2, 3$. Since x_4 is adjacent to x_1, $\phi(x_4) \neq y_1$. Therefore, $\phi(x_4) \in \{y_2, y_3\}$, $\phi(x_5) \in \{y_1, y_3\}$, and $\phi(x_6) \in \{y_1, y_2\}$. For this reason, there is no homomorphism ϕ from H into K_3 for which $\phi(x_7) \neq \phi(x_8)$, and we have a contradiction.

Thus, there is no vertex transitive graph T such that $P_T = P_H$. ∎

From the above discussion, we know that some graphs are PAMI: vertex transitive graphs and (n, n)-graphs. However, not all graphs are PAMI. In the following section, we shall give an example of a non-PAMI graph.

3. AN EXAMPLE OF A NON-PAMI GRAPH

Given two graphs G and H, it is obvious that $\omega(G \times H) = \min\{\omega(G), \omega(H)\}$ and $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$. Note that Hedetniemi [2] conjectured that $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$.

Theorem 3.1. Let G be an (m, n)-graph and H be a (p, q)-graph. If $G \times H$ is an (r, s)-graph then $r = \min\{m, p\}$ and $s \leq \min\{n, q\}$. In particular, G^2 is an (m, n)-graph.

Proof. From above, $\chi(G^2) \leq \chi(G)$. Since $G \subseteq G^2$, $\chi(G) \leq \chi(G^2)$. Thus $\chi(G^2) \leq \chi(G)$. ∎

Lemma 3.1. If G is an (m, n)-graph and H is a (p, q)-graph with $m < p \leq q < n$, then $\gamma_G(G^i \times H^k) = \gamma_H(G^i \times H^k) = 0$ for any positive integers i and k.

Proof. Since $\chi(H^k) = \chi(H)$ for any positive integer k, $\chi(G^i \times H^k) \leq q$. If $\gamma_G(G^i \times H^k) \neq 0$, then $G \subseteq G^i \times H^k$. Thus, $\chi(G^i \times H^k) \geq \chi(G^i) = \chi(G) = n$. This contradicts Theorem 3.1. Thus, $\gamma_G(G^i \times H^k) = 0$.

Similarly, since $\omega(G^i) = \omega(G)$ for any positive integer i, $\omega(G^i \times H^k) = m$. If $\gamma_H(G^i \times H^k) \neq 0$, then $H \subseteq G^i \times H^k$. Thus, $\omega(G^i \times H^k) \geq \omega(H^k) = \omega(H) = p$. Again, this contradicts Theorem 3.1. Thus, $\gamma_H(G^i \times H^k) = 0$. ∎

Theorem 3.2. If G is a connected PAMI (m, n)-graph and H is a connected PAMI (p, q)-graph such that $m < p \leq q < n$, we have $P_{G+H}(aG + bH) = \min\{aP_G(G), bP_H(H)\}$.

Proof. Since G and H are PAMI, we have $P_G(aG) = aP_G(G)$ and $P_H(bH) = bP_H(H)$. By Lemma 3.1, $\gamma_G(G^i \times H^{k-i}) = 0$ for every $0 \leq i \leq k - 1$ and $\gamma_H(G^i \times H^{k-i}) = 0$ for every $1 \leq i \leq k$. Since both G and H are connected, we may apply the Binomial Theorem to obtain $\gamma_G((aG + bH)^k) = \gamma_G(a^k G^k)$, $\gamma_H((aG + bH)^k) = \gamma_H(b^k H^k)$, and $\gamma_{G+H}((aG + bH)^k) = \min\{\gamma_G(a^k G^k), \gamma_H(b^k H^k)\}$. Hence

$$\begin{aligned} P_{G+H}(aG + bH) &= \lim_{k\to\infty}[\gamma_{G+H}((aG + bH)^k)]^{1/k} \\ &= \min\{\lim_{k\to\infty}[\gamma_G(a^k G^k)]^{1/k}, \lim_{k\to\infty}[\gamma_H(b^k H^k)]^{1/k}\} \\ &= \min\{P_G(aG), P_H(bH)\} = \min\{aP_G(G), bP_H(H)\}. \quad \blacksquare \end{aligned}$$

Given two positive integers n and k, we construct a graph $G_{n,k}$ as follows. The vertices of $G_{n,k}$ are the n-subsets of $\{1, 2, \ldots, 2n + k\}$ and two of the vertices are joined by an edge if and only if they are disjoint. These graphs are called *Kneser's graphs*. It is obvious that $G_{n,k}$ is vertex transitive. In [8], Lovász proved that $\omega(G_{n,k}) = \lfloor(2n + k)/n\rfloor$ and $\chi(G_{n,k}) = k + 2$.

Example 3. Let G be the Kneser graph $G_{3,2}$ and H be the cube of K_3, K_3^3. Then $G + H$ is non-PAMI. In particular, we have

$$P_{G+H}((3G + H) + (G + 3H)) \neq P_{G+H}(3G + H) + P_{G+H}(G + 3H);$$

and

$$P_{G+H}((3G + H) \times (G + 3H)) \neq P_{G+H}(3G + H) \cdot P_{G+H}(G + 3H).$$

Proof. Obviously, G is a connected vertex transitive (2,4)-graph. By Lemma 2.1, $P_G(G) = 56$. Since K_3 is vertex transitive, H is a connected vertex transitive (3,3)-graph with $P_H(H) = 27$. It follows from Theorem 3.2 that

$$\begin{aligned} P_{G+H}(3G + H) &= \min\{3P_G(G), P_H(H)\} = 27; \\ P_{G+H}(G + 3H) &= \min\{P_G(G), 3P_H(H)\} = 56; \end{aligned}$$

and

$$P_{G+H}(4G + 4H) = \min\{4P_G(G), 4P_H(H)\} = 108.$$

Thus $P_{G+H}((3G + H) + (G + 3H)) \neq P_{G+H}(3G + H) + P_{G+H}(G + 3H)$.

