MODELING THE ASYMMETRY OF STOCK MOVEMENTS USING PRICE RANGES

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ABSTRACT

It is shown in Chou (2005). Journal of Money, Credit and Banking, 37, 561–582that the range can be used as a measure of volatility and the conditional autoregressive range (CARR) model performs better than generalized auto regressive conditional heteroskedasticity (GARCH) in forecasting volatilities of S&P 500 stock index. In this paper, we allow separate dynamic structures for the upward and downward ranges of asset prices to account for asymmetric behaviors in the financial market. The types of asymmetry include the trending behavior, weekday seasonality, interaction of the first two conditional moments via leverage effects, risk premiums, and volatility feedbacks. The return of the open to the max of the period is used as a measure of the upward and the downward range is defined likewise. We use the quasi-maximum likelihood estimation (QMLE) for parameter estimation. Empirical results using S&P 500 daily and weekly frequencies provide consistent evidences supporting the asymmetry in the US stock market over the period 1962/01/01–2000/08/ 25. The asymmetric range model also provides sharper volatility forecasts than the symmetric range model.

1. INTRODUCTION

It's known for a long time in statistics that range is a viable measure of the variability of random variables. In the recent two decades, applications to finance issues discovered that ranges were useful to construct efficient volatility estimators; e.g., see Parkinson (1980); Garman and Klass (1980); Beckers (1983); Wiggins (1991); Rogers and Satchell(1991); Kunitomo (1992); Rogers (1998); Gallant, Hsu, and Tauchen (1999); Yang and Zhang (2000); and Alizadeh, Brandt, and Diebold (2002). In Chou (2005), we propose the conditional autoregressive range (CARR) model for range as an alternative to the modeling of financial volatilities. It is shown both theoretically and empirically that CARR models are worthy candidates in volatility modeling in comparison with the existing methodologies, say the generalized auto regressive conditional heteroskedasticity (GARCH) models. Empirically, the CARR model performs very satisfactory in forecasting volatilities of S&P 500 using daily and weekly observations. In all four cases with different measures of the ''observed volatility'', CARR dominates GARCH in the Mincer/Markovtz regression of forecasting evaluations. It's a puzzle (see Cox & Rubinstein, 1985) that despite the elegant theory and the support of simulation results, the range estimator has performed poorly in empirical studies. In Chou (2005), we argue that the failure of all the range-based models in the literature is due to its ignorance of the temporal movements of the range. Using a proper dynamic structure for the conditional expectation of range, the CARR model successfully resolves this puzzle and retains its superiority in empirical forecasting powers.

This paper focuses on an important feature in financial data: asymmetry. Conventionally, symmetric distributions are usually assumed in asset pricing models, e.g., normal distributions in CAPM and the Black/Sholes option pricing formula. Furthermore, in calculating various measures of risk, standard deviations (or equivalently, variances) are used frequently, which implicitly assume a symmetric structure of the prices. However, there are good reasons why the prices of speculative assets should behave asymmetrically. For investors, the more relevant risk is generated by the downward price moves rather than the upward price moves; the latter is important in generating the expected returns. For example, the consideration of the value-at-risk only utilizes the lower tail of the return distribution. There are also models of asset prices that utilize the third moment (an asymmetric characteristic feature), for example, Levy and Markowitz (1979). Furthermore, asymmetry can arise in a dynamic setting in models considering time-varying conditional moments. For example, the ARCH-M model of Engle,

Lilien, and Robbins (1987) posits a linkage between the first sample moment and past second sample moments. This model has a theoretical interpretation in finance: the risk premium hypothesis (See Malkiel, 1978; Pindyck, 1984; Poterba & Summers, 1986; Chou, 1988). The celebrated leverage effect of Black (1976) and Christie (1982) is cast into a dynamic volatility model in the form of the linkage between the second sample moment and past first sample moments; See EGARCH of Nelson (1991) and NGARCH of Engle and Ng (1993). Furthermore, the asymmetry can arise in other forms such as the volatility feedback of Campbell (1997). Barberis and Huang (2000) give an example of loss aversion and mental account that would predict an asymmetric structure in the price movements. Tsay (2000) uses only observations of the downward, extreme movements in stock prices to model the crash probability.

Chou (2005) incorporated one form of asymmetry, the leverage effect, into the CARR model and it appeared to be more significant than reported in the literature of GARCH or Stochastic Volatility models. The nature of the CARR model is symmetric because range is used in modeling which treats the maximum and minimum symmetrically. In this paper, a more general form of asymmetry is considered by allowing the dynamic structure of the upward price movements to be different from that of the downward price movements. In other words, the maximum and the minimum of price movements in fixed intervals are treated in separate forms. It may be relevant to suspect that the information in the downward price movements are as relevant as the upward price movements in predicting the upward price movements in the future. Similarly, the opposite case is true. Hence it is worthy to model the CARR model asymmetrically.

The paper is organized as following. It proposes and develops the Asymmetric CARR (ACARR) model with theoretical discussions in section 2. In addition, discussions are given about some immediate natural extensions of the ACARR model. An empirical example is given in section 3 using the S&P 500 daily index. Section 4 concludes with considerations of future extensions.

2. MODEL SPECIFICATION, ESTIMATION, AND PROPERTIES

2.1. The Model Specification, Stochastic Volatilities, and the Range

Let P_t be the logarithmic price of a speculative asset observed at time t, $t = 1, 2, \ldots T$. P_t is a realization of a price process $\{P_t\}$, which is assumed to

be a continuous processf.[1] We further assume that within each time interval, we observe P_t at every fixed time interval dt. Let n denote the number of intervals between each unit time, then $dt = 1/n$. There are hence, $n+1$ observations within each time interval between $t-1$ and t. Let P_t^o, P_t^c, P_t^{HIGH}, P_t^{LOW}, be the opening, closing, high and low prices, in natural logarithm, between $t-1$ and t. The closing price at time t will be identical to the opening price at time $t+1$ in considerations of markets that are operated continuously, say, some of the foreign exchange markets. Further, define UPR_t, the upward range, and $DWNR_t$, the downward range as the differences between the daily highs, daily lows, and the opening price respectively, at time t, in other words,

$$UPR_t = P_t^{HIGH} - P_t^o$$
$$DWNR_t = P_t^{LOW} - P_t^o \tag{2.1}$$

Note that these two variables, UPR_t and $DWNR_t$, represent the maximum and the minimum returns respectively, over the unit time interval $(t-1, t)$. This is related to the range variable in Chou (2005) that R_t, defined to be

$$R_t = P_t^{HIGH} - P_t^{LOW} \tag{2.2}$$

It's clear that the range is also the difference between the two variables, UPR_t, and $DWNR_t$, in other words,

