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GRAVITY-TYPE INTERACTIVE MARKOV MODELS—PART II: LYAPUNOV STABILITY OF STEADY STATES*

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ABSTRACT. In Part I of this paper (Smith and Hsieh, 1997) a programming formulation of steady states was developed for gravity-type interactive Markov chains in terms of their associated spatial-flow chains. These results are here applied to analyze the stability properties of interactive Markov chains. In particular, the objective function for this programming formulation is shown to constitute a Lyapunov function for an appropriately defined continuous-time version of spatial-flow chains. The Lyapunov stability properties of these spatial flows are then shown to yield corresponding stability properties for the continuous-time versions of interactive Markov chains. In particular, these processes always exhibit global convergence to steady states. Finally, it is shown that when steady states are unique, these convergence results are inherited by those interactive Markov chains that are 'sufficiently close' to their continuous-time versions.

1. INTRODUCTION

This paper constitutes the second part of the steady-state analysis of gravity-type interactive Markov chains begun in Smith and Hsieh (1997), here designated as Part I. In that paper a programming formulation of steady states was developed in terms of the spatial-flow chains defined by such processes. The main objective of the present paper is to show that this programming approach can also be used to analyze stability properties. As with most discrete-time processes, interactive Markov chains can be highly unstable when period-to-period adjustments are large. Hence it is appropriate to allow period lengths (adjustment sizes) to become arbitrarily small, and to focus on the limiting continuous-time versions of such chains. These limiting versions, designated as *interactive Markov processes,* were formalized in the final section of Part I, along with their associated *spatial-flow processes.* It was also observed that this correspondence is nonunique, and that there are generally many 'flow versions' of interactive Markov processes.

In the present paper, we develop a *symmetric flow version* of gravity-type interactive Markov processes that can be completely analyzed in terms of the programming formulation in Part I. In particular, it is shown that the corresponding objective function is a global Lyapunov function for this symmetric flow version on the invariant set of symmetric flows (which contains all

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steady states of the process). Hence all standard Lyapunov stability properties can be established for these processes. Such stability properties are in turn shown to be inherited by the interactive Markov processes that they generate. Specifically, it is shown that all gravity-type interactive Markov processes necessarily converge to their set of steady states (so that no 'cycling' behavior is possible). Since this implies that unique steady states must be asymptotically globally stable, it then follows from the results in Part I that gravity-type interactive processes with pure-congestion effects must be globally asymptotically stable. More generally, it follows that the set of locally asymptotically stable steady states for any gravity-type interactive Markov process must correspond precisely to the set of isolated local minima for its associated Lyapunov function. Finally, it is shown that for the case of unique steady states, global convergence is inherited by all gravity-type interactive Markov chains with sufficiently small adjustments, that is, which are 'sufficiently close' to their continuous-time versions.

To establish these results, we begin in the next section by focusing explicitly on flow versions of *gravity-type* interactive Markov processes. In particular, the important class of *symmetric flow versions* is developed. The key Lyapunov property of the objective function in Part I is then established in Section 3, and is employed to obtain a number of stability properties of interactive Markov processes. Finally, global asymptotic stability of the corresponding interactive Markov chains is treated in Section 4. The paper concludes in Section 5 with a brief discussion of possible extensions of the present results. To minimize restatement, we shall refer directly to definitions and results in Part I by including 'I' in their designations [e.g., Definition I.3 and expression (I.5)].

2. GRAVITY-TYPE SPATIAL FLOW PROCESSES

Given the general properties of flow versions for interactive Markov processes developed in Part I, we now focus on the flow versions of *gravity-type interactive Markov processes.* To do so, we first show that all steady-state properties of the spatial flow chains in Part I are inherited by these continuous flow versions. We then develop the important class of *symmetric flow versions* of interactive Markov processes that will form the focus of our subsequent analysis.

Steady States for Flow Versions

We begin by showing that the steady states, $S(\mathbf{\Phi})$, for each flow version $(\mathbf{\Phi}, \sigma)$ of a given interactive Markov process $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$ are necessarily equivalent to steady states of the corresponding spatial-flow chains in Part I. To do so, recall first from Theorem I.3 that if $S(\mathbf{M}^{\mathbf{a},\mathbf{f}})$ denotes the class of *steady states* for $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$, and if $SFD[\mathbf{a}, \mathbf{f}]$ denotes the class of *steady state flow distributions,* $\mathbf{P} \in \mathbb{P}_{q\times q}$, satisfying

$$(1) \qquad P(ij) = \mu a_i[P(\cdot i)]a_j[P(\cdot j)]f_{ij}, \qquad ij \in I \times I$$

for some $\mu > 0$, then the mapping, $\mathbf{P}^{\mathbf{a},\mathbf{f}}: S(\mathbf{M}^{\mathbf{a},\mathbf{f}}) \to \mathbb{P}_{q\times q}$, defined for all $\mathbf{p} \in S(\mathbf{M}^{\mathbf{a},\mathbf{f}})$ by

$$\mathbf{P}_{\mathbf{p}}^{\mathbf{a},\mathbf{f}}(ij) = \mu(\mathbf{p}) a_i(p_i) a_j(p_j) f_{ij}, \qquad ij \in I \times I \tag{2}$$

(with $\mu(\mathbf{p}) = [\Sigma_{ij}\, a_i(p_i) a_j(p_j) f_{ij}]^{-1}$) is a bijection from $S(\mathbf{M}^{\mathbf{a},\mathbf{f}})$ to $SFD[\mathbf{a}, \mathbf{f}]$. This in turn implies from Theorem I.1 that each steady-state flow distribution, $\mathbf{P}$, is equivalently characterized by the condition that

$$P(\cdot j) = \mu a_j [P(\cdot j)] \sum_k a_k [P \cdot k)] f_{kj}, \qquad j \in I \tag{3}$$

Using these results, we now have the following extended version of Theorem I.2:

THEOREM 1: Steady State Flows. *For any flow version* $(\mathbf{\Phi}, \sigma)$ *of a gravity-type interactive Markov process,* $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$

$$S(\mathbf{\Phi}) = SFD[\mathbf{a}, \mathbf{f}] \tag{4}$$

Proof. (*i*) First observe that if $\mathbf{P} \in S(\mathbf{\Phi})$ then it follows at once from (I.5) together with the flow-balance condition (I.41) and transitional consistency condition (I.43) that for all $ij \in I \times I$

$$P(ij) = P(\cdot i) M_{ij}^{\mathbf{a},\mathbf{f}}[\mathbf{P}(I\cdot)] = P(\cdot i) M_{ij}^{\mathbf{a},\mathbf{f}}[\mathbf{P}(\cdot I)] = P(\cdot i) \frac{a_j [P \cdot j)] f_{ij}}{\sum_k a_k [P(\cdot k)] f_{ik}} \tag{5}$$

which is precisely expression (I.16). Thus it follows from the argument in part (*ii*) of the proof of Theorem I.2 that $\mathbf{P} \in SFD[\mathbf{a}, \mathbf{f}]$, and hence that $S(\mathbf{\Phi}) \subseteq SFD[\mathbf{a}, \mathbf{f}]$.

(*ii*) To establish the converse, choose any $\mathbf{P}^0 \in SFD[\mathbf{a}, \mathbf{f}]$ and let $\mathbf{p}^0 = \mathbf{P}^0(\cdot I) = (\mathbf{P}^{\mathbf{a},\mathbf{f}})^{-1}(\mathbf{P}^0) \in S(\mathbf{M}^{\mathbf{a},\mathbf{f}})$. Then, by setting $\mathbb{P}_{q\times q}^0 = \{\mathbf{P} \in \mathbb{P}_{q\times q}: \mathbf{P}(\cdot I) = \mathbf{p}^0\}$, it follows on the one hand from the bijective property of $\mathbf{P}^{\mathbf{a},\mathbf{f}}$ that

$$\{\mathbf{P}^0\} = SFD[\mathbf{a}, \mathbf{f}] \cap \mathbb{P}_{q\times q}^0 \tag{6}$$

(since for any $\mathbf{P} \in SFD[\mathbf{a}, \mathbf{f}] \cap \mathbb{P}_{q\times q}^0$ we must have $\mathbf{P}(\cdot I) = \mathbf{p}^0 \in S(\mathbf{M}^{\mathbf{a},\mathbf{f}}) \Rightarrow (\mathbf{P}^{\mathbf{a},\mathbf{f}})^{-1}(\mathbf{P}) = (\mathbf{P}^{\mathbf{a},\mathbf{f}})^{-1}(\mathbf{P}^0) \Rightarrow \mathbf{P} = \mathbf{P}^0$). But, on the other hand, since it follows from marginal consistency (I.2) that $\dot{\mathbf{P}}(\cdot I) = 0$ for each $\mathbf{P} \in \mathbb{P}_{q\times q}$ with $\mathbf{P}(\cdot I) = \mathbf{p}^0$, we also must have

$$0 = \dot{P}(\cdot j) = \sigma\left[\sum_i \Phi_{ij}(\mathbf{P}) - P(\cdot j)\right], \qquad j \in I$$

$$\Rightarrow \sum_i \Phi_{ij}(\mathbf{P}) = P(\cdot j) = p_j^0, \qquad j \in I$$

$$\Rightarrow \mathbf{\Phi}(\mathbf{P}) \in \mathbb{P}_{q\times q}^0$$

Hence $\mathbf{\Phi}(\mathbb{P}_{q\times q}^0) \subseteq \mathbb{P}_{q\times q}^0$, which, together with the continuity of $\mathbf{\Phi}$ on the compact convex set $\mathbb{P}_{q\times q}^0$, implies (from the Brouwer Fixed-Point Theorem) that $\mathbf{\Phi}(\mathbf{P}) = \mathbf{P}$ for some $\mathbf{P} \in \mathbb{P}_{q\times q}^0$. However, since part (*i*) together with (I.40) then shows that $\mathbf{P} \in SFD[\mathbf{a}, \mathbf{f}]$, it follows from (6) that $\mathbf{P} = \mathbf{P}^0$, and thus that $\mathbf{\Phi}(\mathbf{P}^0) = \mathbf{P}^0$. Hence we may conclude from the arbitrary choice of $\mathbf{P}^0$ that $SFD[\mathbf{a}, \mathbf{f}] \subseteq S(\mathbf{\Phi})$. ■

Given these properties of all flow versions of gravity-type interactive Markov processes, we now develop the class of flow versions that is most useful for the stability analysis to follow.

Symmetric Flow Versions of Gravity-Type Processes

As in Part I, it follows from (1) that each steady-flow distribution, $\mathbf{P} \in SFD[\mathbf{a}, \mathbf{f}]$, is *symmetric,* that is $\mathbf{P} = \mathbf{P}^T$. But in the present context, the mathematical consequences of this symmetry property go much deeper. In particular, it turns out that gravity-type interactive Markov processes have *symmetric flow versions* that exhibit symmetry away from steady states as well. To formalize these flow versions, observe first that if the subset of *positive flow distributions* in $\mathbb{P}_{q\times q}$ is denoted by $\mathbb{P}^+_{q\times q} = \mathbb{P}_{q\times q} \cap \mathbb{R}^{q\times q}_{++}$, so that the set of *positive symmetric distributions* in $\mathbb{P}_{q\times q}$ is given by

$$\mathbb{S} = \{\mathbf{P} \in \mathbb{P}^+_{q\times q}\colon \mathbf{P} = \mathbf{P}^T\} \tag{7}$$

then the desired symmetric flows versions can be defined in terms of invariant sets as follows

DEFINITION 1: *A flow version* $(\mathbf{\Phi}, \sigma)$ *of* $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$ *is said to be* symmetric *iff* $\mathbb{S}$ *is an invariant set for* $(\mathbf{\Phi}, \sigma)$.