Similar to the proof of Theorem 3.2, $P_{G+H}((3G^2 + 10G \times H + 3H^2)) = \min\{3P_G^2(G), 3P_H^2(H)\}$. Thus

$$\begin{aligned} P_{G+H}((3G + H) \times (G + 3H)) &= P_{G+H}(3G^2 + 10G \times H + 3H^2) \\ &= P_{G+H}(3G^2 + 3H^2) = \min\{3P^2(G), 3P^2(H)\} = 3 \cdot 27 \\ &< 27 \cdot 56 = P_{G+H}(3G + H) \cdot P_{G+H}(G + 3H). \end{aligned}$$

Hence $P_{G+H}((3G + H) \times (G + 3H)) \neq P_{G+H}(3G + H) \cdot P_{G+H}(G + 3H)$. $\blacksquare$

Thus, $G_{3,2} + K_3^3$ is a non-PAMI graph with 83 vertices. It is interesting to find the smallest non-PAMI graph. In the following section, we will prove that the 5-wheel graph, W_5, is the smallest non-PAMI graph.

4. THE SMALLEST NON-PAMI GRAPH

Let W_5 be the 5-wheel graph shown in Figure 2. The vertex o is called the *center vertex* of W_5. Let $k, r_1, r_2, \ldots, r_5$ be positive integers and $W_5(k, r_1, r_2, \ldots, r_5)$ be the graph obtained from W_5 by copying $o, x_1, x_2, \ldots, x_5$, the vertices of W_5, $k, r_1, r_2, \ldots, r_5$ times, respectively. More precisely, $V(W_5(k, r_1, r_2, \ldots, r_5)) = \{o_1, o_2, \ldots, o_k\} \cup \{x_{i,j} | 1 \le i \le 5$ and $1 \le j \le r_i\}$, and $E(W_5(k, r_1, r_2, \ldots, r_5)) = \{[o_i, x_{k,l}]|$ for every $i, k, l\} \cup \{[x_{p,q}, x_{m,n}] || p - m| = 1 (\text{mod } 5)\}$. In other words, $W_5(k, r_1, r_2, \ldots, r_5)) = (W_5)^{\vec{r}}$, where $\vec{r} = \{k, r_1, r_2, \ldots, r_5\}$. The set $\{o_1, o_2, \ldots, o_k\}$ is called the *center* of $W_5(k, r_1, r_2, \ldots, r_5)$. Let $C_5(r_1, r_2, \ldots, r_5)$ denote the subgraph of $W_5(k, r_1, r_2, \ldots, r_5)$ induced by all $x_{i,j}$'s. We call $C_5(r_1, r_2, \ldots, r_5)$ the *outside* of $W_5(k, r_1, r_2, \ldots, r_5)$.

Theorem 4.1. $P_{W_5}(W_5(k, r_1, r_2, \ldots, r_5)) = \min\{k, 5 \cdot (\Pi_{i=1}^5 r_i)^{1/5}\}$.

Proof. Let $G = W_5(k, r_1, r_2, \ldots, r_5)$. We claim that the center of each copy of W_5 in G^n is in the center of G^n, i.e., $A = \{(y_1, y_2, \ldots, y_n) | y_j \in \{o_1, o_2, \ldots, o_k\}$ for every $j\}$. If not, there exists a copy G' in G^n with its center not in A. Then G' induces an isomorphism f from W_5 to G'. We have $f(o) = (z_1, z_2, \ldots, z_n)$ with $z_i \notin \{o_1, o_2, \ldots, o_k\}$ for some i. Let i be the index such that $z_i \notin \{o_1, o_2, \ldots, o_k\}$ and let f_i be the ith projection of f. Then f_i is a homomorphism and its image is a subgraph of $K_3^I = K_4 - e$. This contradicts the fact that $\chi(W_5) = 4$ and $\chi(K_3^I) = 3$.

Again, every vertex in A is adjacent only to those vertices in $(C_5(r_1, r_2, \ldots, r_5))^n$. Hence $\gamma_{W_5}(G^n) = \min\{k^n, \gamma_{C_5}((C_5(r_1, r_2, \ldots, r_5))^n)\}$. Thus $P_{W_5}(G) = \min\{k, P_{C_5}(C_5(r_1, r_2, \ldots, r_5)\}$. By Lemma 2.1, $P_{C_5}(C_5(r_1, r_2, \ldots, r_5)) = 5 \cdot (\Pi_{i=1}^5 r_i)^{1/5}$. ∎

Theorem 4.2. Let $A = W_5(k, r_1, r_2, \ldots, r_5)$ and $C = W_5(m, s_1, s_2, \ldots, s_5)$. Then $P_{W_5}(A \times C) = \min\{km, 5 \cdot (\Pi_{i=1}^5 r_i)^{1/5} \cdot 5 \cdot (\Pi_{i=1}^5 s_i)^{1/5}\}$.