$$R_t = UPR_t - DWNR_t \tag{2.3}$$

In Chou (2005) we propose a dynamic model, the CARR model, for the range. It's a conjecture that the extreme value theory can be used to show that the conditional range, or equivalently the disturbance term, has a limiting distribution that is governed by a shifted Brownian bridge on the unit interval.[2] In this paper, we propose a model for the one-sided range, UPR_t, and $DWNR_t$, to follow a similar dynamic structure. In particular,

$$UPR_t = \lambda_t^u \varepsilon_t^u$$
$$DWNR_t = -\lambda_t^d \varepsilon_t^d$$
$$\lambda_t^u = \omega^u + \sum_{i=1}^p \alpha_i^u UPR_{t-i} + \sum_{j=1}^q \beta_j^u \lambda_{t-j}^u$$
$$\lambda_t^d = \omega^d + \sum_{i=1}^p \alpha_i^d DWNR_{t-i} + \sum_{j=1}^q \beta_j^d \lambda_{t-j}^d$$
$$\varepsilon_t^u \sim iid f^u(\cdot), \ \varepsilon_t^d \sim iid f^d(\cdot) \tag{2.4}$$

Model (2.4) is called the asymmetric conditional autoregressive range (Asymmetric CARR or ACARR, henceforth) model. In the following discussions, we will disregard the super-scripts when there is no concern of confusion. In (2.4), λ_t is the conditional mean of the one-sided range based on all information up to time t. The distribution of the disturbance terms ε_t of the normalized one-sided-range, or $OSR_t(= UPR_t$ or $DWNR_t,)$, $\varepsilon_t = OSR_t/\lambda_t$, are assumed to be identically independent with density function $f^i(\cdot)$, where $i = u$ or d. Given that both the one-sided ranges UPR_t and $-DWNR_t$, and their expected values λ_t are both positive hence their disturbances ε_t, the ratio of the two, are also positively valued.

The asymmetric behavior between the market up and down movements can be characterized by different values for the pairs of parameters, (ω^u, ω^d), (α^u, α^d), (β^u, β^d), and from the error distributions $(f^u(\cdot), f^d(\cdot))$.

The equations specifying the dynamic structures for λ_t's characterize the persistence of shocks to the one-sided range of speculative prices or what is usually known as the volatility clustering. The parameters ω, α_i, β_j, characterize respectively, the inherent uncertainty in range, the short-term impact effect and the long-term effect of shocks to the range (or the volatility of return). The sum of the parameters $\sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{q}\beta_j$, plays a role in determining the persistence of range shocks. See Bollerslev (1986) for a discussion of the parameters in the context of GARCH.

The model is called an asymmetric conditional autoregressive range model of order (p,q), or ACARR(p,q). For the process to be stationary, we require that the characteristic roots of the polynomial to be out side the unit circle, or $\sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{q}\beta_j < 1$. The long-term range denoted ω-bar, is calculated as $\omega/(1 - (\sum_{i=1}^{p}\alpha_i + \sum_{j=1}^{q}\beta_j))$. Further, all the parameters in the second equation, are assumed positive, i.e., ω, α_i, $\beta_j > 0$.

It is useful to compare this model with the CARR model of Chou (2005):

$$R_t = \lambda_t \varepsilon_t$$

$$\lambda_t = \omega + \sum_{i=1}^{p} \alpha_i R_{t-i} + \sum_{j=1}^{q} \beta_j \lambda_{t-j}$$

$$\varepsilon_t \sim iid\, f(.) \tag{2.5}$$

Ignoring the distribution functions, the ACARR model reduces to the CARR if all the parameters with superscript u and d are identical pair-wise. Testing these various types of model asymmetry will be of interest because asymmetry can arise in varieties, e.g., size of the range, i.e., level of the volatility $(\omega - bar = \omega/(1 - \alpha - \beta))$, the speed of mean-reversion $(\alpha + \beta)$, and the short-term (α) versus long-term (β) impact of shocks.

Eq. (2.4) is a reduced form for the one-sided ranges. It is straight forward to consider extending the model to include other explanatory variables, $X_{t-1,l}$ that are measurable with respect to the information set up to time $t-1$.

$$\lambda_t = \omega + \sum_{i=1}^{p} \alpha_i R_{t-i} + \sum_{j=1}^{q} \beta_j \lambda_{t-j} + \sum_{l=1}^{L} \gamma_l X_{t-1,l} \qquad (2.6)$$

This model is called the ACARR model with exogenous variables, or ACARRX. Among others, some important exogenous variables are trading volume (see Lamoureux & Lastrapes, 1990; Karpoff, 1987), the lagged returns measuring the leverage effect of Christie (1982), Black (1976) and Nelson (1990) and some seasonal factor to characterize the seasonal pattern within the range interval.

Note that although we have not specified specifically, all the variables and parameters in (2.4) are all dependent on the parameter n, the number of intervals used in measuring the price within each range-measured interval. It is clear that all the range estimates are downward biased if we assume the true data-generating mechanism is continuous or if the sampling frequency is lower than that of the data generating process if the price is discrete. The bias of the size of the one-sided-range, whether upward range (UPR_t) or downward range ($DWNR_t$), like the total range, will be a a non-increasing function of n. Namely, the finer the sampling interval of the price path, the more accurate the measured ranges will be.

It is possible that the highest frequency of the price data is non-constant given the heterogeneity in the trading activities within each day and given the nature of the transactions of speculative assets. See Engle and Russell (1998) for a detailed analysis of the non-constancy of the trading intervals, or the durations. Extensions to the analyses of the ranges of non-fixed interval prices will be an interesting subject for future research.[3] However, some recent literature suggest that it is not desirable to work with the transaction data in estimating the price volatility given the consideration of microstructures such as the bid/ask bounces, the intra-daily seasonality, among others. See Andersen, Bollerslev, Diebold, and Labys, (2000); Bai, Russell, and Tiao (2000), and Chen, Russell, and Tsay (2000).

As is the case for the CARR model, the ACARR model mimics the ACD model of Engle and Russell (1998) for durations between trades. Nonetheless, there are important distinctions between the two models. First, duration is measured at some random intervals but the range is measured at fixed intervals, hence the natures of the variables of interest are different although they share the common property that all observations are positively valued.

Second, in the ACD model, the distribution of the disturbances is usually chosen arbitrarily – a feature also shared by all GARCH models. The ACARR model, on the contrary, has some natural choices from the results of extreme value theories in statistics.[4]

2.2. Properties of ACARR: Estimation and Relationships with Other Models

Given that the ACARR model has exactly the same form as the CARR model, all the statistical results in CARR apply to ACARR. Furthermore, the ACARR model has some unique properties of its own. We illustrate some of the important properties in this subsection. Given that the upward and the downward range evolutions are specified independently, the estimation can hence be performed separately. Further, consistent estimation of the parameters can be obtained by the quasi-maximum likelihood estimation (QMLE) method. The consistency property follows from the ACD model of Engle and Russell (1998) and Chou (2005). It indicates that the exponential distribution can be used in constructing the likelihood to consistently estimate the parameters in the conditional mean equation.