To construct a symmetric flow version of $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$, we first define the functions, $\theta_{ij}\colon \mathbb{P}_q \to \mathbb{R}_+$, $ij \in I \times I$, for all $\mathbf{p} \in \mathbb{P}_q$ by

$$\theta_{ij}(\mathbf{p}) = p_i \frac{a_j(p_j)f_{ij}}{\sum_k a_k(p_k)f_{ik}} \tag{8}$$

and construct the associated matrix-valued function, $\mathbf{Q}\colon \mathbb{P}_{q\times q} \to \mathbb{P}_{q\times q}$, for all $\mathbf{P} \in \mathbb{P}_{q\times q}$ and $ij \in I \times I$ by

$$Q_{ij}(\mathbf{P}) = \theta_{ij}[\mathbf{P}(\cdot I)] = P(\cdot i)\frac{a_j[P(\cdot j)]f_{ij}}{\sum_k a_k[P(\cdot k)]f_{ik}} \tag{9}$$

If the function, $\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}\colon \mathbb{P}_{q\times q} \to \mathbb{P}_{q\times q}$, is then defined for all $\mathbf{P} \in \mathbb{P}_{q\times q}$ by

$$\Phi^{\mathbf{a},\mathbf{f}}_{ij}(\mathbf{P}) = \tfrac{1}{2}Q_{ij}(\mathbf{P}) + \tfrac{1}{2}Q_{ji}(\mathbf{P}), \qquad ij \in I \times I \tag{10}$$

our objective is to show that the desired symmetric flow version of $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$ is given by the process $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, \sigma)$ with $\sigma = 2\lambda$. By (I.39) the differential equation system for $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$ is given by

$$\dot{P}_t(ij) = 2\lambda\{\tfrac{1}{2}[Q_{ij}(\mathbf{P}_t) + Q_{ji}(\mathbf{P}_t)] - P_t(ij)\}, \qquad ij \in I \times I \tag{11}$$

and can also be written in matrix form as:

$$\dot{\mathbf{P}}_t = 2\lambda\{[\tfrac{1}{2}\mathbf{Q}(\mathbf{P}_t) + \tfrac{1}{2}\mathbf{Q}_T(\mathbf{P}_t] - \mathbf{P}_t\}$$

In particular, the *steady states* of $(\Phi^{\mathbf{a},\mathbf{f}}, 2\lambda)$ are precisely the distributions, $\mathbf{P} \in \mathbb{P}_{q\times q}$, satisfying the condition

$$\mathbf{P} = \Phi^{\mathbf{a},\mathbf{f}}(\mathbf{P}) = \tfrac{1}{2}\mathbf{Q}(\mathbf{P}) + \tfrac{1}{2}\mathbf{Q}^T(\mathbf{P}) \tag{12}$$

which together with the symmetry of $\Phi^{\mathbf{a},\mathbf{f}}(\mathbf{P})$ implies that $\mathbf{P}$ must be symmetric. Thus every steady state automatically satisfies the balance condition (I.41), and it follows that $(\Phi^{\mathbf{a},\mathbf{f}}, 2\lambda)$ is a flow process. With these observations we now show that

THEOREM 2: Symmetric Flow Versions. *For any gravity-type interactive Markov process,* $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$, *the flow process* $(\Phi^{\mathbf{a},\mathbf{f}}, 2\lambda)$ *is a symmetric flow version of* $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$.

Proof. To establish that $(\Phi^{\mathbf{a},\mathbf{f}}, 2\lambda)$ is a flow version of $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$, observe first that for all $ij \in I \times I$

$$\begin{aligned} \dot{P}_t(\cdot j) &= \sum_i \dot{P}_t(ij) = 2\lambda\left\{\frac{1}{2}\left[\sum_i Q_{ij}(\mathbf{P}_t) + \sum_i Q_{ji}(\mathbf{P}_t)\right] - P_t(\cdot j)\right\} \\ &= \lambda\left\{\left[\sum_i P_t(\cdot i)\frac{a_j[P_t(\cdot j)]f_{ij}}{\sum_k a_k[P_t(\cdot k)]f_{ik}} + P_t(\cdot j)\right] - 2P_t(\cdot j)\right\} \\ &= \lambda\left\{\sum_i P_t(\cdot i)\frac{a_j[P_t(\cdot j)]f_{ij}}{\sum_k a_k[P_t(\cdot k)]f_{ik}} - P_t(\cdot j)\right\} \end{aligned} \tag{13}$$

which together with (I.5) shows that $(\Phi^{\mathbf{a},\mathbf{f}}, 2\lambda)$ satisfies the marginal consistency condition (I.42). To establish transitional consistency (I.43), observe first that the marginal distribution, $\mathbf{P}(\cdot I)$, for every steady state $\mathbf{P}$ of $(\Phi^{\mathbf{a},\mathbf{f}}, 2\lambda)$ must satisfy (3), which together with (9) and the symmetry of $\mathbf{f}$, implies that

$$\begin{aligned} \frac{P(\cdot i)}{a_i[P(\cdot i)]\sum_k a_k[P(\cdot k)]f_{ik}} &= \mu = \frac{P(\cdot j)}{a_j[P(\cdot j)]\sum_k a_k[P(\cdot k)]f_{jk}} \\ &\Rightarrow P(\cdot i)\frac{a_j[P(\cdot j)]}{\sum_k a_k[P(\cdot k)]f_{ik}} = P(\cdot j)\frac{a_i[P(\cdot i)]}{\sum_k a_k[P(\cdot k)]f_{jk}} \\ &\Rightarrow P(\cdot i)\frac{a_j[P(\cdot j)]f_{ij}}{\sum_k a_k[P(\cdot k)]f_{ik}} = P(\cdot j)\frac{a_i[P(\cdot i)]f_{ji}}{\sum_k a_k[P(\cdot k)]f_{jk}} \\ &\Rightarrow Q_{ij}(\mathbf{P}) = Q_{ji}(\mathbf{P}) \end{aligned}$$

for all $ij \in I \times I$. But since every steady state $\mathbf{P}$ of $(\mathbf{\Phi}^{\mathbf{a,f}}, 2\lambda)$ must also satisfy (12), we may then conclude that for all $ij \in I \times I$

$$P(ij) = \frac{1}{2} Q_{ij}(\mathbf{P}) + \frac{1}{2} Q_{ji}(\mathbf{P}) = Q_{ij}(\mathbf{P})$$

$$= P(\cdot i) \frac{a_j [P(\cdot j)] f_{ij}}{\sum_k a_k [P(\cdot k)] f_{ik}} = P(i\cdot) \frac{a_j [P(\cdot j)] f_{ij}}{\sum_k a_k [P(\cdot k)] f_{ik}}$$

$$\Rightarrow \frac{P(ij)}{P(i\cdot)} = \frac{a_j [P(\cdot j)] f_{ij}}{\sum_k a_k [P(\cdot k)] f_{ik}} = M_{ij}^{\mathbf{a,f}}[\mathbf{P}(\cdot I)]$$

where $\mathbf{M}^{\mathbf{a,f}}$ is the transition function for $(\mathbf{M}^{\mathbf{a,f}}, \lambda)$. Thus $(\mathbf{\Phi}^{\mathbf{a,f}}, 2\lambda)$ satisfies transitional consistency, and must be a flow version of $(\mathbf{M}^{\mathbf{a,f}}, \lambda)$. Finally, to verify that $\mathbb{S}$ is an invariant set for $(\mathbf{\Phi}^{\mathbf{a,f}}, 2\lambda)$, observe first from the symmetry of $\mathbf{Q}(\mathbf{P}) + \mathbf{Q}^T(\mathbf{P})$ that the differential equations for $P_t(ij)$ and $P_t(ji)$ in (11) are identical. But since for any distribution, $\mathbf{P}_0 \in \mathbb{S}$ we must have $P_0(ij) = P_0(ji)$, it follows that the unique adjustment path, $(\mathbf{P}_t)$, for $(\mathbf{\Phi}^{\mathbf{a,f}}, 2\lambda)$ with starting point $\mathbf{P}_0$ must satisfy $P_t(ij) = P_t(ji)$ for all $t \in \mathbb{R}_+$. Hence $\mathbf{P}_t = \mathbf{P}_t^T$ for all $t \in \mathbb{R}_+$, so that by Proposition I.3 it remains only to show that $\mathbf{P}_0 \in \mathbb{S}$ implies $\mathbf{P}_t \in \mathbb{R}_{++}^{q \times q}$ for all $t \in \mathbb{R}_+$. To do so, recall first from part (*ii*) of the proof of Proposition I.2 together with the positivity of $\mathbf{M}^{\mathbf{a,f}}$ that $\alpha = \min_{ij} \{\min_{\mathbf{p} \in \mathbb{P}_q} \mathbf{M}_{ij}^{\mathbf{a,f}}(\mathbf{p})\} > 0$. Moreover, since the marginal consistency condition (I.47) implies that the marginal, $[\mathbf{P}_t(\cdot I)]$, of each adjustment path, $(\mathbf{P}_t)$, must be a solution of (I.37) for $\mathbf{M} = \mathbf{M}^{\mathbf{a,f}}$, it also follows from part (*ii*) of the proof of Proposition I.2 that if for any $\mathbf{P}_0 \in \mathbb{S}$ we set $\epsilon_j = \min \{P_0(\cdot j), \alpha/2\} > 0$ and let $\epsilon = \min_j \epsilon_j > 0$, then $P_t(\cdot j) \geq \epsilon$ for all $t \in \mathbb{R}_+$ and $j \in I$. Hence, observing from (9) that $Q_{ij}(\mathbf{P}_t) = Q_{ji}(\mathbf{P}_t) \geq \epsilon\alpha$ for all t and $ij \in I \times I$, we see from (11) that

$$\dot{P}_t(ij) \geq 2\lambda\{1/2[\epsilon\alpha + \epsilon\alpha] - P_t(ij)\} = 2\lambda[\epsilon\alpha - P_t(ij)], \qquad ij \in I \times I$$

for all $t \in \mathbb{R}_+$. Finally, by setting $\epsilon_{ij} = \min \{P_0(ij), \epsilon\alpha/2\} > 0$, it may be concluded from the argument following (I.38) in the proof of Proposition I.2 that $P_t(ij) \geq \epsilon_{ij} > 0$ for all $t \in \mathbb{R}_+$, and thus that $\mathbf{P}_t \in \mathbb{R}_{++}^{q \times q}$ for all $t \in \mathbb{R}_+$. ■

This process $(\mathbf{\Phi}^{\mathbf{a,f}}, 2\lambda)$ will play an important role in our subsequent analysis, and hence is now designated as the *symmetric flow version* of $(\mathbf{M}^{\mathbf{a,f}}, \lambda)$.[1] In particular, it follows from Theorem 1 above that in analyzing

[1]It is of interest to note that this symmetric flow version is precisely the symmetrization of the *canonical flow version* to $(\mathbf{M}^{\mathbf{a,f}}, \lambda)$, as can be seen from a comparison of (9) with (5) and (I.44). The intensity factor 2λ simply allows this process to be written in standard form (I.39). It is also of interest to note that this symmetrization yields a flow version of gravity-type interactive Markov chains that differs from the spatial-flow chain developed in Part I. In particular, the flow chain defined for any given transition function, $\mathbf{M}$, by $P_{t+1}(ij) = P_t(\cdot i) M_{ij}[\mathbf{P}_t(\cdot I)] + P_t(\cdot j)\mathbf{M}_{ji}[\mathbf{P}_t(\cdot I)]$ for $i \neq j$ and $P_{t+1}(ii) = (2M_{ii}[\mathbf{P}_t(\cdot I)] - 1)P_t(\cdot i)$ is easily seen to yield a symmetric version of the interactive

in analyzing the steady-state flow distributions for $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$, there is no loss of generality in restricting our attention to the steady states of $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$.