Proof. Let $B = C_5(r_1, r_2, \ldots, r_5)$ and $D = C_5(s_1, s_2, \ldots, s_5)$. Using arguments similar to the proof of Theorem 4.1, we have $\gamma_{W_5}((A \times C)^n) = \min\{(km)^n, \gamma_{C_5}((B \times D)^n)\}$. Hence $P_{W_5}(A \times C) = \min\{km, P_{C_5}(B \times D)\}$. Since P_{C_5} is multiplicative, we have $P_{W_5}(A \times C) = \min\{km, P_{C_5}(B) \cdot P_{C_5}(D)\} = \min\{km, 5 \cdot (\Pi_{i=1}^5 r_i)^{1/5} \cdot 5 \cdot (\Pi_{i=1}^5 s_i)^{1/5}\}$. ∎

Using Theorem 4.1 and Theorem 4.2, we have

$$P_{W_5}(W_5(6, 1, 1, 1, 1, 1)) = \min\{6, P_{C_5}(C_5(1, 1, 1, 1, 1))\} = 5,$$
$$P_{W_5}(W_5(1, 2, 1, 1, 1, 1)) = \min\{1, P_{C_5}(C_5(2, 1, 1, 1, 1))\} = 1,$$

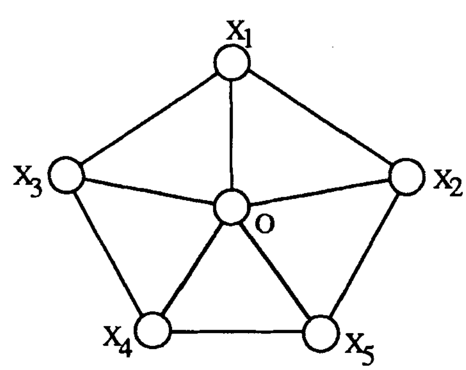

FIGURE 2. The smallest non-PAMI graph, W_5.

and

$$P_{W_5}(W_5(6,1,1,1,1,1) \times (W_5(1,2,1,1,1,1)) = \min\{6, 5 \cdot 5 \cdot 2^{1/5}\} = 6.$$

Thus

$$P_{W_5}(W_5(6,1,1,1,1,1) \times W_5(1,2,1,1,1,1)) \neq P_{W_5}(W_5(6,1,1,1,1,1)) \cdot P_{W_5}(W_5(1,2,1,1,1,1)).$$

Therefore, P_{W_5} is not pseudo-multiplicative.

Theorem 4.3. If P_G is (pseudo-)additive, P_G is (pseudo-)multiplicative.

Proof. Since $P_G(H^2) = \lim_{n\to\infty}[\gamma_G(H^{2n})]^{1/n} = \lim_{n\to\infty}\{[\gamma_G(H^{2n})]^{1/2n}\}^2 = P_G^2(H)$, we have $P_G((H+K)^2) = P_G^2(H+K)$. Then

$$\begin{aligned} P_G((H+K)^2) &= P_G^2(H^2 + 2H \times K + K^2) \\ &= P_G(H^2) + 2P_G(H \times K) + P_G(K^2) \\ &= P_G^2(H) + 2P_G(H \times K) + P_G^2(K). \end{aligned}$$

But $P_G((H+K)^2) = P_G^2(H+K) = (P(H) + P(K))^2 = P_G^2(H) + 2P_G(H) \times P_G(K) + P_G^2(K)$. We have $P_G(H \times K) = P_G(H)P_G(K)$. ∎

Therefore, P_{W_5} is not pseudo-additive.

Theorem 4.4. W_n is a non-PAMI graph for every odd integer $n \geq 5$. In particular, W_5 is the smallest non-PAMI graph.

Proof. By a discussion similar to that above, it is easy to see that every odd wheel graph W_n with $n \geq 5$ is a non-PAMI graph. Since any graph with at most 5 vertices is either C_5 or a graph with its clique number equal to its chromatic number, such a graph is PAMI. Thus W_5 is the smallest non-PAMI graph. ∎

5. CLASSIFICATION OF PAMI GRAPHS

From the above discussion, we know that some graphs are AMI, some PAMI but not AMI, and some non-PAMI. In this and the following sections, we shall classify these graphs.

Let $G = (V, E)$ be any graph. The *homomorphism digraph* $G^* = (V^*, E^*)$ of G is the directed graph with $V^* = V$ and $(a, b) \in E^*$ if there is a homomorphism ϕ from G into itself such that $\phi(a) = b$. Obviously, $(v, v) \in E^*$ for every $v \in V$. Let S be a subset of V. The *out-neighborhood* of S is the set $\Gamma(S) = \{y | (x, y) \in E^* \text{ with } x \in S\}$. Thus, $S \subseteq \Gamma(S)$ for every $\emptyset \neq S \subseteq V$. A nonempty subset S of V is called a *closed set* of G if (1) $\Gamma(S) \subseteq S$ and (2) there is no proper subset S' of S such that $\Gamma(S') \subseteq S'$. It is easy to see that there exists a closed set for every graph.

Lemma 5.1. Suppose that S is a closed set of a graph G and D is a subset of S. The induced directed subgraph $G^*|_D$ in G^* is a complete digraph.

Proof. First, we prove that $G^*|_D$ is strongly connected. Suppose not. Then there exists a proper subset D' of D such that $\Gamma(D') \cap D \subseteq D'$. Let $X = \{x | x \in S - D$ and there exists a homomorphism $f : G \to G$ such that $f(x) \in D - D'\}$.

Suppose that there exists a homomorphism $g : G \to G$ for which $g(y) \in X$ for some $y \in D'$. Since $g(y)$ is in X, there exists a homomorphism $h : G \to G$ such that $h(g(y)) \in D - D'$. Then $h \circ g$ is a homomorphism mapping the element y in D' to an element in $D - D'$. This contradicts $\Gamma(D') \cap D \subseteq D'$. Thus, there is no homomorphism g from G into itself such that $g(y) \in X$ for some $y \in D'$.

It follows from the above discussion that the set $Y = (S - D) - X \cup D'$ is a proper subset of S such that $\Gamma(Y) \subseteq Y$. This contradicts the fact that S is a closed set. Thus, $G^*|_D$ is strongly connected.

Since the composite of homomorphic functions is again a homomorphism, $G^*|_D$ forms a complete digraph. ∎

Corollary 5.1. For any two different closed sets S_1 and S_2 of G, $S_1 \cap S_2 = \varnothing$.