Specifically, given the exponential distribution for the error terms, we can perform the QMLE. Using $R_t, t = 1, 2, \ldots, T$ as a general notation of UPR_t and $DWNR_t$, the log-likelihood function for each of the one-sided range series is

$$L\big(\alpha_i, \, \beta_j; \; R_1, \, R_2, \, \ldots, \, R_T,\big) = -\sum_{t=1}^{T}\left[\log(\lambda_t) + \frac{R_t}{\lambda_t}\right].$$

The intuition of this property relies on the insight that the likelihood function in ACARR with an exponential density is identical to the GARCH model with a normal density function with some simple adjustments on the specification of the conditional mean. Furthermore, all asymptotic properties of GARCH apply to ACARR. Given that ACARR is a model for the conditional mean, the regularity conditions (e.g., the moment condition) are in fact, less stringent then in GARCH.

Note that although QMLE is consistent, it is not efficient. The efficiency can be obtained if the conditional density function is known. This leads us to the limiting distribution of the conditional density of range. The discussion will require a far more complicated theoretic framework, which is worthy of pursuing by an independent work. We hence do not pursue this

route in this paper and follow the strategy of Chou (2005) in relying on the QMLE.[5] Again, it is an empirical question as to how substantial in efficiency such methods can generate. Engle and Russell (1998) reported that deviations from the exponential density function do not offer efficiency gain sufficiently high in justifying the extra computation burdens.

It is important to note that the direct application of QMLE will not yield consistent estimates for the covariance matrix of the parameters. The standard errors of the parameters are consistently estimated by the robust method of Bollerslev and Wooldridge (1992). The efficiency issue related to these estimates is a subject for future investigation.

Another convenient property for ACARR (due to its connection with ACD) is the ease of estimation. Specifically, the QMLE estimation of the ACARR model can be obtained by estimating a GARCH model with a particular specification: specifying a GARCH model for the square root of range without a constant term in the mean equation.[6] This property is related to the above QMLE property by the observation of the equivalence of the likelihood functions of the exponential distribution in ACARR and ACD and of the normal density in GARCH. It indicates that it is almost effortless to estimate the ACARR model if a GARCH software is available.

It will be interesting and important to investigate whether the ACARR model will satisfy a closure property, namely, whether the ACARR process is invariant to temporal and cross-sectional aggregations. This is important given the fact that in financial economics, aggregates are frequently encountered, e.g., portfolios are cross-sectional aggregates and monthly, weekly returns are temporal aggregates of daily returns. It is also a property that is stressed in the literature of time series econometrics.[7]

Another interesting property of the CARR model is the encompassing property. It is interesting that the square-root-GARCH model turns out to be a special case of CARR, and in fact, the least efficient member of the CARR model. This property does not apply to ACARR since there are no analogies of the open to maximum (minimum) in the GARCH model family.

2.3. Robust ACARR

It is suggested in statistics that range is sensitive to outliers. It is useful hence, to consider extension of ACARR to address such considerations. We consider robust measures of range to replace the standard range defined as the difference between the max and the min. A simple naive method is to use

the next-to-max for max and the next-to-min for min. By doing so, the chance of using outliers created by typing errors will be greatly reduced. It will also reduce the impact of some true outliers.

A second alternative is to use the quantile range, for example, a 90% quantile range is defined as the difference between the 95% percentile and the 5% percentile. A frequently adopted robust range is the interquartile range (IQR) which is a 75% quantitle and it can be conveniently obtained by taking the difference of the medians of the top and lower halves of the sampling data. In measuring a robust maximum or minimum likewise, we can use the 75% quantile in both the upward price distribution and the downward price distribution.

Similarly other types of robust extreme values can be adopted like the next-ith-to-max (min) and the average of the top 5% observations and the bottom 5% observations, et.al. There are several important issues relevant in considerations such as the efficiency loss, e.g., the IQR discards 50% of the information while the next-to-max approach discards very little. Another issue is the statistical tractability of the new range measures. For example, the quantile range will have a more complicated distribution than the range and the statistical property for the next-ith-to-extreme approach is less known than the quantile range. Another consideration is the data feasibility. In most cases, none of the information other than the extreme observations are available. For example, the standard data sources such as CRSP, and the Wall Street Journal, the Financial Times only report the daily highs and lows. As a result, the robust range estimators are infeasible unless one uses the intra-daily data. Nonetheless, the robust range estimators are feasible if the target volatility is measured at lower frequency than a day. This is obvious since there are 20 some daily observations available in each given month hence the monthly volatility can be measured by a robust range if the outlier problem is of concern. Given the existence of intra-daily data, daily robust range model is still an important topic for future research.

3. AN EMPIRICAL EXAMPLE USING THE S&P 500 DAILY INDEX, 1962/01/03–2000/08/25

3.1. The Data Set

The daily index data of the Standard and Poor 500 (S&P 500) are used for empirical study in this paper to gauge the effectiveness of the ACARR

model. The data set is downloaded from the finance subdirectory of the website "Yahoo.com". The sample period covered in this paper is 1962/01/03–2000/08/25. The models are estimated by using this daily data set, comparisons are made for various volatility models on the accuracy of the volatility predictions.

Table 1 gives the summary statistics of RANGE, *UPR*, and *DWNR* for the full sample and two sub-sample periods. Sub-samples are considered because there is an apparent shift in the level of the daily ranges roughly on the date 1982/04/20. As is shown in Table 1, the averaged range level was reduced by almost a half since this particular date. Reductions in the level of similar magnitude are seen for the max and min as well. It's likely an institutional change occurred at the above-mentioned date. From a telephone conversation with the Standard and Poor Incorporated, the source of this structural change was revealed. Before this stated date, the index high and index low were compiled by aggregating the highs and lows of individual firm prices for each day. This amounts to assume that the highs and lows for all 500 companies occur at the same time in each day. This is clearly an incorrect assumption and amounts to an overestimate of the highs and an underestimate of the lows. As a result, the ranges are over-estimated. The compiling process was corrected after April 1982.[8] The company computes the index value at some fixed (unknown to me, say, 5 min) intervals within each day and than select the maximum and minimum price levels to be the index highs and index lows.[9]

Figs. 1–4 give the plots of the daily max and min price movements. It is interesting that the upward and downward range are roughly symmetric from the closeness of summary statistics and the seemingly reflective nature of Fig. 1. Another interesting observation (see Figs. 2–4) is that excluding the outlier of the 1987 crash, the two measures have very similar unconditional distributions. Careful inspection of the Figures and Tables however, reveals important differences in these two measures of market movements in the two opposite directions. For example, although both one-sided ranges (henceforth OSRs) have clustering behaviors but their extremely large values occur at different times and with different magnitudes. Further, as Table 1 and Fig. 5 show, the magnitudes of the autocorrelation at some lags for the UPR seem to be substantially different from that of the DWNR indicating different level of persistence. This can be viewed as a primitive indicator of the difference in the dynamic structure of the two processes. The true comparison of the dynamic structures of the two range processes will be made in the next section.

