Weyl invariant black hole

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The properties of the Weyl invariant black hole and its implications are studied. The calculation shows that there is no hair, up to a gauge choice, for a Weyl gauge field outside the event horizon of a black hole in the Weyl invariant limit. We also show that a scalar field will remain constant once the scale symmetry is broken spontaneously by the well-known Higgs potential. As a result, classical hair for the Weyl vector meson and scalar measuring field vanishes strictly in the presence of a spontaneously symmetry-breaking potential. Hence the no-hair theorem holds both in the Weyl invariant limit and the symmetry-breaking phase.

PACS number(s): 04.70.Bw, 04.20.Jb, 04.50.+h

I. INTRODUCTION

The no hair theorem [1–4] has long been known as a conjecture in black hole physics. Evidence indicates that classical hair, except the electric charge Q, the gravitational mass M and the angular momentum J, cannot possibly survive beyond the event horizon of a black hole. For a more complete review, see Ref. [5]. Indeed, it was shown, for example, that the no-hair theorem holds for a neutral meson U_1 and neutrino fields interacting with a classical source on a Schwarzschild background [6, 7] as well as a class of higher spin systems in a nonspherical gravitational collapse [8, 9].

Another approach has been made, for example, in showing [3, 10–12] that the geometry of the stationary rotating black hole is axisymmetric with an event horizon topologically homeomorphic to $\mathbf{S}^2 \times \mathbf{R}^2$. It was then shown [13, 14] that the Kerr solution is the only type of solution to a black hole with a nondegenerate event horizon.

There does not, however, exist a model-independent final proof for the no-hair theorem. Therefore, we are going to discuss, specifically, the possibility of generating a classical hair in the Weyl (local scale) invariant model in this paper.

Evidence also indicates that scale symmetry has to do with the physics in many areas of interest. Therefore, the Weyl invariant model deserves more investigation. Hence we propose to study the implication of the Weyl symmetry in the formation of a massive black hole. Note that the Weyl invariant model is designed to replace all dimensionful coupling constants with dynamical field variables.

In short, we will show that classical hairs for the scalar field ϕ and the Weyl vector meson S_μ will not survive, up to a gauge choice, beyond the event horizon unless the Weyl vector meson is massless, an unphysical model

as we will discuss briefly later in this paper. We will also argue that classical scalar hair may have a chance to survive if the symmetry is broken topologically. Hence we will also discuss possible effects with the presence of a symmetry-breaking potential. It turns out that the scalar field has to remain constant if $V \neq \lambda\phi^4/8$ due to a nontrivial constraint. Hence the no-hair theorem does hold in both cases. Possible implications will be discussed too.

This paper will be organized as follows: (1) in Sec. II, we will briefly review some properties of the Weyl invariant model; (2) in Sec. III, we will show how to obtain field equations in the unitary gauge; various constraints due to the presence of an event horizon will also be analyzed; (3) in Sec. IV, we will show the no-hair theorem in the unitary gauge; (4) the effects of a spontaneous symmetry-breaking (SSB) potential and all other interesting implications will be discussed in Sec. V; (5) for completeness, we will also discuss the black hole solution in the massless limit; (6) finally, in Sec. VII, we will present some concluding remarks. The convention and the redundancy due to the Bianchi identity will be summarized in the Appendix.

II. WEYL INVARIANT MODEL

Weyl proposed the Weyl invariant action [15–19]

$$S = \int d^4x \sqrt{g} \left[-\tfrac{1}{2}\epsilon\phi^2 \tilde{R} - \tfrac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \right.$$
$$\left. - \frac{\lambda}{8}\phi^4 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \right]. \tag{1}$$

Here $\tilde{R}$ is the Weyl invariant scalar curvature obtained by replacing $\partial_\mu g_{\alpha\beta}$ with $\nabla_\mu g_{\alpha\beta} \equiv (\partial_\mu + 2S_\mu)g_{\alpha\beta}$ in the

definition of the Christoffel symbol; namely, it is obtained by replacing $\Gamma^\alpha_{\mu\nu}$ with $\tilde{\Gamma}^\alpha_{\mu\nu} \equiv \frac{1}{2}g^{\alpha\beta}(\nabla_\mu g_{\nu\beta} + \nabla_\nu g_{\mu\beta} - \nabla_\beta g_{\mu\nu})$ in the definition of R. Hence

$$\tilde{R} = R + 6(D_\mu + S_\mu)S^\mu, \tag{2}$$

accordingly. Furthermore, $\nabla_\mu \phi = (\partial_\mu - S_\mu)\phi$, $F_{\mu\nu} \equiv \partial_\mu S_\nu - \partial_\nu S_\mu$, with S_μ denoting the Weyl gauge connection. It is known that the action (1) is invariant under the Weyl transformation

$$g'_{\mu\nu} = \varphi^2 g_{\mu\nu}, \tag{3}$$

$$\phi' = \varphi^{-1}\phi, \tag{4}$$

$$S'_\mu = S_\mu - \partial_\mu \ln \varphi. \tag{5}$$

Here $\varphi = \varphi(x)$ is the gauge parameter. Note that one can gauge away the coordinate dependence of the scalar field ϕ completely by choosing $\varphi = \phi/\phi_0$, with ϕ_0 denoting a nonzero constant. It is also noted that if ϕ vanishes in some gauge, then it will vanish in all gauges. Therefore a trivial ϕ is very different from all nontrivial field values.