Proof. The proof follows from the fact that $G^*|_S$ is a complete digraph for every closed set of S. ∎

Lemma 5.2. Let S be a closed set of a graph G and f be any homomorphism from G into itself. There is exactly one closed set B of $f(G)$ contained in $S \cap f(G)$. Moreover, $f(S)$ is a subset of B.

Proof. We prove this lemma through the following steps.

(1) Let s be any element in $S \cap f(G)$ and g be any homomorphism from $f(G)$ into itself. Since S is a closed set, $g(s) \in S \cap f(G)$. Thus, the out-neighborhood of $S \cap f(G)$ in $f(G)^*$ is a subset of $S \cap f(G)$. Thus, there exists at least one closed subset B of $f(G)$ in $S \cap f(G)$.

(2) Let B be any closed set of $f(G)$ in $S \cap f(G)$ and x be any element of B. Obviously, $f|_{f(G)}$ is a homomorphism from $f(G)$ into itself. Since B is a closed set, $f(x) \in B \subseteq S$. Thus, the set $f(S) \cap B$ contains at least the element $y(= f(x))$.

(3) Let $z = f(w)$ with $w \in S$ be any element of $f(S)$. By Lemma 5.1, there exists a homomorphism $h : G \to G$ such that $h(y) = w$. Then $f \circ h|_{f(G)}$ is a homomorphism from $f(G)$ into itself such that $f \circ h|_{f(G)}(y) = f(w) = z$. Since B is a closed set, z is an element of B. Thus, $f(S) \subseteq B$.

(4) It follows from Corollary 5.1 that there is exactly one closed set B of $f(G)$ contained in $S \cap f(G)$. ∎

Later, we will prove that a graph G is PAMI if and only if G has exactly one closed set. To prove this statement, we need the following discussion. Let $G = (V(G), E(G))$ be a graph. A nonempty subset C of a closed set S is called a *core* if (1) there exists a homomorphism $\phi : G \to G$ satisfying $\phi(S) = C$ and (2) there is no proper subset C' of C such that there exists a homomorphism $\phi' : G \to G$ satisfying $\phi'(S) = C'$. Again it is easy to see that there exists a core for every closed set. A graph G is called a *core graph* if $V(G)$ is a core for G.

Lemma 5.3. Let C be a core of G for some closed set S. The subgraph $G|_C$ in G induced by C is vertex transitive.

Proof. We prove this lemma through the following steps.

(1) Let ϕ by any homomorphism of G such that $\phi(S) = C$. We claim that the restriction of ϕ on C, $\phi|_C$, is an isomorphism for C. First, we prove that $\phi(C) = C$. Suppose not. $\phi(C)$ is a proper subset of C. Since $\phi(S) = C$, $\phi^2(S) = \phi(C)$. In other words, $\phi(C)$ is a proper subset of C having a homomorphism ϕ^2 such that $\phi^2(S) = \phi(C)$. This contradicts the fact

that C is a core of S. Hence $\phi(C) = C$. Since C is a finite set, ϕ is also one to one from C onto C. Thus, $\phi|_C$ is an isomorphism on C.

(2) From step 1, we know that ϕ_C^{-1} is an isomorphism from C onto itself. Let f be any homomorphism from G into itself. Then $f \circ \phi|_C^{-1}(C) \subseteq S$ because S is a closed set. Therefore $\phi \circ f \circ \phi|_C^{-1}(C) \subseteq C$. We claim that $\phi \circ f \circ \phi|_C^{-1}$ is again an isomorphism on C. Suppose not. Then $\phi \circ f \circ \phi|_C^{-1}(C)$ is a proper subset of C. Since $\phi|_C^{-1}(C) = C$, $\phi \circ f(C)$ is a proper subset of C. Note that $\phi \circ f \circ \phi(S) = \phi \circ f(C)$. Thus, $\phi \circ f(C)$ is a proper subset of C such that there exists a homomorphism ϕ', namely $\phi' = \phi \circ f \circ \phi$, satisfying $\phi'(S) = \phi \circ f(C)$. This contradicts the fact that C is a core. Thus, $\phi \circ f \circ \phi|_C^{-1}$ is an isomorphism on C for every homomorphism $f : G \to G$.

(3) Let a and b be any two vertices of C. Since $\phi|_C^{-1}$ is an isomorphism on C, we can find a' and b' in C such that $\phi(a') = a$ and $\phi(b') = b$. By Lemma 5.1, we know that there exists a homomorphism $f : G \to G$ such that $f(a') = b'$. Then $\phi \circ f \circ \phi|_C^{-1}$ is an isomorphism on C such that $\phi \circ f \circ \phi|_C^{-1}(a) = b$. ∎

Thus $G|_C$ is vertex transitive.

Lemma 5.4. If G is a graph with only one closed set, then G is PAMI.

Proof. Let C be a core of G for the closed set S of G. Since there is only one closed set in G, $G|_C$ is a homomorphic image of G. Since $G|_C$ is a subgraph of G, it follows from Theorem 1.2 that $P_{G|_C} \geq P_G$. By Lemma 5.3, $G|_C$ is vertex transitive. Hence $G|_C$ is PAMI. By Theorem 2.3, G is PAMI. ∎

A graph G is called an *n-core graph* if G has exactly n closed sets $C_1, C_2, \ldots, C_n$ with $V(G) = C_1 \cup C_2 \cup \cdots C_n$ such that C_i is a core for every i. For example, the graph $G_{3,2} + K_3$ and the 5-wheel graph W_5 are 2-core graphs. Observe that there is no edge connection between the two cores of $G_{3,2} + K_3$. On the other hand, all the edge connections between the two cores of W_5 form a complete bipartite graph. These properties play vital roles in the proof that $G_{3,2} + K_3$ and W_5 are non-PAMI. However, not all 2-core graphs have these properties.