Table 1. Summary Statistics of the Daily Range, Upward Range, and Downward Range of S&P 500 Index, 1/2/1962–8/25/2000.

		Nobs	Mean	Median	Max	Min	Std Dev	ρ_1	ρ_2	ρ_{12}	Q(12)
Full Sample											
RANGE	1/2/62-8/25/00	9700	1.464	1.407	22.904	0.145	0.76	0.629	0.575	0.443	30874
UPR	1/2/62-8/25/00	9700	0.737	0.636	9.053	0	0.621	0.308	0.147	0.172	3631
DWNR	1/2/62-8/25/00	9700	−0.727	−0.598	0	−22.9	0.681	0.326	0.181	0.162	4320
Before structural shift											
RANGE	1/2/62-4/20/82	5061	1.753	1.643	9.326	0.53	0.565	0.723	0.654	0.554	21802
UPR	1/2/62-4/20/82	5061	0.889	0.798	8.631	0	0.581	0.335	0.087	0.106	1199
DWNR	1/2/62-4/20/82	5061	−0.864	−0.748	0	−6.514	0.559	0.378	0.136	0.163	2427
After structural shift											
RANGE	4/21/82-8/25/00	4639	1.15	0.962	22.904	0.146	0.818	0.476	0.414	0.229	11847
UPR	4/21/82-8/25/00	4639	0.572	0.404	9.053	0	0.622	0.189	0.089	0.125	651
DWNR	4/21/82-8/25/00	4639	−0.578	−0.388	0	−22.9	0.767	0.247	0.147	0.101	994

Note: Summary statistics for the three variables, RANGE, UPR and DWNR, defined to be the differences between the max and min, the max and open, and the min and open of the daily index prices in logarithm are described. The structural shift refers to the day April 20, 1982, at which the Standard and Poor Inc. changed the ways of constructing the daily max and daily min prices. ρ_1, ρ_2 and ρ_{12} are autocorrelation coefficients for lags 1, 2, and 12 respectively, and Q(12) is the Ljung–Box statistics of lag 12.

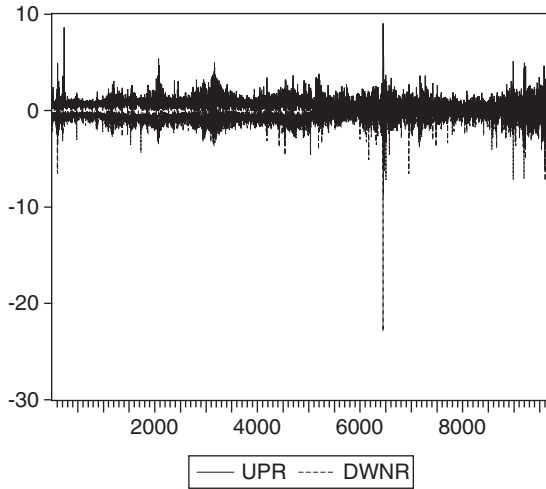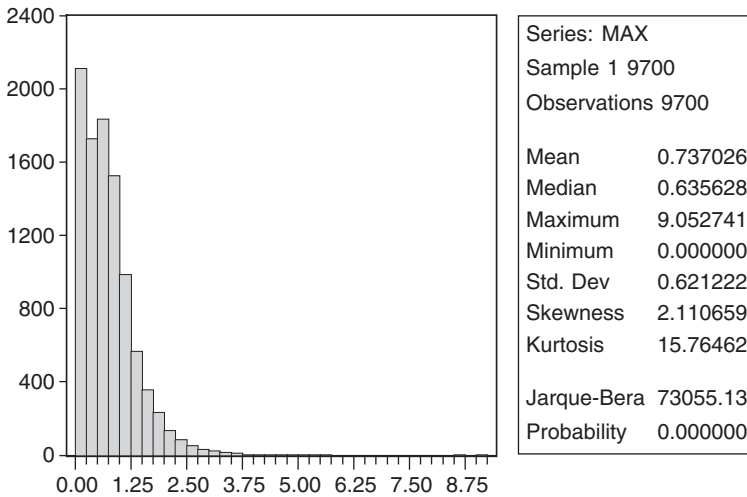*Fig. 1.*   Daily UPR and Daily DWNR, S&P 500, 1962/1–2000/8.

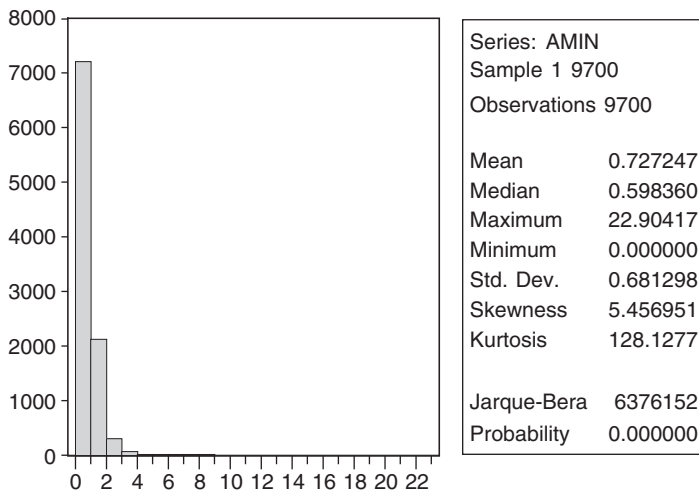*Fig. 2.*   Daily UPR of S&P 500 index, 1962/1–2000/8.

Series: AMIN	
Sample 1 9700	
Observations 9700	
Mean	0.727247
Median	0.598360
Maximum	22.90417
Minimum	0.000000
Std. Dev.	0.681298
Skewness	5.456951
Kurtosis	128.1277
Jarque-Bera	6376152
Probability	0.000000

Fig. 3.   Daily DWNR of S&P 500 index, Unsigned, 1962/1–2000/8.