Note that we will adopt the signature specified by $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$ as the Minkowski metric. The definition of the curvature tensor can be read off from

$$[D_\mu, D_\nu]A_\lambda = R^\delta{}_{\lambda\nu\mu}A_\delta \tag{6}$$

directly. Furthermore, the Ricci tensor $R_{\mu\nu} \equiv R^\lambda{}_{\mu\nu\lambda}$ and the scalar curvature $R \equiv g^{\mu\nu}R_{\mu\nu}$.

It is known that all coupling constants in this theory have to be dimensionless. Therefore $2/\epsilon\phi^2$ and $\lambda\phi^4/8$ represent the well-known gravitational constant and cosmological constant, respectively. Hence ϵ, λ, and all other coupling constants are all dimensionless by construction.

Note that the scale transformation is meant to scale all dynamical fields according to its mass dimension. It is easy to show that $\tilde{\Gamma}$ is invariant under this transfor-

mation since it is of the form $g^{..}\nabla g_{..}$. Hence $\phi^2\tilde{R} \to \varphi^{-4}\phi^2\tilde{R}$, similar to $\phi^2 g^{\mu\nu}$. Hence it is scale invariant as $\sqrt{g} \to \varphi^4\sqrt{g}$ under the same transformation. It is also straightforward to show that each term in the action (1) is scale invariant on its own. Hence the ϵ parameter is nothing more than a free parameter that has been useful in numerical studies.

III. FIELD EQUATIONS

One can obtain the equation of motion for the action (1) from the variational principle with respect to $g_{\mu\nu}$ and other fields. The $g_{\mu\nu}$ equation will be slightly complicate due to the presence of the $\phi^2 R$ and $D_\mu S^\mu$ couplings. We will show briefly how to derive it in a covariant form. Note that it is straightforward to show that

$$\delta\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left[D_\mu\delta g_{\beta\nu} + D_\nu\delta g_{\beta\mu} - D_\beta\delta g_{\mu\nu}\right], \tag{7}$$

$$\delta R_{\mu\nu} = -D_\alpha\delta\Gamma^\alpha_{\mu\nu} + D_\nu\delta\Gamma_\mu. \tag{8}$$

Here $\Gamma_\mu \equiv \Gamma^\lambda_{\mu\nu}g_\lambda{}^\nu$ and note that, in writing terms such as $D_\alpha\delta\Gamma^\alpha_{\mu\nu}$, we have imagined that $\delta\Gamma^\alpha_{\mu\nu}$, $\delta\Gamma_\mu$ and $\delta g_{\mu\nu}$ are tensors of type $T(1,2)$, $T(0,1)$, and $T(0,2)$ respectively. Of course, they are not tensors. But writing Eqs. (7) and (8) as if they are proper tensors will undoubtedly simplify the algebraic structure of the above equations.

Therefore, one can easily show that

$$\sqrt{g}\phi^2 g^{\mu\nu}S_\alpha\delta\Gamma^\alpha_{\mu\nu} = \sqrt{g}[\tfrac{1}{2}g^{\mu\nu}D_\alpha(S^\alpha\phi^2) - D^\mu(S^\nu\phi^2)]\delta g_{\mu\nu}, \tag{9}$$

$$\sqrt{g}\phi^2 g^{\mu\nu}\delta R_{\mu\nu} = \sqrt{g}\left[g^{\mu\nu}D_\alpha D^\alpha\phi^2 - D^\mu D^\nu\phi^2\right]\delta g_{\mu\nu}, \tag{10}$$

after ignoring all total derivative terms. Hence the $g_{\mu\nu}$ equation can be shown to be

$$-\nabla^\mu\phi\nabla^\nu\phi - F^{\mu\alpha}F^\nu{}_\alpha - g^{\mu\nu}\mathcal{L} = \epsilon\Big[\phi^2 R^{\mu\nu} - 3\phi(S^\mu D^\nu\phi^2 + S^\nu D^\mu\phi^2) + 6S^\mu S^\nu\phi^2 + D^\mu D^\nu\phi^2$$
$$-g^{\mu\nu}(D^2\phi^2 + 3D_\alpha S^\alpha\phi^2 + 3S^\alpha D_\alpha\phi^2)\Big]. \tag{11}$$

Furthermore, the S_μ and ϕ equations of motion can be derived by varying the action (1) with respect to S_μ and ϕ, respectively. After some algebra, one has

$$D_\nu F^{\mu\nu} = (1 + 6\epsilon)\phi\nabla^\mu\phi, \tag{12}$$

$$D_\mu D^\mu\phi = \epsilon\phi R + (1 + 6\epsilon)\phi(D_\mu + S_\mu)S^\mu + \frac{\lambda}{2}\phi^3. \tag{13}$$

Since the system is Weyl invariant, one can freely choose a unitary gauge such that $\phi = \phi_0$, with ϕ_0 denoting an arbitrary constant. One would expect that the action is thus reduced to the form

$$S_0 = -\int d^4x\sqrt{g}\left[R + \frac{F^2}{4} + \frac{m^2}{2}S_\mu^2 + \Lambda\right], \tag{14}$$

upon setting $(\epsilon/2)\phi_0^2 = 1$, $\Lambda \equiv (\lambda/8)\phi_0^4$, and writing

$m^2 \equiv (6\epsilon + 1)\phi_0^2$. Note that m is used to denote the mass of the Weyl vector meson.