Lemma 5.5. Let G be a graph with n closed sets. G contains an n-core subgraph $\hat{G}$ as a homomorphic image of G.

Proof. We construct a sequence of graphs $G_0, G_1, \ldots, G_k$ as follows:

Let $G_0 = G$. If there is no homomorphism $f : G_0 \to G_0$ such that $f(G_0) \subset G_0$, the sequence terminates. If there exists a homomorphism $f_0 : G_0 \to G_0$ such that $f(G_0) \subset G_0$, set $G_1 = f(G_1)$. Continue in this way. Let G_i be the newly constructed graph. If there is no homomorphism $f : G_i \to G_i$ such that $f(G_i) \subset G_i$, then the sequence terminates. If there is a homomorphism $f_i : G_i \to G_i$ such that $f_i(G_i) \subset G_i$, then set $G_{i+1} = f_i(G_i)$. Since G is a finite graph, the sequence terminates at some G_k. Let $f = f_{k-1} \circ f_{k-2} \circ \cdots \circ f_0$. Then, f is a homomorphism from G onto the subgraph of G, G_k. It follows from Lemma 5.2 that G_k is a graph with n closed sets. Since there is no homomorphism from G_k into a proper subgraph of itself, G_k is an n-core graph. ∎

We being with the simplest case, 2-core graphs, to prove that those graphs with two or more closed sets are non-PAMI. Let G be a 2-core graph with C_1 and C_2 as its two cores. From Lemma 5.3, the induced subgraph $G|_{C_i}$ is vertex transitive for $i = 1, 2$. By Lemma 5.1, for every $u, v \in C_i$ there exists an isomorphism ϕ in G such that $\phi(u) = v$. From the 2-core graph G, we are going to construct another graph $\tilde{G}$, whose properties we will then discuss.

Let n be a positive integer, $|C_1| = c_1$, and $|C_2| = c_2$. Set $r = c_2 n$ and $s = c_1 n$. Then $\tilde{G} = G^{\bar{r}\bar{s}}$ has two closed sets $\tilde{C}_1 = C_1^{\bar{r}}$ and $\tilde{C}_2 = C_2^{\bar{s}}$ with $|\tilde{C}_1| = |\tilde{C}_2| = c_1 c_2 n = rs/n$ and $V(\tilde{G}) = \tilde{C}_1 \cup \tilde{C}_2$. Let A_i denote the induced subgraph $\tilde{G}|_{\tilde{C}_i}$. Then A_i is a vertex transitive graph for $i = 1, 2$. Moreover, for every $u, v \in \tilde{C}_1$ with $i = 1, 2$ there exists an isomorphism ϕ in $\tilde{G}$ such that $\phi(u) = v$.

Lemma 5.6. For any positive integer x, $P_{\tilde{G}}(\tilde{G}^{\bar{x}\bar{x}}) = xrs/n$.

Proof. Since C_i is a closed set for $i = 1, 2$, every isomorphism from $\tilde{G}$ into $(\tilde{G}^{\bar{x}\bar{x}})^m$ maps $\tilde{C}_i$ into $(\tilde{A}_i^{\bar{x}})^m$. Hence $\gamma_{\tilde{G}}((\tilde{G}^{\bar{x}\bar{x}})^m) \leq \min\{\gamma_{A_i}((A_i^{\bar{x}})^m)|i = 1,2\}$. Thus $P_{\tilde{G}}(\tilde{G}^{\bar{x}\bar{x}}) \leq \min\{P_{A_i}(A_i^{\bar{x}})|i = 1,2\}$. Since A_i is vertex transitive for every i, by Lemma 2.1 we obtain the following equation:

$$P_{\tilde{G}}(\tilde{G}^{\bar{x}\bar{x}}) \leq xrs/n\,. \tag{1}$$

Let $\tilde{C}_1 = \{u_1, u_2, \ldots, u_{rs/n}\}$ and $\tilde{C}_2 = \{v_1, v_2, \ldots, v_{rs/n}\}$. Then $\tilde{C}_1^{\bar{x}} = \{u_{i,j}|1 \leq i \leq rs/n,\ 1 \leq j \leq x\}$ and $\tilde{C}_2^{\bar{x}} = \{v_{i,j}|1 \leq i \leq rs/n,\ 1 \leq j \leq x\}$. We can set a one-to-one correspondence η from $\tilde{C}_1^{\bar{x}}$ to $\tilde{C}_2^{\bar{x}}$ by assigning $\eta(u_{i,j}) = v_{i,j}$ for every i, j. We can then extend η to η', which maps from $(A_1^{\bar{x}})^m$ into $(A_2^{\bar{x}})^m$ by $\eta'(x_1, x_2, \ldots, x_m) = (\eta(x_1), \eta(x_2), \ldots, \eta(x_m))$. Obviously, η' is one-to-one and onto. Let $M_1 = \{\hat{A}_i|1 \leq i \leq \gamma_{A_1}((A_1^{\bar{x}})^m)\}$ be a set of maximum mutually disjoint copies of A_1's in $(A_1^{\bar{x}})^m$. Then the set $M_2 = \{\eta'(\hat{A}_1)|\hat{A}_i \in M_1\}$ forms a set of mutually disjoint A_2's in $(A_2^{\bar{x}})^m$. Moreover, the induced subgraph $(\tilde{G}^{\bar{x}\bar{x}})^m|_{\hat{A}_i \cup \eta'(\hat{A}_i)}$ induces a subgraph isomorphic to $\tilde{G}$. Hence $\gamma_{\tilde{G}}((\tilde{G}^{\bar{x}\bar{x}})^m) \geq \gamma_{A_1}((A_1^{\bar{x}})^m)$. We have the following equation:

$$P_{\tilde{G}}(\tilde{G}^{\bar{x}\bar{x}}) \geq P_{A_1}(A_1^{\bar{x}}) = xrs/n\,. \tag{2}$$

Combining (1) and (2) proves the lemma. ∎

Corollary 5.2. For any positive integers x and y, $P_{\tilde{G}}(\tilde{G}^{\bar{x}\bar{y}}) = \min\{xrs/n, yrs/n\}$.