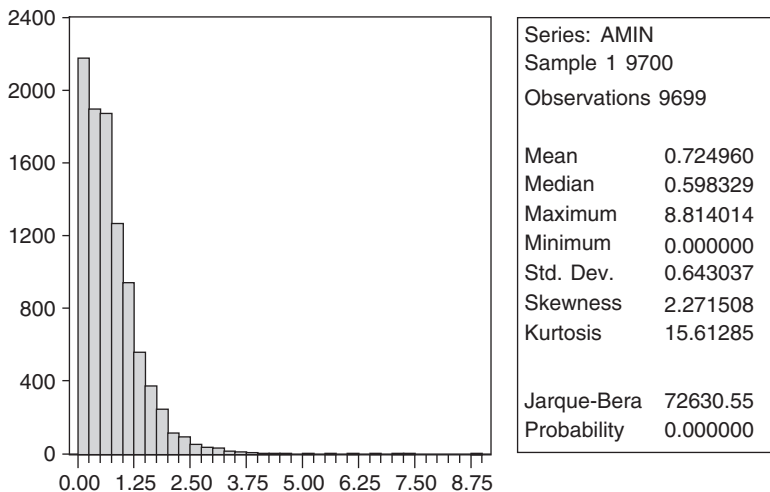| Series: AMIN	
Sample 1 9700	
Observations 9699	
Mean	0.724960
Median	0.598329
Maximum	8.814014
Minimum	0.000000
Std. Dev.	0.643037
Skewness	2.271508
Kurtosis	15.61285
Jarque-Bera	72630.55
Probability	0.000000

Fig. 4.   Daily DWNR w/o crash, Unsigned, 1962/1–2000/8.
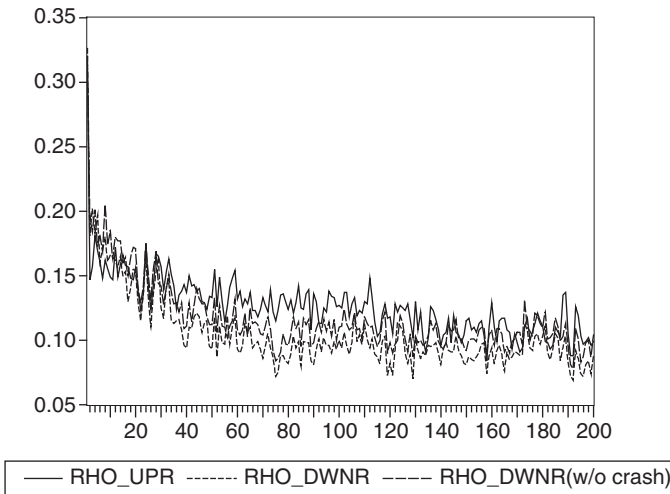
Fig. 5. Correlograms of Daily UPR and DWNR.

3.2. Estimating Results

We use QMLE to estimate the ACARR and ACARRX models with different dynamic specifications and exogenous variables. The exogenous variables considered are lagged return, r_{t-1}, for the leverage effect, a Tuesday (TUE) and a Wednesday dummy (WED), for the weekly seasonal pattern, a structural shift dummy (SD, 0 before 1982/4/20 and 1 otherwise) for capturing the shift in the data compiling method. We also include the lagged opposite range variable, i.e., DWNR in the UPR model and UPR in the DWNR model. This is for the consideration of the volatility clustering effect. Tables 2 and 3 give respectively, the model estimating results for UPR and for DWNR.

It is interesting that for both OSRs, a ACARR(2,1) clearly dominates the simpler alternative of ACARR(1,1) model, which is in contrast of the result in Chou (2005) using CARR to estimate the range variable.[10] This is shown clearly by the difference in the values of the log-likelihood function (LLF) reported for the two models, ACARR(1,1) vs. ACARR(2,1). The ACARR(2,1) model is consistent with the specification of the Component GARCH model of Engle and Kim (1999), in which the volatility dynamics is decomposed into two parts, a permanent component and a temporary component.

Table 2. QMLE Estimation of ACARR Using Daily Upward Range of S&P 500 Index 1/2/1962–8/25/2000.

	ACARR(1,1)	ACARR(2,1)	ACARRX(2,1)-a	ACARR(2,1)-b	ACARR(2,1)-c
LLF	−12035.20	−12011.86	−11955.78	−11949.64	−11950.32
Constant	0.002[3.216]	0.001[3.145]	−0.002[−0.610]	−0.003[−0.551]	−0.004[−0.973]
UPR(t−1)	0.03[8.873]	0.145[10.837]	0.203[14.030]	0.179[11.856]	0.186[12.845]
UPR(t−2)		−0.126[−9.198]	−0.117[−9.448]	−0.112[−8.879]	−0.115[−9.245]
λ(t−1)	0.968[267.993]	0.978[341.923]	0.903[69.643]	0.871[48.426]	0.877[52.942]
r(t−1)			−0.057[−8.431]	−0.018[−1.959]	−0.023[−2.734]
TUE			0.058[3.423]	0.059[3.475]	0.059[3.481]
WED				0.02[1.271]	
SD			0.0000[0.201]	−0.003[−1.647]	
DWNR(−1)				0.046[4.868]	0.042[4.803]
Q(12)	184.4[0.000]	22.346[0.034]	22.304[0.034]	20.282[0.062]	20.503[0.053]

$$UPR_t = \lambda_t \varepsilon_t$$

$$\lambda_t^u = \omega^u + \sum_{i=1}^{p} \alpha_i^u UPR_{t-i} + \sum_{j=1}^{q} \beta_j^u \lambda_{t-j}^u + \sum_{l=1}^{L} \gamma_l X_{t-1}^l$$

$$\varepsilon_t \sim iid\ f(\cdot)$$

Note: Estimation is carried out using the QMLE method hence it is equivalent to estimating an exponencial ACARR(X) (p,q) or and EACARR(X) (p,q) model. Numbers in parentheses are *t*-ratios (*p*-values) with robust standard errors for the model coefficients (*Q* statistics). LLF is the log-likelihood function.

Table 3. QMLE Estimation of ACARR Using Daily Downward Range of S&P500 Index 1/2/1962–8/25/2000.

	ACARR(1,1)	ACARR(2,1)	ACARRX(2,1)-a	ACARRX(2,1)-b	ACARRX(2,1)-c
LLF	−11929.39	−11889.61	−11873.54	−11868.55	−11870.14
Constant	0.014[5.905]	0.004[4.088]	0.017[4.373]	0.017 [3.235]	0.017[4.417]
DWNR(t-l)	0.084[11.834]	0.229[16.277]	0.252[16.123]	0.233[14.770]	0.239[16.364]
DWNR(t-2)		−0.195[−13.489]	−0.189[−12.811]	−0.185[−12.594]	−0.186[−12.897]
$\lambda(t-1)$	0.897[101.02]	0.961[199.02]	0.927[87.639]	0.906[61.212]	0.911[63.893]
$r(t-1)$			0.023[4.721]	−0.009[−1.187]	
TUE				−0.008[0.503]	
WED			−0.051[−3.582]	−0.053[−3.617]	−0.052[−3.587]
SD			−0.002[−2.124]	0.001[0.904]	
UPR(−1)				0.037[4.164]	0.028[5.084]
Q(12)	192.8[0.000]	18.94[0.009]	22.227[0.035]	14.422[0.275]	14.774[0.254]

$$DWNR_t = \lambda_t \varepsilon_t$$

$$\lambda_t^d = \omega^d + \sum_{i=1}^{p} \alpha_i^d DWNR_{t-i} + \sum_{j=1}^{q} \beta_j^d \lambda_{t-j}^d + \sum_{l=1}^{L} \gamma_l X_{t-1}^l$$

$$\varepsilon_t \sim iid\, f(\cdot)$$

Note: Estimation is carried out using the QMLE method hence it is equivalent to estimating an Exponential ACARR(X)(p,q) or and EACARR(X)(p,q) model. Numbers in parentheses are *t*-ratios(*p*-values) with robust standard errors for the model coefficients (*Q* statistics). LLF is the log-likelihood function.