It is not clear at this moment if the action (14) can be considered as a complete and effective action by itself. One has to show first that Eqs. (11)–(13) are identical to the variational equations obtained from the effective action (14).

Note also that we have, however, dropped a term proportional to $\int d^4x\sqrt{g}D_\mu S^\mu$ in bringing the action to the form specified by S_0 given above since $\sqrt{g}D_\mu S^\mu = \partial_\mu(\sqrt{g}S^\mu)$; namely, we have dropped a surface term. This surface term will bring us a constraint on S_μ. It turns out, however, that this term will not affect the equation of motion of the system. Indeed, one can easily show that equations of motion derived from varying the action (14) with respect to $g_{\alpha\beta}$ and S_α are identical to Eqs. (11) and

(12) in the unitary gauge $\phi = \phi_0$. Indeed, one finds that Eqs. (11)–(13) become

$$G_{\mu\nu} = \frac{1}{2}F_{\mu\alpha}F_{\nu}{}^{\alpha} + \frac{m^2}{2}S_\mu S_\nu + \frac{1}{2}g_{\mu\nu}\mathcal{L}_0, \qquad (15)$$

$$D_\mu F^{\mu\nu} = m^2 S^\nu, \qquad (16)$$

$$R = -\frac{m^2}{2}(D_\mu + S_\mu)S^\mu - 2\Lambda, \qquad (17)$$

in the unitary gauge. Here $G_{\mu\nu} \equiv \frac{1}{2}g_{\mu\nu}R - R_{\mu\nu}$ is the Einstein tensor, and $\mathcal{L}_0 \equiv -\frac{1}{4}F^2 - \frac{1}{2}(m^2)S_\mu^2 - \Lambda$ denotes the Lagrangian density of S_μ and Λ in the unitary gauge. By taking the trace of Eq. (15) one obtains, however,

$$R = -\frac{m^2}{2}S_\mu S^\mu - 2\Lambda. \qquad (18)$$

Therefore one has, from comparing with Eq. (17), $D_\mu S^\mu = 0$ as promised. Note this constraint also follows from the covariantly differentiating equation (16). Hence one concludes that the action (14) can be considered as an *effective* action on its own.

Moreover, the extra mass term is of course a very unique feature of the Weyl vector meson. One might suspect that if one tries to tune the parameter ϵ such that $m = 0$, namely, by setting $\epsilon = -1/6$, the effective action (14) may be equivalent to the action of the electromagnetic field. We will show shortly that this is not true by noting that $m^2 = 0$ will require that $\epsilon = -1/6$ which will turn a graviton into a ghost field.

Note that the action (14) is effectively, in the unitary gauge, the massive Proca gauge field minimally coupled to gravity with a cosmological constant term. A similar model has been studied with a method relying on the Green's function[1] [7, 8, 14]. We will try to show clearly what the effect of the mass term is in the proof of the no-hair theorem in the unitary gauge. We will also show clearly why a monopole configuration cannot exist due to the constraint $D_\mu S^\mu = 0$, a reflection of the dropped-out surface term. Our argument, different from the previous proof, will rely on this nontrivial constraint.

Note also that our theory is not exactly the Proca theory minimally coupled to gravity. They are equivalent only in the unitary gauge. Moreover, due to the special coupling, the mass of the massive Weyl vector meson is expected to be of the order of the Planck scale [19]. And the scalar hair may survive, as we shall argue shortly, if the scale symmetry is broken topologically with nontrivial boundary conditions. Furthermore, S_μ is the Weyl vector meson for the connection responding to the scale transformation. It is not the same *gauge* connection responding to the conventional *phase* transformation.

We will study the effect of the Weyl gauge connection S_μ near a spherically symmetric black hole. Therefore, one will assume the spherically symmetric ansatz

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[1] We were informed of the work of Ref. [7] after this paper was completed.

$$ds^2 \equiv g_{ab}dx^a dx^b = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2 d\Omega, \qquad (19)$$

$$S_\mu = S_\mu(r) = (\varphi(r), f(r), 0, 0), \qquad (20)$$

in spherical coordinates. Here $d\Omega$ is the solid angle. Moreover, $\varphi(r)$ and $f(r)$ are both real functions of r. One can show that the constraint equation $D_\mu S^\mu = 0$ becomes

$$\partial_r(e^{A-B}r^2 S_r(r)) = 0, \qquad (21)$$

in spherically symmetric space. Hence one has

$$S_r(r) = \text{const} \times \frac{e^{B-A}}{r^2}. \qquad (22)$$

We will argue that the regularity requirement on S_r will demand the vanishing of S_r. This can be done by observing the behavior of an infalling test particle near the event horizon of a black hole.