Proof. Without loss of generality, we assume that $x \leq y$. Similar to the proof of the above lemma, we have $\gamma_{\tilde{G}}((\tilde{G}^{\bar{x}\bar{y}})^m) \leq \gamma_{A_1}((A_1^{\bar{x}})^m)$. Hence $P_{\tilde{G}}(\tilde{G}^{\bar{x}\bar{y}}) \leq P_{A_1}(A_i^{\bar{x}}) = xrs/n$. However, $\tilde{G}^{\bar{x}\bar{x}}$ is a subgraph of $\tilde{G}^{\bar{x}\bar{y}}$. We have $P_{\tilde{G}}(\tilde{G}^{\bar{x}\bar{y}}) \geq P_{\tilde{G}}(\tilde{G}^{\bar{x}\bar{x}}) = xrs/n$. The corollary follows. ∎

Corollary 5.3. For any positive integers x and y, $P_{\tilde{G}}(\tilde{G}^{\bar{x}\bar{y}}\tilde{G}^{\bar{y}\bar{x}}) = xyr^2s^2/n^2$.

Proof. Similar to the proof of Lemma 5.6, we have $\gamma_{\tilde{G}}((\tilde{G}^{\bar{x}\bar{y}}\tilde{G}^{\bar{y}\bar{x}})^m) \leq \gamma_{A_1}((A_1^{\bar{x}}A_1^{\bar{y}})^m)$ and $P_{\tilde{G}}(\tilde{G}^{\bar{x}\bar{y}}\tilde{G}^{\bar{y}\bar{x}}) \leq P_{A_1}(A_1^{\bar{x}}A_1^{\bar{y}})$. Since P_{A_1} is pseudo-multiplicative, $P_{A_1}(A_1^{\bar{x}}A_1^{\bar{y}}) = P_{A_1}(A_1^{\bar{x}})P_{A_1}(A_1^{\bar{y}}) = (xrs/n)(yrs/n) = xyr^2s^2/n^2$ follows.

Let $\tilde{C}_1^{\bar{x}} = \{u_{i,j}|1 \leq i \leq rs/n,\ 1 \leq j \leq x\}$, $\tilde{C}_1^{\bar{y}} = \{u'_{i,j}|1 \leq i \leq rs/n,\ 1 \leq j \leq y\}$, $\tilde{C}_2^{\bar{x}} = \{v_{i,j}|1 \leq i \leq rs/n,\ 1 \leq j \leq x\}$, and $\tilde{C}_2^{\bar{y}} = \{v'_{i,j}|1 \leq i \leq rs/n,\ 1 \leq j \leq x\}$. We can set a one-to-one correspondence η from $\tilde{C}_1^{\bar{x}} \cup \tilde{C}_1^{\bar{y}}$ to $\tilde{C}_2^{\bar{x}} \cup \tilde{C}_2^{\bar{y}}$ by assigning $\eta(u_{i,j}) = v_{i,j}$ and $\eta(u'_{i,j}) = v'_{i,j}$. We can then extend η to η', which maps from $(A_1^{\bar{x}}A_1^{\bar{y}})^m$ into $(A_2^{\bar{y}}A_2^{\bar{x}})^m$ by $\eta'(x_1, x_2, \ldots, x_m) = (\eta(x_1), \eta(x_2), \ldots, \eta(x_m))$. Obviously, η' is one-to-one and onto. Let $M_1 = \{\hat{A}_i|1 \leq i \leq \gamma_{A_1}((A_1^{\bar{x}}A_1^{\bar{y}})^m)\}$ be a set of maximum mutually disjoint copies of A_1's in $(A_1^{\bar{x}}A_1^{\bar{y}})^m$. Then, the set $M_2 = \{\eta'(\hat{A}_i)|\hat{A}_i \in M_1\}$ forms a set of mutually disjoint A_2's in $(A_2^{\bar{y}}A_2^{\bar{x}})^m$. Moreover, the induced subgraph $(\tilde{G}^{\bar{x}\bar{x}})^m|_{\hat{A}_i \cup \eta'(\hat{A}_i)}$ induces a subgraph isomorphic to

$\tilde{G}$. Hence $\gamma_{\tilde{G}}((\tilde{G}^{\bar{x}\bar{y}}\tilde{G}^{\bar{y}\bar{x}})^m) \geq \gamma_{A_1}((A_1^{\bar{x}}A_1^{\bar{y}})^m)$. We have $P_{\tilde{G}}(\tilde{G}^{\bar{x}\bar{y}}\tilde{G}^{\bar{y}\bar{x}}) \geq P_{A_1}(A_1^{\bar{x}}A_1^{\bar{y}}) = xyr^2s^2/n^2$. The corollary is proved. ∎

Theorem 5.1. A graph G is PAMI if and only if G has exactly one closed set.

Proof. From Lemma 5.4, a graph G is PAMI if it has exactly one closed set. Hence, we need to prove that a graph G is non-PAMI if G has two or more closed sets. We first prove the case where G has exactly two closed sets through the following steps.

(1) It follows from Lemma 5.5 that G contains a 2-core subgraph $\hat{G}$ as a homomorphic image. By Lemma 2.2 and Theorem 1.2, $P_{\hat{G}^{\bar{2}}} \leq P_G \leq P_{\hat{G}}$.