Another conjecture for the inadequacy of the (1,1) dynamic specification is related to the volatility clustering effect. It is known that volatility clusters over time and in the original words of Mandelbrot (1963), "large changes tend to be followed by large changes and small by small, of either sign…". Given that range can be used as a measure of volatility, both UPR and DWNR can be viewed as "signed" measure of volatility. It is hence not surprising that a simple dynamic structure offered by the ACARR(1,1) model is not sufficient to capture the clustering effect. This conjecture is supported by the result of the model specification of ACARRX(2,1)-b where the opposite OSR are included and the coefficients significantly different from zero.

The dynamic structures for the UPR and DWNR variable are different as is revealed in comparing the values of the coefficients. The coefficient of $\beta 1$, measuring the long-term persistence effect, is (0.927, 0.906, 0.911) respectively, for the three different ACARRX specifications for DWNR in Table 3. They are all higher than their corresponding elements (0.903, 0.871, 0.877) in the ACARRX models for UPR in Table 2. This suggests that volatility shocks in the downside are more long-lived than in the upside. Further the impact coefficient $\alpha 1$ is equal to (0.252, 0.233, 0.239) in the DWNR models and is (0.203, 0.179, 0.186) in the UPR models. Volatility shock effects in the short-run are also higher for the downside shocks than for the upward surges. Both of these findings are new in the literature of financial volatility models as all existing literatures do not distinguish the shock asymmetry in this fashion.

Another interesting comparison between the two OSR models is on the leverage effect. This coefficient is statistically negative (positive) for the ACARRX(2,1)-a specifications for the UPR (DWNR). It is however, less significant or insignificant in models ACARRX(2,1)-b and ACARRX(2,1)-c, when the lagged opposite OSR is included. My conjecture is that the opposite sided ranges are correlated with the returns and hence multicollinearity reduces some explanatory power of the leverage effect. It remains, however, to be explained why such a phenomenon is more severe for the DWNR model than the UPR models. We leave this issue for future studies.

A different weekly seasonality also emerges from the comparison of the estimation result of the two OSRs. For reasons unknown to me, a positive Tuesday effect is found for the upward range while a negative Wednesday effect is present for the downward range. The dummy variable SD, measuring the effect of the structuring change in the data compiling method, are not significant for UPR models but are negatively significant for one of the DWNR models. It is not clear why there should be such difference in the

results. Again leave these as empirical puzzles to be explored in future studies.

Model specification tests are carried out in two ways, the Ljung–Box-Q statistics and the Q–Q plots. The Ljung–Box Q statistics measure the overall significance of the autocorrelations in the residuals for the fitted models.[11] The evidence shown in the two tables are consistent that a pure ACARR model is not sufficient and exogenous variables are necessary to warrant the model to pass the model misspecification tests. Using a 5% significance level for the test, the model is satisfactory once the lagged returns, the weekly dummies and the opposite-sided range are included in the specifications.

Figs. 6 and 7 provide the expected and observed daily UPR and DWNR respectively. It is interesting to note that the ACARR model gives smoother yet very adaptive estimates of the two one-sided ranges. Figs. 8–11 are histograms and Q–Q plots of the estimated residuals in the two models. It seems to indicate that the exponential distribution is more satisfactory for the UPR than for the DWNR as the degree of fitness of fit can be measured by the deviations of the Q–Q plot from the 45 degree lines. This fact further indicates the difference in the characteristics of the two variables in addition to the results reported above. Whether a different error distribution will be more useful warrants more investigation. For example in Chou (2005) I found that a more general error distribution such as Weibull might improve the goodness of fit substantially in the CARR model.

The message from this section is clear: the market dynamics for the upward swing and the downward plunge are different. They are different in their dynamics of the volatility shocks, i.e., the short-term impact and long-term persistence. They are also different in the forces that have effects on them, the leverage effect, the weekly seasonal effect and the volatility clustering effect. Finally, even the error structures of the two variables are different.

3.3. Comparing ACARR and CARR in Forecasting Volatility

Although the above results shows important differences in the models for the upward and the downward range, we further ask a question on the value of the modeling of asymmetries. How much difference does this modeling consideration make to improve the power of the model in forecasting volatilities? In Chou (2005) we proposed the CARR model, where the upward and downward movements of the stock price are treated symmetrically. We showed that the CARR model provides a much sharper tool in forecasting
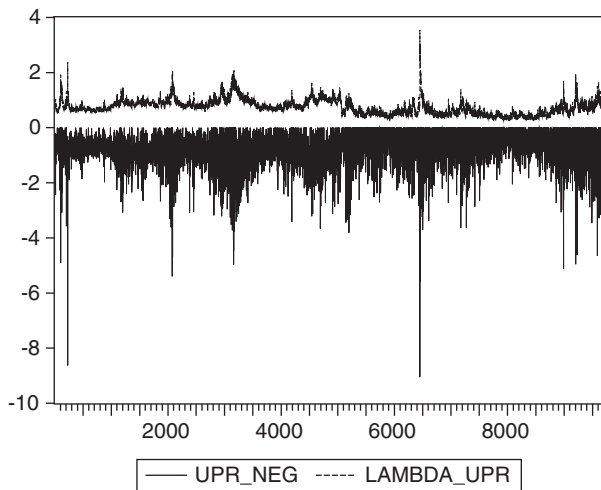
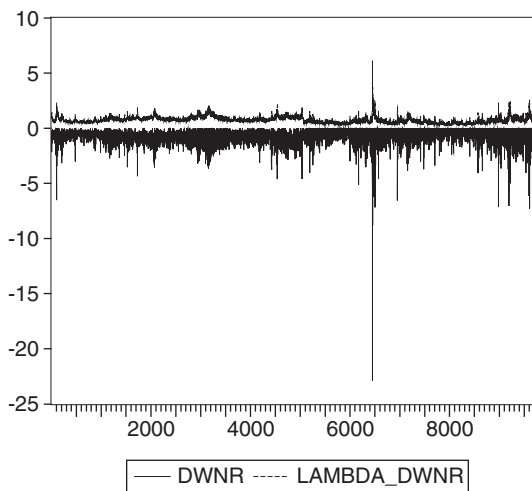*Fig. 6*.   Expected and Observed Daily UPR, 1962/1–2000/8.