3. ASYMPTOTIC STABILITY FOR THE CONTINUOUS CASE

As mentioned in the Introduction, the single most important property of the programming problem $\mathscr{P}[\mathbf{a}, \mathbf{f}]$ in Part I is that the objective function, Z, exhibits a Lyapunov property with respect to the symmetric flow version of $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$. Hence our present objective is to establish this property, and to develop its stability consequences for $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$. To do so, it is important to begin by observing that when analyzing the stability behavior of the interactive Markov process $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$ in terms of its symmetric flow version $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$, there is a *many-to-one* correspondence between the adjustment paths for $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$ and those of $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$. To see this, observe simply that for all starting points, $\mathbf{P}_0 \in \mathbb{P}_{q\times q}$, with a given marginal distribution, $\mathbf{P}_0(\cdot I) = \mathbf{p}^0 \in \mathbb{P}_q$, the marginal consistency condition (I.16) implies that the resulting flow-adjustment paths $(\mathbf{P}_t)$ for $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$ must all generate the same adjustment path for $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$ with starting point $\mathbf{p}^0$. But since there are infinitely many flow distributions $\mathbf{P}_0$ with $\mathbf{P}_0(\cdot I) = \mathbf{p}^0$, it follows that infinitely many flow-adjustment paths of $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$ correspond to the same adjustment path for $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$. This is true even if the starting points are restricted to *symmetric* flow distributions in $\mathbb{S}$ (which, for each *row* vector, $\mathbf{p}^0$, always includes the adjustment path with 'product' starting point, $\mathbf{P}_0 = (\mathbf{p}^0)^T\mathbf{p}^0 \in \mathbb{S}$, in Example 1 of Part I).

Hence, to analyze the stability of adjustment paths $\mathbf{p}(\cdot)$ in terms of their corresponding flow-adjustment paths $(\mathbf{P}_t)$, it is convenient to construct an explicit set of unique representative flow-adjustment paths. To do so, we begin by observing that there is one additional condition that a 'reasonable' choice of representative flow-adjustment paths should satisfy. In particular, if a given adjustment path $\mathbf{p}(\cdot)$ for $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$ starts at a steady state, that is, with $\mathbf{p}^0$ satisfying (I.8), then $\mathbf{p}(\cdot)$ will be a stationary path in the sense that $\mathbf{p}(t) \equiv \mathbf{p}^0$. However, flow-adjustment paths (P_t) with stationary marginals $\mathbf{p}(\cdot)$ need not themselves be stationary. For example, since the 'product' distribution, $(\mathbf{p}^0)^T\mathbf{p}^0$, will generally not satisfy (12) when $\mathbf{p}^0$ satisfies (I.8), it follows that the flow-adjustment process $(\mathbf{P}_t)$ with starting point $\mathbf{P}_0 = (\mathbf{p}^0)^T\mathbf{p}^0$ will not be stationary, even though its marginal process is. In other words, such flow-adjustment paths will continue to 'move' even though their marginal adjustment paths do not.

Markov chain generated by $\mathbf{M}$ (i.e., symmetry of $\mathbf{P}_0$ implies symmetry of $\mathbf{P}_t$ for all t). Moreover, this process also satisfies the marginal consistency condition, $\mathbf{P}_{t+1}(\cdot I) = \mathbf{P}_t(\cdot I)\mathbf{M}[\mathbf{P}_t(\cdot I)]$, and hence generates the given interactive Markov chain. Note however that these spatial flows violate the 'consistency' condition (I.12) in Part I that gave rise to the spatial-flow chain representation. Note also that such flows are only guaranteed to be nonnegative if $M_{ii}(\cdot) \geq 1/2$. Hence this particular model is meaningful only when periods are short enough (i.e., decision intensities are low enough) to ensure that the majority of individuals do not consider migrating in each period.

Flow Correspondence Mapping

With these observations, our first objective is to construct a 'continuous selection' of flow-adjustment paths $(\mathbf{P}_t)$ for adjustment paths $\mathbf{p}(\cdot)$ satisfying the condition that stationary adjustment paths always correspond to stationary flow-adjustment paths. To do so, we begin by adopting an appropriate topology for analyzing continuous paths in $\mathbb{S}$ as follows. If the *affine hull* of $\mathbb{S}$ (i.e., the smallest affine space containing $\mathbb{S}$), is denoted by

$$(14) \qquad H(\mathbb{S}) = \left\{\mathbf{M} = (M_{ij}) \in \mathbb{R}^{q\times q}: \mathbf{M} = \mathbf{M}^t \text{ and } \sum_{ij} M_{ij} = 1\right\}$$

then the appropriate topology for our purposes is taken to be the topology on $H(\mathbb{S})$ induced by the Euclidean metric topology on $\mathbb{R}^{q\times q}$, with open sets given by the intersections of $H(\mathbb{S})$ with open sets in $\mathbb{R}^{q\times q}$. In particular, since $\mathbb{S}$ is seen to be the intersection of $H(\mathbb{S})$ with the positive orthant, that is, $\mathbb{S} = H(\mathbb{S}) \cap \mathbb{R}^{q\times q}_{++}$, it follows that $\mathbb{S}$ is open in $H(\mathbb{S})$. More generally, all subsequent references to open (closed, compact) subsets of $\mathbb{S}$ or to continuity of functions from or to $\mathbb{S}$ will be with respect to this topology on $H(\mathbb{S})$. Finally, it should also be noted that this topology is homeomorphic to the Euclidean space, $\mathbb{R}^{n(q)}$, with exponent, $n(q) = [q(q + 1)/2] - 1$, denoting the number of 'degrees of freedom' remaining after the symmetry and normalization conditions in (14) have been imposed. Hence we may also treat $\mathbb{S}$ as an open subset of $\mathbb{R}^{n(q)}$ when appropriate.

Within this framework, our present objective is to construct for each gravity-type transition function, $\mathbf{M}^{\mathbf{a,f}}$, a continuous matrix-valued function, $\Psi^{\mathbf{a,f}}: \mathbb{P}_q^+ \to \mathbb{S}$, with images $\Psi^{\mathbf{a,f}}_{\mathbf{p}}$ satisfying the condition that marginals of $\Psi^{\mathbf{a,f}}_{\mathbf{p}}$ always agree with $\mathbf{p}$,[2] that is,

$$(15) \qquad \Psi^{\mathbf{a,f}}_{\mathbf{p}}(\cdot I) = \mathbf{p}, \qquad \mathbf{p} \in \mathbb{P}_q$$

and that $\Psi^{\mathbf{a,f}}_{\mathbf{p}}$ be a steady state for $(\Phi^{\mathbf{a,f}}, 2\lambda)$ whenever $\mathbf{p}$ is a steady state for $(\mathbf{M}^{\mathbf{a,f}}, \lambda)$, that is

$$(16) \qquad \mathbf{p} \in S(\mathbf{M}^{\mathbf{a,f}}) \Rightarrow \Psi^{\mathbf{a,f}}_{\mathbf{p}} \in S(\Phi^{\mathbf{a,f}}), \qquad \mathbf{p} \in \mathbb{P}_q$$

To construct such a function, we first let the function $\Theta: \mathbb{P}_q \to \mathbb{S}$, be defined for all $\mathbf{p} \in \mathbb{P}_q$ and $ij \in I \times I$ by

$$(17) \qquad \Theta_{\mathbf{p}}(ij) = \tfrac{1}{2}\theta_{ij}(\mathbf{p}) + \tfrac{1}{2}\theta_{ji}(\mathbf{p})$$

where θ_{ij} is given by (8) above. Next, if the *closure* of $\mathbb{S}$ in $H(\mathbb{S})$ is denoted by $\overline{\mathbb{S}}$, and if the mapping, $D: \overline{\mathbb{S}} \times \mathbb{S} \to \mathbb{R}_+$, denotes the restriction of the divergence function D, defined in (I.25) of Part I, to the set, $\overline{\mathbb{S}} \times \mathbb{S} \subset \mathbb{P}_{q\times q} \times \mathbb{R}^{q\times q}_{++}$, then for each probability vector, $\mathbf{p} \in \mathbb{R}^+_q$, we now consider the programming problem:

$$(18) \qquad \min: D(\mathbf{P}, \Theta_{\mathbf{p}}) \qquad \text{subject to: } \mathbf{P} \in \overline{\mathbb{S}}(\mathbf{p}) \equiv \{\mathbf{P} \in \overline{\mathbb{S}}: \mathbf{P}(\cdot I) = \mathbf{p}\}$$

[2]Recall that for the mapping $\mathbf{P}^{\mathbf{a,f}}$ in (2) this marginal correspondence holds only at *steady states.*

Observe that $\mathbf{p}^T\mathbf{p} \in \bar{\mathbb{S}}(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{P}_q$, so that the feasibility space, $\bar{\mathbb{S}}(\mathbf{p})$, is always a nonempty compact convex subset of $\bar{\mathbb{S}}$. Moreover, since $\mathbf{p} \in \mathbb{P}_q^+$ implies that $\mathbf{p}^T\mathbf{p} \in \mathbb{P}_{q\times q}^+$, it follows by letting $\mathbf{Q} = \mathbf{p}^T\mathbf{p}$ in the argument of (I.28) in Part I that there is a unique positive solution, $\mathbf{\Psi}_{\mathbf{p}}^{\mathbf{a,f}} \in \mathbb{S}$, to problem (18). These positive solutions thus yield a well-defined function, $\mathbf{\Psi}^{\mathbf{a,f}}: \mathbb{P}_q^+ \rightarrow \mathbb{S}$, which we now designate as the *flow-correspondence mapping* for $\mathbf{M}^{\mathbf{a,f}}$. The following result, proved in Smith and Hsieh (1996), shows that the mapping $\mathbf{\Psi}^{\mathbf{a,f}}$ satisfies the desired correspondence properties, together with certain additional mapping properties. In particular, if the range of $\mathbf{\Psi}^{\mathbf{a,f}}$ is restricted to the *image set*, $\mathbf{\Psi}^{\mathbf{a,f}}(\mathbb{P}_q^+) = \{\mathbf{\Psi}_{\mathbf{p}}^{\mathbf{a,f}}: \mathbf{p} \in \mathbb{P}_q^+\}$, then the resulting map, $\mathbf{\Psi}^{\mathbf{a,f}}: \mathbb{P}_q^+ \rightarrow \mathbf{\Psi}^{\mathbf{a,f}}(\mathbb{P}_q^+)$, is *a homeomorphism,* that is, a continuous bijection with continuous inverse:

PROPOSITION 1: *The flow-correspondence mapping,* $\mathbf{\Psi}^{\mathbf{a,f}}: \mathbb{P}_q^+ \rightarrow \mathbb{S}$, *satisfies* (15), (16), *and its restriction,* $\mathbf{\Psi}^{\mathbf{a,f}}: \mathbb{P}_q^+ \rightarrow \mathbf{\Psi}^{\mathbf{a,f}}(\mathbb{P}_q^+)$, *is a continuous bijection with continuous inverse,* $[\mathbf{\Psi}^{\mathbf{a,f}}]^{-1}: \mathbf{\Psi}^{\mathbf{a,f}}(\mathbb{P}_q^+) \rightarrow \mathbb{P}_q^+$, *given by*

$$[\mathbf{\Psi}^{\mathbf{a,f}}]^{-1}(\mathbf{P}) = \mathbf{P}(\cdot I), \qquad \mathbf{P} \in \mathbf{\Psi}^{\mathbf{a,f}}(\mathbb{P}_q^+) \tag{19}$$

As a direct consequence of this result, it follows that by restricting the domain of $\mathbf{\Psi}^{\mathbf{a,f}}$ to $S(\mathbf{M}^{\mathbf{a,f}})$, we obtain precisely the bijection $\mathbf{P}^{\mathbf{a,f}}$ from $S(\mathbf{M}^{\mathbf{a,f}})$ to $S(\mathbf{\Phi}^{\mathbf{a,f}})$ (=SFD[**a**, **f**]) in (2) above:

PROPOSITION 2. *The restriction,* $\mathbf{\Psi}^{\mathbf{a,f}}: S(\mathbf{M}^{\mathbf{a,f}}) \rightarrow \mathbf{\Psi}^{\mathbf{a,f}}[S(\mathbf{M}^{\mathbf{a,f}})]$, *is equal to* $\mathbf{P}^{\mathbf{a,f}}$.