(2) Let C_1 and C_2 be the two cores of $\hat{G}$. Assume that $|C_1| = c_1$, $|C_2| = c_2$, $r = 2c_2$, and $s = 2c_1$. Let $\tilde{G} = \hat{G}^{\bar{r}\bar{s}}$. Since $\hat{G}^{\bar{2}}$ is a subgraph of $\tilde{G}$, $P_{\hat{G}^{\bar{2}}} \geq P_{\tilde{G}}$. By Theorem 1.3, $P_{\hat{G}^{\bar{2}}} \leq P_{\tilde{G}}$. We have $P_{\hat{G}^{\bar{2}}} = P_{\tilde{G}}$.

(3) Let H be any graph such that $P_{\hat{G}^{\bar{2}}}(H) \neq 0$. From step 1, $\hat{G}$ is a homomorphic image of $\hat{G}^{\bar{2}}$ and $P_{\hat{G}} \geq P_{\hat{G}^{\bar{2}}}$. By Theorem 2.3, $P_{\hat{G}}(H) = P_{\hat{G}^{\bar{2}}}(H)$.

(4) Let x and y be any two positive integers with $x \leq y$. Obviously, $\hat{G}^{\bar{2}} \subseteq \tilde{G}^{\bar{x}\bar{y}}$. We have $P_{\hat{G}^{\bar{2}}}(\tilde{G}^{\bar{x}\bar{y}}) \neq 0$. By steps 2 and 3, $P_{\hat{G}}(\tilde{G}^{\bar{x}\bar{y}}) = P_{\hat{G}^{\bar{2}}}(\tilde{G}^{\bar{x}\bar{y}}) = P_{\tilde{G}}(\tilde{G}^{\bar{x}\bar{y}})$. Since $P_{\hat{G}^{\bar{2}}} \leq P_G \leq P_{\hat{G}}$, $P_G(\tilde{G}^{\bar{x}\bar{y}}) = P_{\tilde{G}}(\tilde{G}^{\bar{x}\bar{y}})$. By Corollary 5.2, $P_G(\tilde{G}^{\bar{x}\bar{y}}) = xrs/2$. Similarly, $P_G(\tilde{G}^{\bar{y}\bar{x}}) = xrs/2$, and $P_G(\tilde{G}^{\bar{x}\bar{y}}\tilde{G}^{\bar{y}\bar{x}}) = xyr^2s^2/4$. Hence $P_G(\tilde{G}^{\bar{x}\bar{y}}) = P_G(\tilde{G}^{\bar{y}\bar{x}}) = xrs/2$ and $P_G(\tilde{G}^{\bar{x}\bar{y}}\tilde{G}^{\bar{y}\bar{x}}) = xyr^2s^2/4$. Therefore, P_G is not pseudo-multiplicative. By Theorem 4.3, G is non-PAMI.

Now, we prove the case where G has exactly three closed sets. As in step 1, G contains a 3-core subgraph $\hat{G}$ as a homomorphic image. We have $P_{\hat{G}^{\bar{2}}} \leq P_G \leq P_{\hat{G}}$. Let C_1, C_2, and C_3 be the cores of $\hat{G}$. Assume that $|C_1| = c_1$, $|C_2| = c_2$, and $|C_3| = c_3$. We set $r = 2c_2c_3$, $s = 2c_1c_3$, and $t = 2c_1c_2$. Let $\tilde{G} = \hat{G}^{\bar{r}\bar{s}\bar{t}}$. As in step 2, we have $P_{\hat{G}^{\bar{2}}} = P_{\tilde{G}}$. Let $x \leq y \leq z$ be three positive integers. As in steps 3 and 4, we have $P_G(\tilde{G}^{\bar{x}\bar{y}\bar{z}}) = P_G(\tilde{G}^{\bar{y}\bar{z}\bar{x}}) = P_G(\tilde{G}^{\bar{z}\bar{x}\bar{y}}) = xrst/2$, and $P_G(\tilde{G}^{\bar{x}\bar{y}\bar{z}}\tilde{G}^{\bar{y}\bar{z}\bar{x}}\tilde{G}^{\bar{z}\bar{x}\bar{y}}) = xyzr^3s^3t^3/8$. Hence, G is non-PAMI.

Now, we discuss the general case where G has exactly $n\,(n \geq 2)$ closed sets. As in the above cases, we can construct $H_1, H_2, \ldots, H_n$ and find that $P_G(H_1)P_G(H_2)\cdots P_G(H_n) \neq P_G(H_1H_2\cdots H_n)$. Thus, G is non-PAMI. ∎

6. CLASSIFICATION OF AMI GRAPHS

In the above section, we classified PAMI graphs. A graph G is PAMI if and only if it has exactly one closed set. Obviously, a graph G is PAMI if it is AMI. As mentioned earlier, not all PAMI graphs are AMI. Therefore the classification of AMI graphs is also an interesting topic.

Theorem 6.1. A PAMI graph G is AMI if and only if (1) $P_G = P_{G|_C}$, where C is a core in the unique closed set in G, and (2) $G|_C$ is primary.

Proof. Suppose graph G is PAMI, such that (1) $P_G = P_{G|_C}$, where C is a core in the unique closed set in G, and (2) $G|_C$ is primary. Obviously $G|_C$ is vertex transitive and primary. Thus $P_{G|_C}$ is AMI. Since $P_G = P_{G|_C}$, G is AMI.

Assume G is AMI. Let C be a core in the unique closed set in G. We use A to denote the induced subgraph $G|_C$. Since A is a subgraph of G, $P_G \leq P_A$. On the other hand, G is a subgraph of AG, because A is a homomorphic image of G. Thus $P_G(AG) \neq 0$. Since G is AMI, $P_G(AG) = P_G(A)P_G(G) \neq 0$. We have $P_G(A) \neq 0$. Thus $G \subseteq A^t$ for some integer t. By Theorem 1.3, $P_G \geq P_A$. Hence $P_G = P_A$. Now, we prove that A is primary. Let B be a homomorphic image of A. We have $P_A(AB) \neq 0$, because $A \subseteq AB$. Since

$P_A(AB) = P_A(A)P_A(B)$, $P_A(B) \neq 0$ follows. Thus A is a subgraph of B^k for some integer k. The graph A is primary. ∎

With the above theorem, all AMI graphs are classified. The author tested several examples of AMI graphs, and all the examples tested indicate that condition (2) above is redundant. We therefore have the following conjecture.