Indeed, given a test particle running on the geodesic parametrized by its proper length, i.e., $x^\mu = x^\mu(s)$, the geodesic equation reads [20]

$$\frac{dV^\mu}{ds} + \Gamma^\mu_{\nu\alpha}V^\nu V^\alpha = 0. \qquad (23)$$

Here $V^\mu \equiv dx^\mu/ds$ is the four-velocity of the test particle. In the case of a radially infalling test particle, i.e., $V^\mu = (V^t, V^r, 0, 0)$, one finds

$$V^t = V_\infty e^{-2A}, \qquad (24)$$

from the $\frac{dV^t}{ds}$ equation. Here $V_\infty \equiv V^t(r = \infty)$ is an integration constant. Hence, one has

$$V^r = \pm|V_\infty|e^{-A-B}(1 + V_\infty^{-2}e^{2A})^{1/2}, \qquad (25)$$

due to the fact that $V_\mu V^\mu = 1$. Therefore one has, from $dt = \frac{V^t}{V^r}dr$,

$$\Delta t = \left| \int_r^{r_H} \frac{e^{B-A}dr'}{(1 + e^{2A}V_\infty^{-2})^{1/2}} \right|. \qquad (26)$$

Here Δt is the coordinate time the test particle needs to fall from r to r_H. Note that we have assumed that a horizon exists at $r = r_H$. Once one gets close enough to $r = r_H$, the integral can be expanded as a power series in $(r - r_H)$, namely, $e^{B-A} \propto (r - r_H)^a$, while $1 + e^{2A}V_\infty^{-2} = 1 + 1/(V^t V_\infty)$ remains finite as one approaches r_H. In fact, V^t is expected to slow down exponentially near r_H. Here we keep only the leading power which is enough for our purpose. Note that e^{B-A} has to go to infinity at $r = r_H$ in order that a horizon can exist. Thus one easily finds that $a = -(1 + \epsilon_0)$ with $\epsilon_0 > 0$. In other words,

$$e^{A-B} \propto (r - r_H)^{1+\epsilon_0}. \qquad (27)$$

Note that the boundary condition $\phi_A(r_H) = \infty$ is not acceptable as any regular physical field value. Hence the regularity of S_μ demands, from Eq. (22), that $S_r = 0$. Therefore the only nonvanishing Weyl gauge connection is $S_t \equiv \varphi(r)$. Hence the only nonvanishing field strength $F_{\mu\nu}$ is F_{tr} accordingly.

In fact, the only nonvanishing $F_{\mu\nu}$ component is $F_{rt} =$

φ' in spherically symmetric space specified by Eq. (20). The S_r component cannot show up in F_{rt} as the system is assumed to be static. Equation (22) shows that a black hole exists in a fully consistent manner that the S_r component should decouple completely. Note also that a monopole of the form $F \equiv 2Q_m(r)\sin\theta d\theta \wedge d\varphi$ cannot satisfy Eq. (16) in the massive limit. Indeed, $D_\mu F^{\mu\varphi}$ can be shown to vanish identically in the presence of the monopole field tensor given above. Therefore, the monopole term cannot survive in the massive limit.

IV. NO-HAIR THEOREM IN THE UNITARY GAUGE

Note that, by using the equation

$$[D_\mu, D_\nu]A_\alpha = R^\beta{}_{\alpha\nu\mu}A_\beta \tag{28}$$

and the identity $D_\nu S^\nu = 0$, one can reduce Eq. (12) to the form

$$(D_\mu D^\mu)S_\nu + R_{\nu\mu}S^\mu = m^2 S_\nu. \tag{29}$$

Therefore, one has

$$\partial^2 S_\mu = m^2 S_\mu, \tag{30}$$

asymptotically, since $g_{\mu\nu} \to \eta_{\mu\nu}$ as $r \to \infty$. Hence Eq. (30) indicates that

$$S_\mu \to \frac{e^{-mr}}{r}, \tag{31}$$

at spatial infinity. This indicates that the no-hair theorem for the Weyl vector meson may have to do with the presence of the mass term. This is a common feature of massive gauge fields. We will discuss the role played by the mass term shortly.

Defining $S_\varphi = \int dr \sqrt{g_r}\mathcal{L}_\varphi$ as the effective action for φ, one finds

$$S_\varphi = \int dr \left[\frac{1}{2}r^2\varphi'^2 e^{-A-B} + \frac{m^2}{2}r^2\varphi^2 e^{B-A} \right]. \tag{32}$$

Here we have denoted $\partial_r A$ as A' while $\sqrt{g_r} \equiv r^2 e^{A+B}$ denotes the radially relevant part of g. Also $\mathcal{L}_\varphi \equiv -F^2/4 - (m^2/2)S_\mu^2$. Note that in defining the effective action S_φ, we have ignored the irrelevant angular part as well as the t integral. The field Eq. of φ can be obtained either by varying S_φ or by solving Eq. (16) directly. It is easy to show that both ways give the same result. That is the reason we say S_φ represents the effective action of A, B, and φ. After some algebra, one finds

$$\left[r^2\varphi' e^{-A-B} \right]' = m^2 r^2 e^{B-A}\varphi. \tag{33}$$

Equation (33) can be rearranged as

$$\left[r^2 e^{-A-B}\varphi\varphi' \right]' = r^2 e^{-A-B}\varphi'^2 + m^2 r^2 e^{B-A}\varphi^2. \tag{34}$$