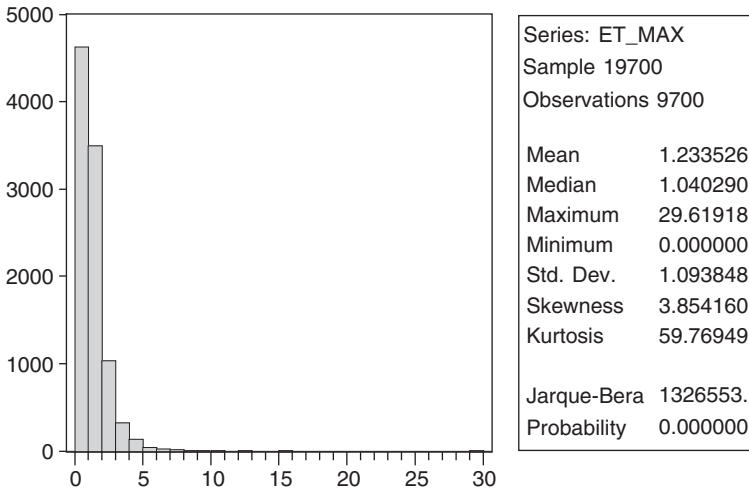*Fig. 7*.   Expected and Observed Daily DWNR, 1962/1–2000/8.

	Series: ET_MAX
	Sample 19700
	Observations 9700
	Mean 1.233526
	Median 1.040290
	Maximum 29.61918
	Minimum 0.000000
	Std. Dev. 1.093848
	Skewness 3.854160
	Kurtosis 59.76949
	Jarque-Bera 1326553.
	Probability 0.000000

Fig. 8.   Histogram of Daily et_UPR, 1962/1–2000/8.

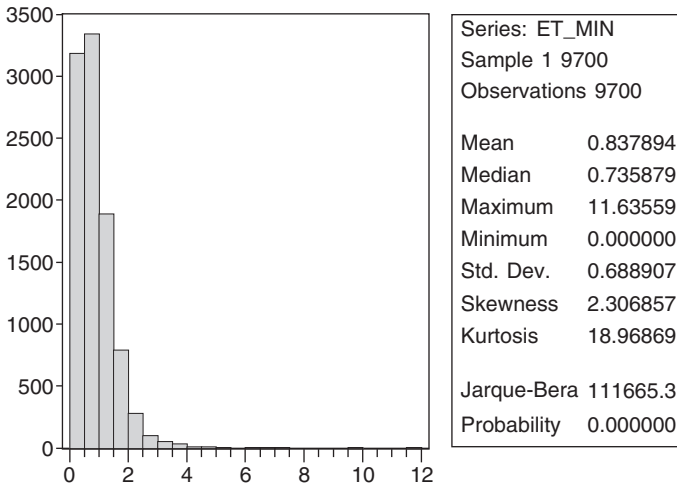|	
	Series: ET_MIN
	Sample 1 9700
	Observations 9700
	Mean 0.837894
	Median 0.735879
	Maximum 11.63559
	Minimum 0.000000
	Std. Dev. 0.688907
	Skewness 2.306857
	Kurtosis 18.96869
	Jarque-Bera 111665.3
	Probability 0.000000

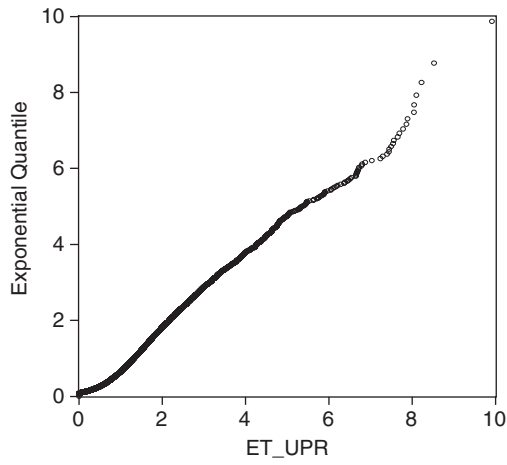*Fig. 9.*   Histogram of Daily et_DWNR, 1962/1–2000/8.
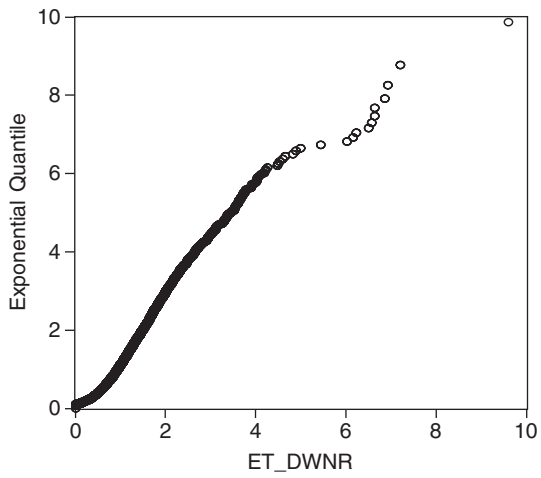
Fig. 10.   Q–Q Plot of et_UPR.

Fig. 11.   Q–Q Plot of et_DWNR.

volatility than the GARCH model. In this section, we further compare the forecasting power for volatilities of the CARR model, which ignores the asymmetry, and the ACARR model which give explicit considerations to the asymmetric structures. Given our finding of the importance of modeling asymmetry in the above section, we would expect the ACARR model to provide more accurate volatility forecast comparing with the CARR model.

Since volatility is an unobservable variable, we employ three proxies as measures of volatility (henceforth MVs). They are the daily high/low range (RNG) as defined in (2.2), the daily return squared (RETSQ) as is commonly used in the literature of volatility forecast comparisons and the absolute value of the daily returns (ARET) which is more robust to outliers than the second measure. We then use the following regressions to gauge the forecasting powers of the CARR and the ACARR models.

$$\text{MV}_t = a + b\,\text{FV}_t(\text{CARR}) + u_t \tag{3.1}$$

$$\text{MV}_t = a + b\,\text{FV}_t(\text{ACARR}) + u_t \tag{3.2}$$

$$\text{MV}_t = a + b\,\text{FV}_t(\text{CARR}) + c\,\text{FV}(\text{ACARR}) + u_t \tag{3.3}$$

$\text{FV}_t(\text{CARR})$ is the forecasted volatility using the CARR model in (2.5). $\text{FV}_t(\text{ACARR})$ is computed as the sum of the forecasted UPR and forecasted DWNR as is shown in (2.4). Proper transformations are made to adjust the difference between a variance estimator and a standard deviation estimator. Table 4 gives the estimation result.