Proof. If for any $\mathbf{p} \in S(\mathbf{M}^{\mathbf{a,f}})$ we let $\mathbf{P}_{\mathbf{p}}^{\mathbf{a,f}} = \mathbf{P} \in SFD[\mathbf{a}, \mathbf{f}]$, then it follows by summing (I.24) over i and applying Theorem I.1 that $\mathbf{p} = \mathbf{P}(\cdot I)$. Moreover, since $\mathbf{\Phi}^{\mathbf{a,f}}(\mathbf{P}) = \mathbf{P}$ by Theorem 1, it then follows by combining (1), (2), (9), (10), and (17) that

$$\mathbf{\Psi}_{\mathbf{p}}^{\mathbf{a,f}}(ij) = \tfrac{1}{2}\theta_{ij}(\mathbf{p}) + \tfrac{1}{2}\theta_{ji}(\mathbf{p}) = \tfrac{1}{2}\theta_{ij}[\mathbf{P}(\cdot I)] + \tfrac{1}{2}\theta_{ji}[\mathbf{P}(\cdot I)]$$

$$= \tfrac{1}{2}Q_{ij}(\mathbf{P}) + \tfrac{1}{2}Q_{ji}(\mathbf{P}) = \Phi_{ij}^{\mathbf{a,f}}(\mathbf{P}) = P(ij)$$

and hence that $\mathbf{\Psi}_{\mathbf{p}}^{\mathbf{a,f}} = \mathbf{P} = \mathbf{P}_{\mathbf{p}}^{\mathbf{a,f}}$ for all $\mathbf{p} \in S(\mathbf{M}^{\mathbf{a,f}})$. ■

Given this flow-correspondence mapping, $\mathbf{\Psi}^{\mathbf{a,f}}$, observe next from Theorem 2 together with Definition 1 that for each choice of starting points, $\mathbf{p}^0 \in \mathbb{P}_q^+$ and $\mathbf{P}_0 = \mathbf{\Psi}_{\mathbf{p}^0}^{\mathbf{a,f}} \in \mathbb{S}$, the resulting adjustment path $(\mathbf{P}_t)$ for $(\mathbf{\Phi}^{\mathbf{a,f}}, 2\lambda)$ must lie in $\mathbb{S}$. Moreover, since (15) also implies that $\mathbf{\Psi}_{\mathbf{p}^0}^{\mathbf{a,f}}(\cdot I) = \mathbf{p}^0$, it follows from the marginal consistency condition (I.42) that the adjustment paths $[\mathbf{P}_t(\cdot I)]$ and $\mathbf{p}(\cdot)$ must be *identical.* Hence if $(\mathbf{P}_t)$ converges to a steady-state flow distribution, $\mathbf{P}$, then it follows at once that $\mathbf{p}(\cdot)$ must converge to the marginal steady-state distribution, $\mathbf{P}(\cdot I)$. More generally it follows that all stability properties of $(\mathbf{\Phi}^{\mathbf{a,f}}, 2\lambda)$ in the invariant set $\mathbb{S}$ are inherited by $(\mathbf{M}^{\mathbf{a,f}}, \lambda)$. Hence, by employing the flow-correspondence mapping, $\mathbf{\Psi}^{\mathbf{a,f}}$, we can study the stability properties of $(\mathbf{M}^{\mathbf{a,f}}, \lambda)$ in terms of its symmetric flow version, $(\mathbf{\Phi}^{\mathbf{a,f}}, 2\lambda)$.

Lyapunov Property

With this end in mind, our primary purpose is to show that the objective function, Z, in Part I [expression (21) below] is a Lyapunov function for $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$ on $\mathbb{S}$. To do so, we first observe that for any continuously differentiable function, H: $\mathbb{R}^{q\times q}_{++} \rightarrow \mathbb{R}$, and adjustment path, $\mathbf{P}$: $\mathbb{R}_+ \rightarrow \mathbb{S}$, for $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$, the composite function, $[H \circ \mathbf{P}]$: $\mathbb{R}_+ \rightarrow \mathbb{R}$, defined for all $t \in \mathbb{R}_+$ by $[H \circ \mathbf{P}](t) = H(\mathbf{P}_t)$ is also continuously differentiable, so that the time derivative

$$(20) \qquad \frac{d}{dt}[H \circ \mathbf{P}](t) = \sum_{ij} \nabla_{ij}H(\mathbf{P}_t)\dot{P}_t(ij) = \sum_{ij} \nabla_{ij}H(\mathbf{P}_t)[\Phi^{\mathbf{a},\mathbf{f}}_{ij}(\mathbf{P}_t) - P_t(ij)]$$

is well defined and continuous. Thus letting the function $\dot{H}$: $\mathbb{P}^+_{q\times q} \rightarrow \mathbb{R}$, be defined for all $\mathbf{P} = [P(ij): ij \in I \times I] \in \mathbb{S}$ by

$$\dot{H}(\mathbf{P}) = \sum_{ij} \nabla_{ij}H(\mathbf{P})[\Phi^{\mathbf{a},\mathbf{f}}_{ij}(\mathbf{P}) - P(ij)]$$

so that by definition,

$$\frac{d}{dt}[H \circ \mathbf{P}](t) = \dot{H}(\mathbf{P}_t)$$

it follows that the sign of $(d/dt)[H \circ \mathbf{P}](t)$ on every adjustment path for $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$ is determined by the function H, which is completely independent of paths. With these observations, we now define the appropriate notion of a Lyapunov function for our purposes as follows:

DEFINITION 2: *If $\Omega \subseteq \mathbb{S}$ is an open invariant set for $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$, then a continuously differentiable function, H: $\Omega \rightarrow \mathbb{R}$, is designated as a* Lyapunov function *(or* strict Lyapunov function*) for $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$ on Ω iff H satisfies the following two conditions for all $\mathbf{P} \in \Omega$*
(i) $\dot{H}(\mathbf{P}) \leq 0$,
(ii) $\dot{H}(\mathbf{P}) = 0 \Leftrightarrow \mathbf{\Phi}^{\mathbf{a},\mathbf{f}}(\mathbf{P}) = \mathbf{P}$

In particular, since $\mathbb{S}$ is itself an open invariant set for $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$ it follows that any function, H: $\mathbb{S} \rightarrow \mathbb{R}$, satisfying *(i)* and *(ii)* for all $\mathbf{P} \in \mathbb{S}$ is a Lyapunov function for $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$ on $\mathbb{S}$. With this definition, our main result can be stated as follows:

THEOREM 3: Lyapunov Property. *For any gravity-type interactive Markov process $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$ the continuously differentiable function, Z: $\mathbb{R}^{q\times q}_{++} \rightarrow \mathbb{R}$, defined for all $\mathbf{V} \in \mathbb{R}^{q\times q}_{++}$ by*

$$(21) \qquad Z(\mathbf{V}) = D(\mathbf{V}, \mathbf{f}) - \sum_i \int_0^{V(i\cdot)} \log[a_i(x)]\,dx - \sum_j \int_0^{V(\cdot j)} \log[a_j(x)]\,dx$$

is a Lyapunov function for the symmetric flow version $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$ of $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$ on $\mathbb{S}$.[3]

[3]Although the domain $\mathbb{R}^{q\times q}_{++}$ is more convenient for partial differentiation of Z, the relevant Lyapunov function in Definition 2 is implicitly taken to be the restriction of Z to $\mathbb{S} \subset \mathbb{R}^{q\times q}_{++}$.

Proof. By using (I.31) in Part I, we see from (20) and (21) together with the identity $\Sigma_{ij}\,[\Phi^{\mathbf{a},\mathbf{f}}_{ij}(\mathbf{P}) - P(ij)] = 0$ on $\mathbb{S}$ that for any $\mathbf{P} \in \mathbb{S}$

$$\dot{Z}(\mathbf{P}) = \sum_{ij} \nabla_{ij} Z(\mathbf{P})[\Phi^{\mathbf{a},\mathbf{f}}_{ij}(\mathbf{P}) - P(ij)] \tag{22}$$

$$= \sum_{ij} \left\{\log\left[\frac{P(ij)}{a_i[P(i\cdot)]a_j[P(\cdot j)]f_{ij}}\right] + 1\right\}[\Phi^{\mathbf{a},\mathbf{f}}_{ij}(\mathbf{P}) - P(ij)]$$

$$= \sum_{ij} \log\left[\frac{P(ij)}{a_i[P(i\cdot)]a_j[P(\cdot j)]f_{ij}}\right][\Phi^{\mathbf{a},\mathbf{f}}_{ij}(\mathbf{P}) - P(ij)]$$

But since the symmetry of $\mathbf{P} \in \mathbb{S}$ implies

$$P(i\cdot) = \sum_j P(ij) = \sum_j P(ji) = P(\cdot i) \tag{23}$$

for all $i \in I$, we see that if the distribution $\mathbf{W} \in \mathbb{S}$ is defined for all $ij \in I \times I$ by

$$W(ij) = Ka_i[P(\cdot i)]a_j[P(\cdot j)]f_{ij} \tag{24}$$

with $K = [\Sigma_{ij}\, a_i[P(\cdot i)]a_j[P(\cdot j)]f_{ij}]^{-1}$, then it follows from (10) [with $\mathbf{Q} = \mathbf{Q}(\mathbf{P})$ and $\mathbf{Q}^T = \mathbf{Q}^T(\mathbf{P})$] together with (23) and the identity, $\Sigma_{ij}\,[\Phi^{\mathbf{a},\mathbf{f}}_{ij}(\mathbf{P}) - P(ij)] = 0$, that (22) may be rewritten as

$$\dot{Z}(\mathbf{P}) = \sum_{ij} \log\left[\frac{P(ij)}{W(ij)}\right]\left[\frac{1}{2}Q(ij) + \frac{1}{2}Q^T(ij) - P(ij)\right] \tag{25}$$

$$= \frac{1}{2}\left\{\sum_{ij}[Q(ij) - P(ij)]\log\left[\frac{P(ij)}{W(ij)}\right]\right\}$$

$$+ \frac{1}{2}\left\{\sum_{ij}[Q^T(ij) - P(ij)]\log\left[\frac{P(ij)}{W(ij)}\right]\right\}.$$