Upon integrating Eq. (34) by $\int_{r_H}^\infty dr$, the left-hand side (LHS) of Eq. (34) becomes a surface term while the RHS of Eq. (34) is a positive integral. One can hence reach the conclusion $\varphi = \varphi' = 0$ provided that the LHS of Eq. (34) vanishes. Indeed, its LHS

$$\left. r^2 e^{-A-B}\varphi\varphi' \right|_{r_H}^\infty \tag{35}$$

can be shown to vanish by noting that $r^2\varphi\varphi' \to r^{-2-2\delta}$. This is because the regularity of S_φ demands $\varphi \to r^{-\frac{3}{2}-\delta}$, and hence $\varphi' \to r^{-\frac{5}{2}-\delta}$ accordingly at spatial infinity. Note that in the absence of the mass term, the regularity only requires that $\varphi' \to r^{-\frac{3}{2}+\delta}$ and hence $\varphi \to r^{-\frac{1}{2}+\delta}$ accordingly. Therefore, $r^2\varphi\varphi' \to r^{-2\delta}$ in the massless limit. Here δ is a small positive constant. In both cases $m = 0$ or $m \neq 0$, the contributions from the infinity vanish accordingly. Note that the expression $f(r) \to r^{-n\pm\delta}$ means that the function $f(r)$ goes to 0 slower $(+)$ or faster $(-)$ than r^{-n} at spatial infinity. Here $2n$ is an integer.

Moreover, the regularity of S_φ at r_H gives

$$\left. \varphi'^2 e^{-A-B} \right|_{r_H} < \infty, \tag{36}$$

$$\left. \varphi^2 e^{B-A} \right|_{r_H} < \infty. \tag{37}$$

Careful analysis shows that $\varphi^2|_{r_H} \to (r - r_H)^{1+\epsilon_1}$ with $\epsilon_1 \geq \epsilon_0$ from Eq. (37). Hence $\varphi'|_{r_H} \to (r - r_H)^{-\frac{1}{2}+\frac{\epsilon_1}{2}}$. Therefore, $\varphi\varphi'|_{r_H} \to (r - r_H)^{\epsilon_1}$. Furthermore, Eq. (36) indicates that $e^{-A-B}|_{r_H} \to (r - r_H)^{1+\epsilon_2}$ with $\epsilon_1 + \epsilon_2 \geq 0$. Hence $\varphi\varphi' e^{-A-B}|_{r_H} \to (r - r_H)^{1+\epsilon_1+\epsilon_2}$. Therefore, one finds that

$$\left. \varphi\varphi' e^{-A-B} \right|_{r_H} = 0. \tag{38}$$

Hence the LHS of the Eq. (34) vanishes as promised.

Note, however, that this means that

$$\phi = \psi(r)\phi_0, \tag{39}$$

$$S_\mu = \partial_\mu \ln\psi(r) \tag{40}$$

are black hole solutions of the system too. Therefore one concludes that there is no hair, up to a gauge choice, for massive Weyl gauge fields.

Hence the only solution of the Weyl invariant model is

$$S_\mu = 0, \tag{41}$$

$$\phi = \phi_0, \tag{42}$$

in the unitary gauge. Therefore, the rest of the equations of motion become

$$G_{\mu\nu} = -\frac{\lambda}{8\epsilon}\phi_0^2 g_{\mu\nu}. \tag{43}$$

The solution to above equation is the well-known Schwarzschild solution with a cosmological constant. Hence one concludes that there is no classical hair, up to a gauge choice, for the scalar measuring field and the Weyl vector meson. Note that choosing a different gauge will in general turn on a gauge hair. This is, however, not a physical hair in the scale invariant limit.

V. NO-HAIR THEOREM WITH A SPONTANEOUSLY SYMMETRY-BREAKING POTENTIAL

A nontrivial physical hair may survive if the scale symmetry is broken, dynamically or topologically. As a result, the field values of ϕ at the event horizon may be different from its value at the spatial infinity if they are

frozen at different values topologically. Indeed, if one assumes that $\phi(\infty) \neq \phi(r_H)$, similar to a spring with fixed ends assuming different energy states at each end, then a nontrivial hair has to exist in order to compromise the gauge choice. One has to admit that it is not clear how to realize the symmetry-breaking effect appropriately. One popular way to break the symmetry is to introduce the Higgs potential. A similar model has also been studied in its application to cosmology [21]. One hopes that the scalar field may have to take a nontrivial value in order to compromise the symmetry-breaking effect. This does not, however, turn out to be the case if one adopts a Higgs mechanism to achieve SSB.

Note that the $g_{\mu\nu}$ equation and S_μ equation of the system in the presence of a symmetry-breaking potential V can be shown to be the same as Eqs. (11) and (12) except that one should replace the $\frac{\lambda}{8}\phi^4$ term with V in Eq. (11). Moreover, the ϕ equation becomes

$$D_\mu D^\mu \phi = \epsilon \phi R + (1 + 6\epsilon)\phi(D_\mu + S_\mu)S^\mu + \partial_\phi V. \quad (44)$$

It is then straightforward to compute the trace of Eq. (11) which gives

$$\nabla^\mu \phi \nabla_\mu \phi + 4V = -\epsilon(\phi^2 R + 6\phi^2 S^2 + 3D^2\phi^2 - 6S \cdot D\phi^2). \quad (45)$$

Comparing it with the ϕ equation (44), one finds that

$$\frac{1}{2}(1 + 6\epsilon)D^\mu(\nabla_\mu \phi^2) = \phi\frac{\partial V}{\partial \phi} - 4V. \quad (46)$$