Conjecture 1. Every core graph is primary.

7. RADICAL GRAPHS AND NON-RADICAL GRAPHS

In this, the final section, we discuss another property of graph capacity functions. In light of Theorem 1.2 (2), it is very natural to ask if $P_{kH} = P_H$. This statement is not true in general. Let us call a graph G a *radical graph* if $P_G(G) = 1$ and a *non-radical graph* otherwise. A vertex v in graph G is called a *fixed point* if $f(v) = v$ for every homomorphism f from G into itself. Obviously, K_1 is a radical graph and its only vertex is a fixed point.

Theorem 7.1. A graph G is radical if and only if G has at least one fixed point.

Proof. Let $V(G) = \{x_1, x_2, \ldots, x_u\}$. Assume that G has no fixed points. We can find a homomorphism $\phi_i : G \longrightarrow G$ such that $\phi_i(x_i) \neq x_i$ for every i. Let $\vec{y}_i = (x_i, x_i, \ldots, x_i)$ ($u + 1$ times) and $\vec{z}_i = (x_i, \phi_1(x_i), \phi_2(x_i), \ldots, \phi_u(x_i))$ for $i = 1, 2, \ldots, u$ that are vertices in G^{u+1}. It is easy to check that $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_u\}$ and $\{\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_u\}$ induce two disjoint G's in G^{u+1}. Hence $\gamma_G(G^{u+1}) \geq 2$. We have $P_G(G) \geq (\gamma_G(G^{u+1}))^{1/u+1} > 1$.

Assume G has a fixed point, say x_1. Let G' be any copy of G in G^m with $V(G') = \{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_u\}$, where $\vec{y}_i = (y_{i,1}, y_{i,2}, \ldots, y_{i,m})$ corresponds to x_i for every i. We can define m homomorphisms $\phi_1, \phi_2, \ldots, \phi_m$ from G into itself by $\phi_j(x_i) = y_{i,j}$ for every i and j. Since x_1 is a fixed point, $y_{1,j} = x_1$ for every j. Each copy of G in G^m contains the vertex $\vec{y}_1 = (x_1, x_1, \ldots, x_1)$ (m times) in common. Thus, $\gamma_G(G^m) \leq 1$ for every m. Since $G \subseteq G^m$ for every m, we have $\gamma_G(G^m) = 1$. Therefore $P_G(G) = 1$. ∎

With the above theorem, it is easy to check that all odd wheel graphs W_n, with $n \geq 5$, and the Grötzsch-Mycielski graph are radical and that all the (n,n)-graphs, with $n \geq 2$, are non-radical. Moreover, the graph G^2 is non-radical for any graph G. It can be proved that there exists a radical (m,n)-graph for all integers m and n with $1 < m < n$.

Lemma 7.1. $P_{2H}(G) = P_H(G)$ if $P_H(G) > 1$ and $P_{2H}(G) = 0$ if $P_H(G) \leq 1$.

Proof. It is easy to see that $\gamma_{2H}(G) = \lfloor \frac{1}{2}\gamma_H(G) \rfloor$ for any graph G. We have $(\lfloor \frac{1}{2}\gamma_H(G^n) \rfloor)^{1/n} = (\gamma_{2H}(G^n))^{1/n}$. Thus, $P_{2H}(G) = P_H(G)$ if $P_H(G) > 1$ and $P_{2H}(G) = 0$ if $P_H(G) \leq 1$. ∎

With the above lemma, we have the following theorem.

Theorem 7.2. Let k be an integer greater than 1. If H is a non-radial graph, then $P_{kH} = P_H$. If H is a radical graph, then $P_{kH}(G) = P_H(G)$ for $P_H(G) \neq 1$ and $P_{kH}(G) = 0$ otherwise. ∎

ACKNOWLEDGMENT

The author wishes to thank an anonymous referee for a very thorough review and many helpful suggestions.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph theory with applications,* Elsevier Science Publishing, New York (1976).

[2] S. T. Hedetniemi, Homomorphisms of Graphs and Automata, Doctoral Dissertation, University of Michigan, Ann Arbor (1966).

[3] L. H. Hsu, On a multiplicative graph function conjecture, *Discrete Math.* **45** (1983), 245–253.

[4] L. H. Hsu, Generalized homomorphism graph functions, *Discrete Math.* **83** (1990), 31–37.

[5] L. H. Hsu, C. Chen, and E. Y. Jean, A class of additive multiplicative graph functions, *Discrete Math.* **65** (1987), 53–63.

[6] L. H. Hsu and S. F. Hwang, Graph capacity equivalence relation, *Chinese J. Math.* **17** (1989), 191–205.

[7] S. F. Hwang and L. H. Hsu, Capacity equivalent class for graphs with fixed odd girth, *Tamkang J. Math.* **20** (1989), 159–167.

[8] L. Lovász, Kneser's conjecture, chromatic number, and homotoph, *J. Combin. Theory, Ser. A* **25** (1978), 319–324.

[9] L. Lovász, Personal communication to J. Spenser.

[10] J. Mycielski, Sur le coloriage des graphs, *Colloq. Math.* **3** (1955), 161–162.

[11] C. E. Shannon, The zero-error capacity of a noisy channel, *Comp. Information Theory, IRE Trans.* (1956), 3–15.

Received November 10, 1994