But recall from the definition of D in (I.25) that the first bracketed term of (25) can be written as

$$\sum_{ij}[Q(ij) - P(ij)]\log[P(ij)/W(ij)]$$

$$= \sum_{ij} Q(ij)\log[P(ij)/W(ij)] - \sum_{ij} P(ij)\log[P(ij)/W(ij)]$$

$$= \left\{\sum_{ij} Q(ij)\log[Q(ij)/W(ij)] - \sum_{ij} Q(ij)\log[Q(ij)/P(ij)]\right\} - D(\mathbf{P},\mathbf{W})$$

$$= [D(\mathbf{Q},\mathbf{W}) - D(\mathbf{P},\mathbf{W})] - D(\mathbf{Q},\mathbf{P})$$

A similar argument applied to the second bracketed term shows that (25) can be written as

$$\text{(26)}\quad \dot{Z}(\mathbf{P}) = \tfrac{1}{2}\{[D(\mathbf{Q},\mathbf{W}) - D(\mathbf{P},\mathbf{W})] - D(\mathbf{Q},\mathbf{P})\} + \tfrac{1}{2}\{[D(\mathbf{Q}^T,\mathbf{W}) - D(\mathbf{P},\mathbf{W})] - D(\mathbf{Q}^T,\mathbf{P})\}$$

In this form, observe that if for any $\mathbf{P} \in \mathbb{S}$ we consider the *programming problem,* $\mathscr{P}_1[\mathbf{P}]$:

$$\text{min: } D(\mathbf{V},\mathbf{W}) \qquad \text{subject to: } \mathbf{V} \in \mathbb{P}_{q\times q} \text{ and } \mathbf{V}(I\cdot) = \mathbf{P}(I\cdot)$$

then the argument in (I.28) of Part I again shows that the solution to $\mathscr{P}_1[\mathbf{P}]$ must lie in $\mathbb{P}^+_{q\times q}$. Hence the only binding constraint is $\mathbf{V}(I\cdot) = \mathbf{P}(I\cdot)$, and it follows that the solution to $\mathscr{P}_1(\mathbf{P})$ is given by the first-order conditions of the Lagrangian function

$$\mathscr{L}_1[\mathbf{V},(\alpha_i)] = D(\mathbf{V},\mathbf{W}) + \sum_i \alpha_i[V(i\cdot) - P(i\cdot)]$$

In particular, the solution $\mathbf{V}$ satisfies

$$0 = \nabla_{ij}\mathscr{L}_1[\mathbf{V},(\alpha_i)] = 1 + \log V(ij) - \log W(ij) - \alpha_i$$

so that we must have

$$\text{(27)}\qquad V(ij) = \lambda_i W(ij), \qquad ij \in I \times I$$

where $\lambda_i = \exp[\alpha_i - 1] > 0$. But since $P(\cdot i) = P(i\cdot) = V(i\cdot) = \Sigma_j V(ij) = \lambda_i \Sigma_j W(ij) = \lambda_i W(i\cdot)$ together with (27) implies that

$$V(ij) = P(\cdot i)\frac{W(ij)}{W(i\cdot)} = P(\cdot i)\frac{a_j[P(\cdot j)]f_{ij}}{\sum_k a_k[P(\cdot k)]f_{ik}}$$

it then follows from (9) that $\mathbf{V} = \mathbf{Q}$. Finally, since $\mathbf{P}$ automatically satisfies all the constraints of $\mathscr{P}_1[\mathbf{P}]$ and hence is feasible, we may conclude that

$$\text{(28)}\qquad D(\mathbf{Q},\mathbf{W}) \le D(\mathbf{P},\mathbf{W})$$

In a parallel manner, if we consider the *programming problem,* $[\mathscr{P}_2(\mathbf{P})]$:

$$\text{min: } D(\mathbf{V},\mathbf{W}) \qquad \text{subject to: } \mathbf{V} \in \mathbb{P}_{q\times q} \text{ and } \mathbf{V}(\cdot I) = \mathbf{P}(\cdot I)$$

then the same argument as above shows that the solution to $\mathscr{P}_2[\mathbf{P}]$ is given by the first-order conditions of the Lagrangian function

$$\mathscr{L}_2[\mathbf{V},(\beta_i)] = D(\mathbf{V},\mathbf{W}) + \sum_j \beta_j[V(\cdot j) - P(\cdot j)]$$

which for each $V(ij)$ takes the form

$$0 = \nabla_{ij}\mathscr{L}_2[\mathbf{V},(\beta_i)] = 1 + \log V(ij) - \log W(ij) - \beta_j$$

Again for $V(ij)$ we obtain

(29) $$V(ij) = \mu_j W(ij), \qquad ij \in I \times I$$

with $\mu_j = \exp[\beta_j - 1] > 0$, so that $P(\cdot j) = V(\cdot j) = \mu_j W(\cdot j)$ together with (23), (29) and the symmetry of $\mathbf{f}$ now yields

(30) $$V(ij) = P(\cdot j)\frac{W(ij)}{W(\cdot j)} = P(\cdot j)\frac{a_i[P(\cdot i)]f_{ji}}{\sum_k a_k[P(\cdot k)]f_{jk}}$$

Hence (30) together with (9) now implies that $\mathbf{V} = \mathbf{Q}^T$. But since $\mathbf{P}$ automatically satisfies the constraints of $\mathscr{P}_2(\mathbf{P})$, we may also conclude that

(31) $$D(\mathbf{Q}^T, \mathbf{W}) \leq D(\mathbf{P}, \mathbf{W})$$

Finally, observing from the well-known properties of the divergence function (Kullback, 1968, Theorem 3.1) that $D(\mathbf{V}, \mathbf{W}) \geq 0$ for all $\mathbf{V} \in \mathbb{P}_{q\times q}$ and $\mathbf{W} \in \mathbb{P}^+_{q\times q}$, and $D(\mathbf{V}, \mathbf{W}) = 0$ iff $\mathbf{V} = \mathbf{W}$, it follows from (28) and (31) that the right-hand side of (26) is always nonpositive and is zero iff $\mathbf{Q} = \mathbf{P} = \mathbf{Q}^T$. Hence we may conclude from (26) that $\dot{Z}(\mathbf{P}) \leq 0$ for all $\mathbf{P} \in \mathbb{S}$ and from (10) that $\dot{Z}(\mathbf{P}) = 0$ iff $\Phi^{\mathbf{a},\mathbf{f}}(\mathbf{P}) = \mathbf{P}$. Thus, Z is a Lyapunov function for $(\Phi^{\mathbf{a},\mathbf{f}}, 2\lambda)$ on $\mathbb{S}$, and the result is established. ■

Remark. There is a striking parallel between this result and the classical Lyapunov property exhibited by those 'master equations' with (*i*) time-invariant transition kernels, $\mathbf{K}$, and (*ii*) steady states, $\mathbf{p}^*$, satisfying the 'detailed balance' condition, $p_i^* K_{ij} = p_j^* K_{ji}$, for all i and j. For such systems, it is well known that the corresponding steady states, $\mathbf{p}^*$, are both unique and globally stable, with a Lyapunov function given by the divergence function, $D(\mathbf{p}, \mathbf{p}^*) = \Sigma_i p_i \ln(p_i/p_i^*)$ (see for example Schnakenberg, 1976 and Weidlich, 1988). In the present case, while our transition kernel, $\mathbf{M}^{\mathbf{a},\mathbf{f}}[\mathbf{P}(\cdot I)]$, is state dependent (and hence not time invariant), the marginal process in (13) is indeed equivalent to a master equation system with steady states satisfying the detailed balance condition above (as was shown in footnotes 20 and 28 of Part I). Moreover, for the special case in which time invariance holds (i.e., in which the attraction weights, a_i, are constant), it follows from (21) that $Z(\mathbf{P}) = \Sigma_{ij} P_{ij} \log(P_{ij}/f_{ij}) - \Sigma_{ij} P_{ij} \log(a_i) - \Sigma_{ij} P_{ij} \log(a_j) = D(\mathbf{P}, \mathbf{P}^*)$, where $\mathbf{P}^*$ has the form, $P^*_{ij} = \mu a_i a_j f_{ij}$. In addition it follows from Theorems I.2 and I.4 (together with the fact that every constant function is nonincreasing) that $\mathbf{P}^*$ must be the unique steady-state flow distribution for this case. Hence Theorem 3 may in this sense be viewed as an extension of the classical result above to those master-equation systems with gravity-type interactive Markov kernels. ■

Stability Properties of Steady-State Flows

To apply this result, we require a number of stability concepts for differential equation systems. For any nonempty open set, $X \subseteq \mathbb{R}^n$, and continuously differentiable function, $\mathbf{F}\colon X \to \mathbb{R}^n$, we now suppose that X is an invariant set for the (autonomous) differential equation system

(32) $$\dot{\mathbf{x}}(t) = \mathbf{F}[\mathbf{x}(t)], \qquad t \in \mathbb{R}_+$$

Then every *solution,* $\mathbf{x}(\cdot)$, to (32) defines a continuously differentiable function, $\mathbf{x}: \mathbb{R}_+ \rightarrow X$. If for any $z \in \mathbb{R}^n$ and nonempty set $\Omega \subseteq \mathbb{R}^n$ we define the *distance,* $d(z, \Omega)$, from z to Ω by

$$d(z, \Omega) = \inf_{y \in \Omega} \|z - y\|$$

then a solution $\mathbf{x}(\cdot)$ is said to *converge to* Ω, written $\mathbf{x}(t) \rightarrow \Omega$, iff $\lim_{t\rightarrow\infty} d[\mathbf{x}(t), \Omega] = 0$. Note also that since $d(\cdot, \Omega)$ is a continuous function (Dugundji, 1966, Theorem IX.4.2), it follows that for *closed* sets, Ω, there always exists a point in Ω closest to z, so that

(33) $$d(z, \Omega) = \alpha \Rightarrow \|\mathbf{z} - \mathbf{y}\| = \alpha \text{ for some } \mathbf{y} \in \Omega.$$

Finally, if for any subset, $\Omega \subseteq X$, we define the (closed) ϵ*-neighborhood* of Ω in X for each $\epsilon > 0$ by

$$X(\Omega, \epsilon) = \{\mathbf{x} \in X: d(\mathbf{x}, \Omega) \leq \epsilon\}$$

then we may define the following stability concepts for system (32):

DEFINITION 2: *For any nonempty compact subset,* $\Omega \subseteq X$, *we say that*
(*i*) Ω *is an* attractor *in X iff there exists some* $\epsilon > 0$ *such that for every solution,* $\mathbf{x}(\cdot)$,

(34) $$\mathbf{x}(0) \in X(\Omega, \epsilon) \Rightarrow \mathbf{x}(t) \rightarrow \Omega;$$

(*ii*) Ω *is a* global attractor *in X iff* (34) *holds for all* $\epsilon > 0$;
(*iii*) Ω *is* stable *in X iff for each* $\epsilon > 0$ *there exists some* $\delta_\epsilon \in (0, \epsilon)$ *such that for every solution,* $\mathbf{x}(\cdot)$,

(35) $$\mathbf{x}(0) \in X(\Omega, \delta_\epsilon) \Rightarrow \{\mathbf{x}(t): t \in \mathbb{R}_+\} \subseteq X(\Omega, \epsilon):$$

(*iv*) Ω *is* asymptotically stable *in X iff* Ω *is both an attractor and stable in X*;
(*v*) Ω *is* globally asymptotically stable *in X iff* Ω *is both a global attractor and stable in X.*

When the domain X is understood, we say simply that Ω is *stable* (*asymptotically stable*). Also when Ω is a singleton, say, $\Omega = \{\mathbf{x}\}$, we say that $\mathbf{x}$ is *stable* (*globally asymptotically stable,* etc.). To relate the above stability concepts to the Lyapunov function in (21), we begin with the following basic convergence result, where $\dot{Z}^{-1}(0) = \{\mathbf{P} \in \mathbb{S}: \dot{Z}(\mathbf{P}) = 0\}$.