Note that Eq. (46) becomes

$$(1 + 6\epsilon)D_\mu S^\mu = \left[4V - \phi\frac{\partial V}{\partial \phi}\right]_{\phi_0} \quad (47)$$

if $\phi = \phi_0$. If $V = \lambda\phi^4/8$, one then reproduces the result

$$D_\mu S^\mu = 0, \quad (48)$$

in the unitary gauge. Note that one also has a current conservation law from Eq. (12): namely,

$$D_\mu D_\nu F^{\mu\nu} = 0 = \frac{1}{2}(1 + 6\epsilon)D^\mu \nabla_\mu \phi^2. \quad (49)$$

Equations (46) and (49) hence show that

$$\phi\frac{\partial V}{\partial \phi} = 4V. \quad (50)$$

If one wishes to study the effect of a small symmetry-breaking potential term such as

$$V = \frac{\lambda}{8}(\phi^2 - v^2)^2, \quad (51)$$

Eq. (50) will simply demand that

$$\lambda v^2(\phi^2 - v^2) = 0. \quad (52)$$

Hence one has $\lambda v^2 = 0$ or $\phi^2 = v^2$. Note that $\lambda = 0$ represents the $V = 0$ case, while $v^2 = 0$ represents the scale invariant limit $V = \lambda\phi^4/8$. These two cases do not

represent the deviated effect we wish to study.

Therefore, one has $\phi^2 = v^2$ in the presence of the SSB potential. This indicates that the only solution to the SSB Weyl model is the same as the solution of the Weyl invariant model in the unitary gauge with $\phi_0 = \pm v$. Hence one has $\Lambda = V(\phi = \phi_0) = 0$. Note that the equation of motion for the metric $g_{\mu\nu}$ now reads

$$G_{\mu\nu} = 0, \quad (53)$$

in the broken phase. We will present the solution to above equation for completeness. Note that the G_{tt} and G_{rr} equations become

$$1 - 2rB' - e^{2B} = 0, \quad (54)$$

$$1 + 2rA' - e^{2B} = 0. \quad (55)$$

Hence (54) and (55) give

$$(A + B)' = 0. \quad (56)$$

Hence $A + B = $ const. Let $A + B = k$ with k denoting the constant. Equation (55) then becomes

$$2rB' = 1 - e^{2B}, \quad (57)$$

which can be solved to give

$$e^{-2B} = 1 - \frac{\ell}{r}. \quad (58)$$

Here ℓ is an integration constant too. Hence

$$e^{2A} = e^k\left(1 - \frac{\ell}{r}\right). \quad (59)$$

Note that the Weyl invariant system can only admit the above solution in the unitary gauge. In general, one can impose a constraint by requiring $e^{2A} \to 1$ at spatial infinity. Hence one has

$$e^{2A} = 1 - \frac{2M}{r}, \quad (60)$$

$$e^{2B} = \left(1 - \frac{2M}{r}\right)^{-1}, \quad (61)$$

with M denoting the gravitational Arnowitt-Deser-Misner (ADM) mass. Note that, in contrast to the Weyl invariant model, there is no gauge degree of freedom in the symmetry-breaking model here. Hence there is no classical hair for the Weyl vector meson and the scalar measuring field even if a symmetry-breaking term is introduced. In short, our result presents further evidence of the no-hair theorem.

VI. THE MASSLESS LIMIT

Note that one can directly solve the equation of φ and obtain

$$\varphi' = \text{const} \times \frac{e^{A+B}}{r^2}, \quad (62)$$

in the massless limit where $\epsilon = -1/6$. Hence the regularity of $F_{\mu\nu}$ only requires $e^{A+B}\big|_{r_H}$ being regular. In other

words, $e^{A+B}\big|_{r_H} < \infty$. Therefore the value of φ at the horizon does not need to vanish. Indeed, when $\epsilon = -1/6$, the action (1) becomes

$$S = \int d^4x \sqrt{g} \left[\frac{1}{12}\phi^2 R - \frac{1}{2}D_\mu\phi D^\mu\phi - \frac{\lambda}{8}\phi^4 - \frac{1}{4}F^2 \right],$$
(63)

plus a surface term. Note that, in the massless limit, the Weyl gauge connection decoupled completely. This is because the action (63) is known to be Weyl invariant since the combination $\phi^2 R/12 - D_\mu\phi D^\mu\phi/2$ is scale invariant by itself. Also note that, to be a physical theory of gravitons, one should write the action (63) as

$$S \equiv \int d^4x \sqrt{g} \left(-\frac{1}{12}\phi^2 R + \frac{1}{2}D_\mu\phi D^\mu\phi + \frac{\lambda}{8}\phi^4 + \frac{F^2}{4} \right),$$
(64)

in order that the gravitational constant remain positive. This turns, however, ϕ and S_μ into ghost fields with negative kinetic energy. Although this is not a physical model any longer, similar systems have been studied by Bekenstein [22]. Therefore, we will proceed to obtain its solution for completeness.

By choosing the unitary gauge $\phi = \phi_0$, one finds that the action (64) becomes

$$S = \int d^4x \sqrt{g} \left[-\frac{1}{12}\phi_0{}^2 R + \frac{\lambda}{8}\phi_0^4 + \frac{F^2}{4} \right].$$
(65)

Hence the equation of motion becomes

$$G_{\mu\nu} = -\frac{6}{\phi_0{}^2}\left[F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F^2 \right] + \frac{6}{\phi_0{}^2}\Lambda g_{\mu\nu},$$
(66)

$$D_\mu F^{\mu\nu} = 0.$$
(67)

Here $\Lambda \equiv \lambda\phi_0^4/8$ as usual.