The results are consistent for the three measures of volatility. In all cases, the forecasted volatility using ACARR dominates the forecasted volatility using CARR. In the three measures, the corresponding t-ratios for the two models are (21.83, 0.46) using RNG, (7.61, −2,32) using RETSQ and (8.09, −1.91) using ARET. Once the forecasted volatility using ACARR is included, CARR provides no additional explanatory power. Another interesting observation is that the results using range as the measured volatility look particularly favorable for the ACARR model and the absolute results are a bit weaker. In other words, the adjusted R^2 of the regression using these two measures are much smaller than that using RNG. This result is consistent with the observation in Chou (2005) that both RETSQ and ARET are based on close-to-close return data and are much more noisier than RNG which is based on the extreme values of the price.

Table 4. ACARR versus CARR.

Measured Volatility	Explanatory Variables			Adj. R^2	S.E.
	Constant	FV(CARR)	FV(ACARR)		
RNG	−0.067[−0.366]	1.005[96.29]		0.489	0.543
RNG	−0.006[−4.148]		1.047[101.02]	0.513	0.531
RNG	−0.067[0.632]	0.021[0.46]	1.026[21.83]	0.513	0.531
RETSQ	−1.203[−1.35]	0.397[14.25]		0.02	5.725
RETSQ	−0.265[−2.94]		0.459[16.02]	0.026	5.709
RETSQ	−0.249[−2.76]	−0.191[−2.32]	0.644[7.61]	0.026	5.708
ARET	1.142[7.41]	0.334[27.07]		0.07	0.642
ARET	0.113[5.85]		0.354[28.28]	0.076	0.639
ARET	0.115[5.95]	−0.106[−1.91]	0.458[8.09]	0.076	0.639

Note: In-sample Volatility Forecast Comparison Using Three Measured Volatilities as Benchmarks. The three measures of volatility are RNG, RETSQ and ARET: respectively, daily ranges, squared-daily-returns, and absoulte daily return. ACARR(1,1) model is fitted for the range series and a ACARR models are fitted for the upward range and the downward range series. FV(CARR) (FV(ACARR)) is the forecasted volatility using CARR (ACARR). FV(ACARR) is the forecasted range using the sum of the forcasted upward range and downward range. Proper transformations are made for adjusting the difference between a variance estimator and a standard-deviation estimator. Numbers in parentheses are *t*-ratios.

$$MV_t = a + b\, FV_t(CARR) + u_t$$
$$MV_t = a + c\, FV_t(ACARR) + u_t$$
$$MV_t = a + b\, FV_t(CARR) + c\, FV_t(ACARR) + u_t$$

4. CONCLUSION

The ACARR model provides a simple, yet efficient and natural framework to analyze the asymmetry of the price movement in financial markets. Applications can be used in computing the option prices where the upward (downward) range (or the maximum (minimum) return) is more relevant for computing the price of a call (put) option. Value-at-Risk is another important area for applications using the downward range dynamic model. The ACARR model is related to studies like the duration between a threshold high or low price level. Further more, the ACARR model can be used to forecast volatilities comparing with the symmetric model, CARR GARCH, and SV models, or other asymmetric volatility models like EGARCH, GJR-GARCH models. Further Monte Carlo analysis will be useful as well as

applications to other financial markets such as foreign exchanges, bonds, and commodities. Applications of the ACARR model to other frequency of range interval, say every 30 min, every hour, or every quarter, and other frequencies, will provide further understanding of the usefulness/limitation of the model. Other generalization of the ACARR model will be worthy subjects of future research, for example, the generalization of the univariate to a multivariate framework, models simultaneously treat the price return and the range data, long memory ACARR models.[12]

The ACARR model in this paper can be seen as an example of an emerging literature: applications of extreme value theory in finance. Embrecht, Kluppelberg, and Mikosch (1999) and Smith (1999), among others, are strong advocates of such an approach in studying many important issues in financial economics. Noticeable examples are Embrechts, McNeil, and Straumann (2002) for correlation of market extreme movements, McNeil and Frey (2000) for volatility forecasts, and Tsay (2000) for modeling crashes. In fact, all the static range literature (Parkinson, 1980) and the long-term dependence literature using re-scaled range (Mandelbrot, 1972; Lo, 1991) can be viewed as earlier examples of this more general broader approach to the study of empirical finance.

NOTES

1. A general data generating process for P_t can be written as

$$dP_t = \mu_t + \sigma_t dW_t$$
$$d\sigma_t = \theta_t + \kappa dV_t$$

where W_t and V_t are two independent standard Wiener processes, or Brownian motions.

2. See Lo (1991) for a similar case and a proof.

3. It is not clear to me yet how the daily highs/lows of asset prices are compiled reported on the public or private data sources such as the Wall Street Journal, Financial Times, and in CRSP. They may be computed from a very high, fixed frequency. Alternatively, they may be computed directly from the transaction data, a sampling frequency with non-fixed intervals.

4. Although in this paper we follow the approach of Engle and Russell (1998) in relying on the QMLE for estimation, it is important to recognize the fact that the limiting distribution of CARR is known while it is not the case for ACD. This issue is dealt within the later section.

5. It is of course a worthy topic for future research as to how much of efficiency gain can be obtained by utilizing the FIML with the limiting distribution to estimate the parameters. Alternatively, one can estimate the density function using non-parametric methods.

6. See Engle and Russell (1998) for a proof.

7. It is noteworthy that the closure property holds only for the weak-GARCH processes. Namely, in general, the GARCH process is not closed under aggregation. See Drost and Nijman (1993) for the discussion of the closure property of GARCH process.

8. The exact date is unknown since this change in compiling process was not documented by the company. However, from a detailed look at the data, the most likely date is April 20, 1982.

9. Mathematically, these two compiling methods are respectively, index of the highs (lows) and highs (lows) of the index. The Jensen inequality tells us that these two operations are not interchangeable.

10. For the range variable, it is consistently found that a CARR(1,1) model is sufficient to capture the dynamics for daily and weekly and for different sub-sample periods.

11. In the GARCH literature a Q-statistics for the squared normalized residuals is usually included as well to account for the remaining ARCH effect in the residual. Here we do not include such a statistics because range is by itself a measure of volatility and this statistics will be measuring the persistence of the volatility of volatility. For formal tests of the distribution, some tests can be incorporated to complement the Q–Q plots.

12. In the daily ACARR models, as is suggested by the Portmanteau statistics, the memory in range (hence in the volatility) seems to be longer than can be accounted for using the simple ACARR(1,1) or ACARR(2,1) models with short memories. However, such a phenomenon disappears in the weekly model. Given our empirical results, it is questionable whether such an attempt is useful in practice.

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