PROPOSITION 3: Path Convergence. *For any nonempty subset,* $\Omega \subseteq \mathbb{S}$, *with closure,* $\overline{\Omega} \subseteq \overline{\mathbb{S}}$, *and any adjustment path,* $\mathbf{P}: \mathbb{R}_+ \rightarrow \mathbb{S}$, *for* $(\mathbf{\Phi}^{\mathbf{a,f}}, 2\lambda)$,

$$\{\mathbf{P}_t: t \in \mathbb{R}_+\} \subseteq \Omega \Rightarrow \mathbf{P}_t \rightarrow \overline{\Omega} \cap \dot{Z}^{-1}(0)$$

Proof. Since the Lyapunov function, Z, is continuous on $\overline{\mathbb{S}}$ and since the boundedness of $\mathbb{S}$ implies that $\{\mathbf{P}_t: t \in \mathbb{R}_+\}$ is bounded, the assertion follows at once from the more general result in Theorem X.1.3 of Hale (1980).[4] ■

To state the desired stability properties for $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$, it is convenient to introduce two additional concepts. First, if for any subset, $\Omega \subseteq \mathbb{S}$, we denote the set of minimizers of Z in Ω by

$$\min_Z [\Omega] = \{\mathbf{P} \in \Omega: Z(\mathbf{P}) = \min_{\mathbf{V} \in \Omega} Z(\mathbf{V})\}$$

then a set, $\Omega \subseteq \mathbb{S}$, is designated as a locally *Z-minimal set* in $\mathbb{S}$ iff there exists some $\epsilon > 0$ such that

$$\Omega = \min_Z [\overline{\mathbb{S}}(\Omega, \epsilon)]$$

(Observe from the continuity of Z and boundedness of $\mathbb{S}$ that each locally Z-minimal set, $\Omega \subseteq \mathbb{S}$, is compact.) Second, we say that subset of steady states, $\Omega \subseteq S(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}})$, is *isolated in* $S(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}})$ iff no other steady states are 'arbitrarily close' to Ω, that is, iff there is some $\epsilon > 0$ such that $\Omega = \overline{\mathbb{S}}(\Omega, \epsilon) \cap S(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}})$. With these concepts, we now have the following stability properties of steady-state flow distributions:

THEOREM 4: Stability of Steady-State Flow Distributions: *For each gravity-type interactive Markov process,* $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$, *and nonempty compact set of steady-state flow distributions,* $\Omega \subseteq S(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}})$,
(*i*) *If* Ω *is locally Z-minimal in* $\mathbb{S}$, *then* Ω *is stable in* $\mathbb{S}$;
(*ii*) *If* Ω *is stable in* $\mathbb{S}$ *and isolated in* $S(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}})$, *then* Ω *is asymptotically stable in* $\mathbb{S}$;
(*iii*) *The set of all steady state flows,* $S(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}})$, *is a global attractor in* $\mathbb{S}$;
(*iv*) *If* $S(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}) = \{\mathbf{P}^*\}$ *then* $\mathbf{P}^*$ *is a globally asymptotically stable in* $\mathbb{S}$.

Proof. (*i*) If $\Omega = \min_Z [\overline{\mathbb{S}}(\Omega, \epsilon_0)]$ then it suffices to establish (35) for all $\epsilon < \epsilon_0$, since (35) will then hold for any $\sigma > \epsilon$ by setting $\delta_\sigma = \delta_\epsilon$. To do so, observe first that if there is any adjustment path $(\mathbf{P}_t)$ for $(\mathbf{\Phi}^{\mathbf{a},\mathbf{f}}, 2\lambda)$ with $\mathbf{P}_0 \in \mathbb{S}(\Omega, \epsilon)$ and $\mathbf{P}_t \notin \mathbb{S}(\Omega, \epsilon)$ for some t, then by continuity of $d(\cdot, \Omega)$ and $(\mathbf{P}_t)$ there is some $t' \in (0, t)$ with $d(\mathbf{P}_t, \Omega) = \epsilon$. This together with the compactness of $\overline{\mathbb{S}}$ implies that the boundary set, $bd\mathbb{S}(\Omega, \epsilon) = \{\mathbf{P} \in \overline{\mathbb{S}}: d(\mathbf{P}, \Omega) = \epsilon\}$, is nonempty and compact. Hence by the continuity of Z there exists a minimum value, $V_\epsilon = \min\{Z(\mathbf{P}): \mathbf{P} \in bd\mathbb{S}(\Omega, \epsilon)\}$. Moreover, since $\Omega \cap bd\mathbb{S}(\Omega, \epsilon) = \varnothing$ implies that $V_\epsilon > V_0 = \min\{Z(\mathbf{P}): \mathbf{P} \in \Omega\}$ there must exist some $\delta_\epsilon \in (0, \epsilon)$ with $\max\{Z(\mathbf{P}): \mathbf{P} \in \overline{\mathbb{S}}(\Omega, \delta_\epsilon)\} < V_\epsilon$. To see that this δ_ϵ satisfies (35), observe that if for any adjustment path $(\mathbf{P}_t)$ with $\mathbf{P}_0 \in \mathbb{S}(\Omega, \delta_\epsilon)$ $[\subseteq \overline{\mathbb{S}}(\Omega, \delta_\epsilon)]$ it is true that $\mathbf{P}_t \notin \mathbb{S}(\Omega, \epsilon)$ for some t, then $d(\mathbf{P}_0, \Omega) \leq \delta_\epsilon < \epsilon$ and $d(\mathbf{P}_t, \Omega) > \epsilon$ again imply the existence of some $t' \in$

[4]Note that the result in Hale should be interpreted with respect to the Euclidean space, $\mathbb{R}^{n(q)}$, defined following expression (14).

$(0, t)$ with $d(\mathbf{P}_{t'}, \Omega) = \epsilon$. Hence $\mathbf{P}_{t'} \in bd\mathbb{S}(\Omega, \epsilon)$ and we must have $Z(\mathbf{P}_{t'}) \geq V_\epsilon$. Finally, since $\dot{Z} \leq 0$ on $\mathbb{S}$ by Theorem 3, it also follows that

$$(36) \qquad Z(\mathbf{P}_{t'}) = Z(\mathbf{P}_0) + \int_0^{t'} \dot{Z}(\mathbf{P}_s)\, ds \leq Z(\mathbf{P}_0) < V_\epsilon$$

and we obtain a contradiction. Hence no such adjustment path can exist, and it follows that (35) holds for this choice of δ_ϵ.

(*ii*) Observe first from Theorem 3 together with property (*ii*) of Definition 2 that the steady state flow distributions, $\mathbf{P}$, for $\mathbf{\Phi}^{\mathbf{a,f}}$ are given precisely by the condition that $\dot{Z}(\mathbf{P}) = 0$, so that

$$(37) \qquad S(\mathbf{\Phi}^{\mathbf{a,f}}) = \dot{Z}^{-1}(0) = \overline{\mathbb{S}} \cap \dot{Z}^{-1}(0)$$

Hence if Ω is a stable set in $\mathbb{S}$ and if $\Omega = \overline{\mathbb{S}}(\Omega, \alpha) \cap S(\mathbf{\Phi}^{\mathbf{a,f}})$ for some $\alpha > 0$, then by choosing $\epsilon \in (0, \alpha)$ sufficiently small to ensure that $\{\mathbf{P}_t : t \in \mathbb{R}_+\} \subseteq \overline{\mathbb{S}}(\Omega, \alpha)$ for all adjustment paths $(\mathbf{P}_t)$ with $\mathbf{P}_0 \in \mathbb{S}(\Omega, \epsilon)$, it follows from Proposition 3 and Equation (37) that

$$\mathbf{P}_0 \in \mathbb{S}(\Omega, \epsilon) \Rightarrow \mathbf{P}_t \rightarrow \overline{\mathbb{S}}(\Omega, \alpha) \cap \dot{Z}^{-1}(0) = \overline{\mathbb{S}}(\Omega, \alpha) \cap S(\mathbf{\Phi}^{\mathbf{a,f}}) = \Omega$$

and hence that Ω satisfies (34) for this choice of $\epsilon > 0$.

(*iii*) Since the continuity of $\mathbf{\Phi}^{\mathbf{a,f}}$ and boundedness of $\mathbb{S}$ imply that $S(\mathbf{\Phi}^{\mathbf{a,f}})$ is compact, and since $\mathbb{S}$ is an invariant set for $(\mathbf{\Phi}^{\mathbf{a,f}}, 2\lambda)$, it follows at once from (37) and Proposition 3 that for all adjustment paths $(\mathbf{P}_t)$ and $\epsilon > 0$,

$$\mathbf{P}_0 \in \mathbb{S}[S(\mathbf{\Phi}^{\mathbf{a,f}}), \epsilon] \subseteq \overline{\mathbb{S}} \Rightarrow \{\mathbf{P}_t : t \in \mathbb{R}_+\} \subseteq \overline{\mathbb{S}} \Rightarrow \mathbf{P}_t \rightarrow \overline{\mathbb{S}} \cap \dot{Z}^{-1}(0) = S(\mathbf{\Phi}^{\mathbf{a,f}})$$

and hence that $S(\mathbf{\Phi}^{\mathbf{a,f}})$ is a global attractor in $\mathbb{S}$.

(*iv*) If $S(\mathbf{\Phi}^{\mathbf{a,f}}) = \{\mathbf{P}^*\}$ then $\mathbf{P}^*$ is the (unique) global attractor in $\mathbb{S}$ by (*iii*). This in turn implies that for any $\mathbf{P} \neq \mathbf{P}^*$ we must have $Z(\mathbf{P}) > Z(\mathbf{P}^*)$. For if $Z(\mathbf{P}) \leq Z(\mathbf{P}^*)$, then $\dot{Z}(\cdot) \leq 0$ would imply from the argument in (36) that the path $(\mathbf{P}_t)$ with $\mathbf{P}_0 = \mathbf{P}$ does not converge to $\mathbf{P}$. Hence $\mathbf{P} \neq \mathbf{P}^* \Rightarrow Z(\mathbf{P}) > Z(\mathbf{P}^*)$, and we may conclude that $\mathbf{P}^*$ is the unique Z-minimum in $\mathbb{S}$. Thus, $\mathbf{P}^*$ is stable in $\mathbb{S}$ by (*i*), and must therefore be globally asymptotically stable in $\mathbb{S}$. ■