It is then straightforward to find that $D_\mu F^{\mu\nu} = 0$ gives

$$\partial_r(e^{-2A-2B}\varphi') + e^{-2A-2B}\left(A' + B' + \frac{2}{r} \right)\varphi' = 0,$$
(68)

which can be integrated directly. The result is

$$F_{rt} = \frac{q^2}{r^2}e^{A+B}.$$
(69)

Here q is the Weyl charge associated with the Weyl vector meson. Note that the $g_{\mu\nu}$ equation (66) now reads

$$G_{\mu\nu} = -\frac{6}{\phi_0^2}(T_{\mu\nu} - \Lambda g_{\mu\nu}),$$
(70)

with $T_{tt} = \frac{q^2}{2r^4}e^{2A}$ and $T_{rr} = -\frac{q^2}{2r^4}e^{2B}$.

Hence one has the G_{tt} and G_{rr} equations

$$1 - 2rB' - e^{2B} = \frac{6}{\phi_0^2}\left(\frac{1}{2}\frac{q^2}{r^2} + \Lambda r^2 \right)e^{2B},$$
(71)

$$1 + 2rA' - e^{2B} = \frac{6}{\phi_0^2}\left(\frac{1}{2}\frac{q^2}{r^2} + \Lambda r^2 \right)e^{2B}.$$
(72)

Note that one can follow a similar procedure, as in the massive case, to find

$$e^{2A} = e^{-2B} = 1 - \frac{2M}{r} + \frac{1}{\phi_0^2}\left(2\Lambda r^2 - \frac{3q^2}{r^2} \right).$$
(73)

Note also that one can of course make an arbitrary gauge transformation to obtain all gauge equivalent solutions. Moreover, the above solution is similar to the Reissner-Nordstrom solution except that the Weyl vector meson is not quite like the Maxwell photon.

VII. CONCLUSION

We have studied the general properties of a Weyl invariant model in the presence of the black hole event horizon. We have also shown that there is no hair, up to a gauge choice, for the scalar measuring field ϕ and the massive Weyl vector meson S_μ in the scale invariant limit. It was also argued that nontrivial hair may exist if the scale symmetry is broken topologically. Once a symmetry breaking term is introduced, it was shown, however, that ϕ has to be equal to one of the roots of the algebraic equation (50). For example, $\phi = \pm v$ if the SSB potential $V = \frac{\lambda}{8}(\phi^2 - v^2)^2$ is introduced. Hence ϕ hair cannot exist once the symmetry is broken in this way. Possible and interesting implications have also been discussed.

In summary, we have shown that the classical hair of ϕ and S_μ cannot survive, up to a gauge choice in the scale invariant limit, whether the local scale symmetry is broken or not. The only chance is to break the symmetry topologically, although it is not clear how to realize different constraints imposed at the horizon or at spatial infinity. We also present the black hole solution in the massless limit, an unphysical model as it turns out.

ACKNOWLEDGMENTS

APPENDIX: NOTATION AND REDUNDANCY

Note that the curvature tensor $R^\beta_{\mu\nu\alpha}(g_{\mu\nu})$ used in this paper is defined by the equation

$$[D_\mu, D_\nu]A_\alpha = R^\beta_{\alpha\nu\mu}A_\beta,$$
(A1)

i.e., $R^\beta_{\mu\nu\alpha} = -\partial_\alpha\Gamma^\beta_{\mu\nu} - \Gamma^\lambda_{\mu\nu}\Gamma^\beta_{\alpha\lambda} - (\nu \leftrightarrow \alpha)$. Here $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol or spin connection of the covariant derivative, namely, $D_\mu A_\nu \equiv \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu}A_\alpha$. To be more specific, $\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu})$. Here we use $\mu, \nu, \alpha = 0, 1, 2, 3$ and $i, j, k = 1, 2, 3$ to denote time-space and spatial indices, respectively. The Ricci tensor is defined as $R_{\mu\nu} \equiv R^\alpha_{\mu\nu\alpha}$ and the scalar curvature is defined as $R \equiv R_{\mu\nu}g^{\mu\nu}$. Note that our definition of $R_{\mu\nu}$ is the same as the one in Ref. [20].

Note that the only nonvanishing spin connections, in the spherically symmetric space specified by Eq. (19), are

$$\Gamma^t_{tr} = A', \tag{A2}$$

$$\Gamma^r_{tt} = A' e^{2(A-B)}, \tag{A3}$$

$$\Gamma^r_{rr} = B', \tag{A4}$$

$$\Gamma^r_{\theta\theta} = -e^{-2B} r, \tag{A5}$$

$$\Gamma^r_{\phi\phi} = \Gamma^\phi_{\theta\theta} \sin^2\theta, \tag{A6}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r}, \tag{A7}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \tag{A8}$$

$$\Gamma^\phi_{\theta\phi} = \cot\theta. \tag{A9}$$

Therefore the only nonvanishing $\Gamma_\mu \equiv \Gamma^\alpha_{\mu\alpha}$ and $\Gamma^\mu \equiv \Gamma^\mu_{\alpha\beta} g^{\alpha\beta}$ are

$$\Gamma_r = (A+B)' + \frac{2}{r}, \tag{A10}$$

$$\Gamma_\theta = \cot\theta, \tag{A11}$$

$$\Gamma^r = e^{-2B}\left(B' - A' - \frac{2}{r}\right), \tag{A12}$$

$$\Gamma^\theta = -\frac{1}{r^2}\cot\theta. \tag{A13}$$

Note that the Ricci curvature tensor is given by the equation

$$R_{\mu\nu} = -\partial_\alpha \Gamma^\alpha_{\mu\nu} + \partial_\nu \Gamma_\mu - \Gamma^\alpha_{\mu\nu}\Gamma_\alpha + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}. \tag{A14}$$