Stability Properties of Steady-State Population Distributions

Given these stability properties of $(\mathbf{\Phi}^{\mathbf{a,f}}, 2\lambda)$, we can now employ the flow-correspondence mapping, $\mathbf{\Psi}^{\mathbf{a,f}}$, to obtain stability results for the interactive Markov process, $(\mathbf{M}^{\mathbf{a,f}}, \lambda)$. As a parallel to the topological conventions for flow distributions, we now denote the *affine hull* of $\mathbb{P}_q$ by

$$(38) \qquad H(\mathbb{P}_q) = \left\{\mathbf{x} \in \mathbb{R}^q : \sum_i x_i = 1\right\}$$

and let the appropriate topology on $H(\mathbb{P}_q)$ be that induced by $\mathbb{R}^q$, so that the set of positive distributions, $\mathbb{P}_q^+ \ (= \mathbb{P}_q \cap \mathbb{R}^q_{++})$, is now seen to be open in $H(\mathbb{P})$ with compact closure, $\mathbb{P}_q$. This topology is homeomorphic to the Euclidean space, $\mathbb{R}^{q-1}$, so that $\mathbb{P}_q^+$ can be regarded as an open set in $\mathbb{R}^{q-1}$. In particular, it follows from Proposition I.2 that $\mathbb{P}_q^+$ is an open invariant set for the interactive Markov process, $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$, and hence that $\mathbb{P}_q^+$ now plays the role of $\mathbb{S}$ in the above analysis. With these observations, if we again say that a subset of steady-state population distributions, $\Omega \subseteq S(\mathbf{M}^{\mathbf{a},\mathbf{f}})$, is *isolated in* $S(\mathbf{M}^{\mathbf{a},\mathbf{f}})$ iff $\Omega = \mathbb{P}_q(\Omega, \epsilon) \cap S(\mathbf{M}^{\mathbf{a},\mathbf{f}})$ for some $\epsilon > 0$, then we now have:

THEOREM 5: Stability of Steady-State Population Distributions. *For each gravity-type interactive Markov process,* $(\mathbf{M}^{\mathbf{a},\mathbf{f}}, \lambda)$, *with flow-correspondence mapping,* $\Psi^{\mathbf{a},\mathbf{f}}$, *and each nonempty set of steady-state population distributions,* $\Omega \subseteq S(\mathbf{M}^{\mathbf{a},\mathbf{f}})$,
(*i*) *If* $\Psi^{\mathbf{a},\mathbf{f}}(\Omega)$ *is locally Z-minimal in* $\mathbb{S}$, *then* Ω *is stable in* $\mathbb{P}_q^+$;
(*ii*) *If* $\Psi^{\mathbf{a},\mathbf{f}}(\Omega)$ *is locally Z-minimal in* $\mathbb{S}$ *and* Ω *is isolated in* $S(\mathbf{M}^{\mathbf{a},\mathbf{f}})$, *then* Ω *is asymptotically stable in* $\mathbb{P}_q^+$;
(*iii*) *The set of all steady states,* $S(\mathbf{M}^{\mathbf{a},\mathbf{f}})$, *is a global attractor in* $\mathbb{P}_q^+$;
(*iv*) *If* $S(\mathbf{M}^{\mathbf{a},\mathbf{f}}) = \{\mathbf{p}^*\}$ *then* $\mathbf{p}^*$ *is globally asymptotically stable in* $\mathbb{P}_q^+$.[5]

Proof. First observe that if $\Psi^{\mathbf{a},\mathbf{f}}(\Omega)$ is locally Z-minimal in $\mathbb{S}$, then $\Psi^{\mathbf{a},\mathbf{f}}(\Omega)$ is compact in $H(\mathbb{S})$. Hence it follows from the continuity of the inverse function, $[\Psi^{\mathbf{a},\mathbf{f}}]^{-1}$, in Proposition 1 that Ω must be a compact subset of the open set, $\mathbb{P}_q^+$, in $H(\mathbb{P}_q)$. In particular, this implies from the continuity of $d(\cdot, \Omega)$ that $\mathbb{P}_q(\Omega, \sigma) \subseteq \mathbb{P}_q^+$ for some $\sigma > 0$.[6] Hence to study the stability properties of Ω, it suffices (by the argument following (37) above) to consider neighborhoods, $\mathbb{P}_q^+(\Omega, \epsilon)$, of Ω in $\mathbb{P}_q^+$ with $\epsilon < \sigma$, [so that in particular, $\mathbb{P}_q^+(\Omega, \epsilon) \subseteq \mathbb{P}_q(\Omega, \sigma)$]. Next observe from the compactness of $\mathbb{P}_q(\Omega, \sigma)$ that the restriction $\Psi^{\mathbf{a},\mathbf{f}}$: $\mathbb{P}_q(\Omega, \sigma) \rightarrow \mathbb{S}$, is uniformly continuous (Theorem XI.4.6 in Dugundji, 1966), so that for each $\epsilon > 0$ there is some $\alpha(\epsilon) \in (0, \epsilon)$ such that for all $\mathbf{p}, \mathbf{p}' \in \mathbb{P}_q(\Omega, \epsilon)$,

$$(39) \qquad \|\mathbf{p} - \mathbf{p}'\| \leq \alpha(\epsilon) \Rightarrow \|\Psi_{\mathbf{p}}^{\mathbf{a},\mathbf{f}} - \Psi_{\mathbf{p}}^{\mathbf{a},\mathbf{f}}\| \leq \epsilon$$

Hence, if the 'marginal' function, ψ: $\mathbb{P}_{q \times q} \rightarrow \mathbb{P}_q$, is defined for all $\mathbf{P} \in \mathbb{P}_{q \times q}$ by

$$(40) \qquad \psi(\mathbf{P}) = \mathbf{P}(\cdot I)$$

so that $\psi(\mathbf{P}) = [\Psi^{\mathbf{a},\mathbf{f}}]^{-1}(\mathbf{P})$ for all $\mathbf{P} \in S(\Phi^{\mathbf{a},\mathbf{f}})$ by Proposition 1, then it again follows from the compactness of $\mathbb{P}_{q \times q}$ that ψ is uniformly continuous, so that for each $\epsilon > 0$ there is also some $\beta(\epsilon) \in (0, \epsilon)$ such that for all $\mathbf{P}, \mathbf{P}' \in \mathbb{P}_{q \times q}$

$$(41) \qquad \|\mathbf{P} - \mathbf{P}'\| \leq \beta(\epsilon) \Rightarrow \|\mathbf{P}(\cdot I) - \mathbf{P}(\cdot I)\| = \|\psi(\mathbf{P}) - \psi(\mathbf{P}')\| \leq \epsilon$$

With these preliminary observations, we can now establish assertions (*i*), (*ii*), and (*iii*):
(*i*) To establish stability of Ω in $\mathbb{P}_q^+$ it suffices to show that for each $\epsilon < \sigma$ there

[5]It should be noted in both assertions (*iii*) and (*iv*) that $\mathbb{P}_q^+$ can be extended to all of $\mathbb{P}_q$. In particular, the positivity of $\mathbf{p}\mathbf{M}^{\mathbf{a},\mathbf{f}}(\mathbf{p})$ for each $\mathbf{p} \in \mathbb{P}_q$ implies from (I.37) and together with Proposition I.2 that each adjustment path $\mathbf{p}(\cdot)$ in $\mathbb{P}_q$ must satisfy $\mathbf{p}(t) \in \mathbb{P}_q^+$ for all $t > 0$.

[6]Since $bd\mathbb{P}_q$ is compact and $\Omega \cap bd\mathbb{P}_q \neq \emptyset$, an admissible choice of σ is given by $\sigma = \frac{1}{2} \min \{d(\mathbf{p}, \Omega): \mathbf{p} \in bd\mathbb{P}_q\} > 0$.

exists some $\theta_\epsilon \in (0, \epsilon)$ such that for all adjustment paths $\mathbf{p}(\cdot)$ for $(\mathbf{M}^{\mathbf{a,f}}, \lambda)$,

$$\mathbf{p}(0) \in \mathbb{P}_q^+(\Omega, \theta_\epsilon) \Rightarrow \mathbf{p}(t) \in \mathbb{P}_q^+(\Omega, \epsilon), \qquad t \in \mathbb{R}_+ \tag{42}$$

To do so, let $\theta_\epsilon = \alpha[\delta_{\beta(\epsilon)}]$ for α and β in (39) and (41), respectively, and for δ in part (*i*) in the proof of Theorem 4. Then by definition, $\theta_\epsilon < \epsilon < \sigma$ implies that $\mathbb{P}_q^+(\Omega, \theta_\epsilon) \subseteq \mathbb{P}_q(\Omega, \sigma)$, so that by (39) and (33)

$$\mathbf{p}(0) \in \mathbb{P}_q^+(\Omega, \theta_\epsilon) \Rightarrow \|\mathbf{p}(0) - \mathbf{p}\| \leq \theta_\epsilon = \alpha[\delta_{\beta(\epsilon)}] \text{ for some } \mathbf{p} \in \Omega \tag{43}$$

$$\Rightarrow \|\boldsymbol{\Psi}^{\mathbf{a,f}}_{\mathbf{p}(0)} - \boldsymbol{\Psi}^{\mathbf{a,f}}_{\mathbf{p}}\| \leq \delta_{\beta(\epsilon)} \Rightarrow \boldsymbol{\Psi}^{\mathbf{a,f}}_{\mathbf{p}(0)} \in \mathbb{S}[\boldsymbol{\Psi}^{\mathbf{a,f}}(\Omega), \delta_{\beta(\epsilon)}]$$

Now let $(\mathbf{P}_t)$ denote the unique adjustment path for $(\boldsymbol{\Phi}^{\mathbf{a,f}}, 2\lambda)$ with $\mathbf{P}_0 = \boldsymbol{\Psi}^{\mathbf{a,f}}_{\mathbf{p}(0)}$, so that $\mathbf{P}_0(\cdot I) = \mathbf{p}(0)$ by (15), and observe from the marginal consistency condition (I.42) together with the uniqueness of the adjustment path $\mathbf{p}(\cdot)$ for $(\mathbf{M}^{\mathbf{a,f}}, \lambda)$ with starting point $\mathbf{p}(0)$ that

$$\mathbf{P}_0(\cdot I) = \mathbf{p}(0) \Rightarrow \mathbf{P}_t(\cdot I) = \mathbf{p}(t), \qquad t \in \mathbb{R}_+ \tag{44}$$

But since Theorem 4 (*i*) together with the hypothesized local *Z*-minimality of $\boldsymbol{\Psi}^{\mathbf{a,f}}(\Omega)$ implies that $\mathbf{P}_t \in \mathbb{S}[\boldsymbol{\Psi}^{\mathbf{a,f}}(\Omega), \beta(\epsilon)]$ for all $t \in \mathbb{R}_+$, and since for any $\mathbf{P} \in \mathbb{P}_{q\times q}$ it must be true that

$$\mathbf{P} \in \boldsymbol{\Psi}^{\mathbf{a,f}}(\Omega) \Rightarrow \mathbf{P}(\cdot I) = [\boldsymbol{\Psi}^{\mathbf{a,f}}]^{-1}(\mathbf{P}) \in [\boldsymbol{\Psi}^{\mathbf{a,f}}]^{-1}\{\boldsymbol{\Psi}^{\mathbf{a,f}}(\Omega)\} = \Omega$$

we may conclude from (43) and (44) together with (33) that for each $t \in \mathbb{R}_+$

$$\mathbf{p}(0) \in \mathbb{P}_q^+(\Omega, \theta_\epsilon) \Rightarrow \mathbf{P}_t \in \mathbb{S}[\boldsymbol{\Psi}^{\mathbf{a,f}}(\Omega), \beta(\epsilon)]$$

$$\Rightarrow \|\mathbf{P}_t - \mathbf{V}_t\| \leq \beta(\epsilon) \text{ for some } \mathbf{V}_t \in \boldsymbol{\Psi}^{\mathbf{a,f}}(\Omega)$$

$$\Rightarrow \|\mathbf{p}(t) - \mathbf{V}_t(\cdot I)\| \leq \epsilon \text{ and } \mathbf{V}_t(\cdot I) \in \Omega \Rightarrow \mathbf{p}(t) \in \mathbb{P}_q^+(\Omega, \epsilon)$$

so that (42) holds for this choice of θ_ϵ.