Hence one has

$$R_{tt} = e^{2(A-B)}\left(-A'' + A'B' - A'^2 - \frac{2}{r}A'\right), \tag{A15}$$

$$R_{rr} = A'' - A'B' + A'^2 - 2\frac{B'}{r}, \tag{A16}$$

$$R_{\theta\theta} = e^{-2B}(1 + rA' - rB') - 1, \tag{A17}$$

$$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta, \tag{A18}$$

as all the nonvanishing Ricci curvature components. Moreover, the scalar curvature is defined as $R \equiv R_{\mu\nu}g^{\mu\nu}$. Therefore, one has

$$R = 2e^{-2B}\left[A'' - A'B' + A'^2 + \frac{2}{r}(A' - B') + \frac{1}{r^2}\right] - \frac{2}{r^2}. \tag{A19}$$

The Einstein tensor is defined as $G_{\mu\nu} \equiv \frac{1}{2}g_{\mu\nu}R - R_{\mu\nu}$. Hence one can list all nonvanishing Einstein tensor components:

$$G_{tt} = \frac{1}{r^2}e^{2A}[1 + e^{-2B}(2rB' - 1)], \tag{A20}$$

$$G_{rr} = \frac{1}{r^2}(1 + 2rA' - e^{2B}), \tag{A21}$$

$$G_{\theta\theta} = r^2 e^{-2B}\left[A'' - A'B' + A'^2 + \frac{1}{r}(A' - B')\right], \tag{A22}$$

$$G_{\phi\phi} = G_{\theta\theta}\sin^2\theta. \tag{A23}$$

Note that the field equation obtained by varying the metric can always be written as the form

$$G_{\mu\nu} = T_{\mu\nu}. \tag{A24}$$

This equation can be rewritten as

$$\tilde{G}_{\mu\nu} \equiv G_{\mu\nu} - T_{\mu\nu} = 0. \tag{A25}$$

One knows that some of the ten equations in $\tilde{G}_{\mu\nu} = 0$ are redundant in many cases due to the fact $D^\mu \tilde{G}_{\mu\nu} = 0$. Here $D^\mu G_{\mu\nu} = 0$ is the Bianchi identity and $D^\mu T_{\mu\nu} = 0$ is the on shell energy-momentum tensor conservation law. Therefore, one should find out what parts of the equations are redundant before one sets out to solve the equations of motion. This can be done by noting that $D^\mu \tilde{G}_{\mu\nu} = 0$ becomes

$$\partial^\mu \tilde{G}_{\mu\nu} = \Gamma^\mu \tilde{G}_{\mu\nu} + \Gamma^\alpha_{\mu\nu}\tilde{G}^\mu_\alpha. \tag{A26}$$

Because $\tilde{G}_{\mu\nu} \neq 0$ only when $\mu = \nu$, one then finds

$$\partial^r \tilde{G}_{rr} = \Gamma^r \tilde{G}_{rr} + \Gamma^t_{tr}\tilde{G}^t_t + \Gamma^r_{rr}\tilde{G}^r_r, \tag{A27}$$

$$e^{-2B}\partial_r \tilde{G}_{rr} = e^{-2B}\left(-A' - \frac{2}{r}\right)\tilde{G}_{rr} - e^{-2A}A'\tilde{G}_{tt}, \tag{A28}$$

$$\tilde{G}_{\phi\phi} = \sin^2\theta\tilde{G}_{\theta\theta}, \tag{A29}$$

in the space with a metric given by Eq. (19). Note that Eq. (A28) indicates that $\tilde{G}_{rr} = 0$ will imply $\tilde{G}_{tt} = 0$. On the contrary, Eq. (A28) only implies

$$\partial_r(e^A r^2 \tilde{G}_{rr}) = 0 \tag{A30}$$

if $\tilde{G}_{tt} = 0$. Hence $\tilde{G}_{tt} = 0$ merely implies

$$\tilde{G}_{rr} = \text{const} \times \frac{e^{-A}}{r^2}. \tag{A31}$$

Moreover, Eq. (A29) indicates that $\tilde{G}_{\phi\phi}$ and $\tilde{G}_{\theta\theta}$ are related to each other linearly. Hence one should manage to solve either the combinations of equations (a) $\tilde{G}_{rr} = 0$ and $\tilde{G}_{\theta\theta} = 0$ or (b) $\tilde{G}_{rr} = 0$ and $\tilde{G}_{\phi\phi} = 0$ since $\tilde{G}_{tt}$ is in fact redundant, but $\tilde{G}_{rr} = 0$ is not redundant.

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