(*ii*) By (*i*) it follows that Ω is stable in $\mathbb{P}_q^+$, so that we need only show that Ω is an attractor in $\mathbb{P}_q^+$. To do so, observe that since Ω isolated in $S(\mathbf{M}^{\mathbf{a,f}})$ there must exist some $\epsilon > 0$ with

$$\Omega = \mathbb{P}_q(\Omega, \epsilon) \cap S(\mathbf{M}^{\mathbf{a,f}}) \tag{45}$$

Hence we first show that this must imply that $\boldsymbol{\Psi}^{\mathbf{a,f}}(\Omega)$ is isolated in $S(\boldsymbol{\Phi}^{\mathbf{a,f}})$. For if not, then for each n there would exist some $\mathbf{P}_n \in \{\overline{\mathbb{S}}[\boldsymbol{\Psi}^{\mathbf{a,f}}(\Omega), 1/n] \cap S(\boldsymbol{\Phi}^{\mathbf{a,f}})\} - \boldsymbol{\Psi}^{\mathbf{a,f}}(\Omega)$. But for any n with $(1/n) < \beta(\epsilon)$, it would then follow from the compactness of $\boldsymbol{\Psi}^{\mathbf{a,f}}(\Omega)$ together with (33) and (19) that

$$\mathbf{P}_n \in \overline{\mathbb{S}}\left[\boldsymbol{\Psi}^{\mathbf{a,f}}(\Omega), \frac{1}{n}\right] \Rightarrow \|\mathbf{P}_n - \mathbf{P}\| \leq \beta(\epsilon) \text{ for some } \mathbf{P} \in \boldsymbol{\Psi}^{\mathbf{a,f}}(\Omega) \tag{46}$$

$$\Rightarrow \|\mathbf{P}_n(\cdot I) - \mathbf{P}(\cdot I)\| \leq \epsilon \text{ and } \mathbf{P}(\cdot I) \in \Omega \Rightarrow \mathbf{P}_n(\cdot I) \in \mathbb{P}_q(\Omega, \epsilon)$$

which together with $\mathbf{P}_n \in S(\boldsymbol{\Phi}^{\mathbf{a,f}})$ and Proposition 2 implies that $\mathbf{P}(\cdot I) \in \mathbb{P}_q(\Omega, \epsilon) \cap S(\mathbf{M}^{\mathbf{a,f}})$. However, since $\mathbf{P}_n \notin \boldsymbol{\Psi}^{\mathbf{a,f}}(\Omega)$ together with the bijective property of

$\Psi^{a,f}$ in Proposition 1 implies that $\mathbf{P}_n(\cdot I) = [\Psi_{a,f}]^{-1}(\mathbf{P}_n) \notin \Omega$ we would then have $\{\mathbb{P}_q(\Omega, \epsilon) \cap S(\mathbf{M}^{a,f})\} - \Omega \neq \emptyset$, which contradicts (45). Hence $\Psi^{a,f}(\Omega)$ is isolated in $S(\Phi^{a,f})$, which together with the local Z-minimality of $\Psi^{a,f}(\Omega)$ in $\mathbb{S}$ implies from parts (i) and (ii) of Theorem 4 that $\Psi^{a,f}(\Omega)$ is asymptotically stable in $\mathbb{S}$. Thus $\Psi^{a,f}(\Omega)$ is an attractor in $\mathbb{S}$ and there is some $\tau > 0$ such that for all adjustment paths $(\mathbf{P}_t)$ for $(\Phi^{a,f}, 2\lambda)$

$$\mathbf{P}_0 \in \mathbb{S}[\Psi^{a,f}(\Omega), \tau] \Rightarrow \mathbf{P}_t \to \Psi^{a,f}(\Omega) \tag{47}$$

Finally, letting $\epsilon = \alpha(\tau)$, and observing from (39) that for any adjustment path $\mathbf{p}(\cdot)$

$$\mathbf{p}(0) \in \mathbb{P}_q^+(\Omega, \epsilon) \Rightarrow \|\mathbf{p}(0) - \mathbf{p}\| \leq \epsilon = \alpha(\tau) \text{ for some } \mathbf{p} \in \Omega \tag{48}$$
$$\Rightarrow \|\Psi^{a,f}_{\mathbf{p}(0)} - \Psi^{a,f}_{\mathbf{p}}\| \leq \tau \Rightarrow \Psi^{a,f}_{\mathbf{p}(0)} \in \mathbb{S}[\Psi^{a,f}(\Omega), \tau]$$

it follows from (47) and (44) together with the continuity of $[\Psi^{a,f}]^{-1}$ that for the unique adjustment path $(\mathbf{P}_t)$ with $\mathbf{P}_0 = \Psi^{a,f}_{\mathbf{p}(0)}$ we must have

$$\mathbf{p}(0) \in \mathbb{P}_q^+(\Omega, \epsilon) \Rightarrow \mathbf{P}_0 = \Psi^{a,f}_{\mathbf{p}(0)} \in \mathbb{S}[\Psi^{a,f}(\Omega), \tau] \Rightarrow \mathbf{P}_t \to \Psi^{a,f}(\Omega) \tag{49}$$
$$\Rightarrow [\Psi^{a,f}]^{-1}(\mathbf{P}_t) \to [\Psi^{a,f}]^{-1}\{\Psi^{a,f}(\Omega)\} = \Omega \Rightarrow \mathbf{P}_t(\cdot I) \to \Omega \Rightarrow \mathbf{p}(t) \to \Omega$$

Hence Ω is an attractor in $\mathbb{P}_q^+$ and the result is established.
(iii) Observe that since $S(\Phi^{a,f})$ is global attractor in $\mathbb{S}$, if we now set $\Omega = S(\mathbf{M}^{a,f})$ in (49) and recall from Proposition 2 that $\Psi^{a,f}[S(\mathbf{M}^{a,f})] = S(\Phi^{a,f})$, then the same argument now shows that $S(\mathbf{M}^{a,f})$ is a global attractor in $\mathbb{P}_q^+$.
(iv) Finally, if $S(\mathbf{M}^{a,f}) = \{\mathbf{p}^*\}$ then $\mathbf{p}^*$ is a global attractor by (iii). Also from Proposition 2 it follows that $S(\Phi^{a,f}) = \{\Psi^{a,f}(\mathbf{p}^*)\}$, and hence (by the proof of Theorem 4 (iv)) that $\Psi^{a,f}(\mathbf{p}^*)$ is the unique Z-minimum in $\mathbb{S}$. Thus $\mathbf{p}^*$ is stable by (i), and the result is established. ■

As an immediate consequence of this result, we have the following global stability property for the case of pure congestion effects:

PROPOSITION 4. Global Stability of Pure Congestion Processes: *If* $\mathbf{M}^{a,f}$ *is a gravity-type transition function with pure congestion effects, then the unique steady state for each interactive Markov process,* $(\mathbf{M}^{a,f}, \lambda)$, *is globally asymptotically stable*

Proof. The result follows at once from Theorem I.5 in Part I together with part (iv) of Theorem 5. ■

4. ASYMPTOTIC STABILITY FOR THE DISCRETE CASE

Given these results for interactive Markov processes (M, λ), with $\mathbf{M} = \mathbf{M}^{a,f}$, our final objective is to show that in 'almost all' cases in which global asymptotic stability holds with respect to a unique steady state, this property is inherited by the associated interactive Markov chains $(\mathbf{M}, \alpha, \Delta)$ when Δ is sufficiently small. To do so, we require the following stronger condition on admissible participation functions, α. In particular, we now require that that participation rates, $\alpha(\Delta)/\Delta$, not only have positive limit, λ, but also that convergence to λ is 'sufficiently fast'. More precisely, $\alpha: \mathbb{R}_{++} \rightarrow (0, 1]$, is now designated as a *participation function* iff

$$\alpha(\Delta) = \lambda\Delta + O(\Delta^2), \qquad \Delta > 0 \tag{50}$$

The presence of the (Landau) residual, $O(\Delta^2)$, implies the existence of some $B > 0$ such that $|\alpha(\Delta) - \lambda\Delta| \leq B\Delta^2$ for all small values of $\Delta > 0$, which may equivalently be written as[7]

$$\left| \frac{\alpha(\Delta)}{\Delta} - \lambda \right| \leq B\Delta \tag{51}$$

Given this restriction on α, the only particular feature of gravity-type interactive Markov chains which is needed for the present analysis is the *positivity* of $\mathbf{M}^{\mathbf{a},\mathbf{f}}$. Hence for each positive-valued transition function, $\mathbf{M}: \mathbb{P}_q \rightarrow \mathbb{R}^{q\times q}_{++}$, we now designate all associated interactive Markov processes, $(\mathbf{M}, \lambda)$, and interactive Markov chains, $(\mathbf{M}, \alpha, \Delta)$, as *positive*. In addition, for each positive $\mathbf{M}$, it is convenient to define the associated function, $\boldsymbol{\phi}: \mathbb{P}_q \rightarrow \mathbb{P}_q^+$, for all $\mathbf{p} \in \mathbb{P}_q$ by

$$\phi_j(\mathbf{p}) = \sum_i p_i M_{ij}(\mathbf{p}) - p_j, \qquad j \in I \tag{52}$$

so that (I.37) can be written more simply as

$$\dot{\mathbf{p}}(t) = \lambda\boldsymbol{\phi}[\mathbf{p}(t)], \qquad t \in \mathbb{R}_+ \tag{53}$$

Similarly, the adjustment paths associated with each interactive Markov chain can be written as

$$\mathbf{p}^{t+\Delta} = \mathbf{p}^t + \alpha(\Delta)\boldsymbol{\phi}(\mathbf{p}^t) \tag{54}$$

If for any fixed value of $\Delta > 0$ we now let $t_n = n\Delta$ for each $n \in \mathbb{Z}_+$, then it follows from (54) that the probability values, $\mathbf{p}^n_\Delta$, at time points t_n satisfy the difference equation

$$\mathbf{p}^{n+1}_\Delta = \mathbf{p}^n_\Delta + \alpha(\Delta)\boldsymbol{\phi}(\mathbf{p}^n_\Delta) = \mathbf{p}^n_\Delta + \Delta\boldsymbol{\Phi}_\Delta(\mathbf{p}^n_\Delta) \tag{55}$$

where

$$\boldsymbol{\Phi}_\Delta(\mathbf{p}) = [\alpha(\Delta)/\Delta]\boldsymbol{\phi}(\mathbf{p}) \tag{56}$$

These adjustment paths form the main objects of interest in the present analysis. Hence where specific reference to $(\mathbf{M}, \alpha, \Delta)$ is not necessary, we shall simply refer to the class of solutions to (55) as the associated *adjustment process*, $(\mathbf{p}^n_\Delta)$. In particular, such adjustment processes are seen from (55) are seen to be instances of 'one-step' methods for approximating solutions to (53), as

[7]As is shown in Lemma 2 below, condition (51) ensures that adjustment paths for the interactive Markov chain (M, α, Δ) converge to those of the Markov process $(\mathbf{M}, \lambda)$ *uniformly* on each finite time interval.

detailed for example in Gear (1971) and Stetter (1973). Hence the following analysis relies heavily on known results for this class of procedures. Our first objective, is to show for 'almost all parameterizations' of ϕ in (53) there is a neighborhood $\mathbb{P}_q(\mathbf{p}^*, \epsilon)$ of $\mathbf{p}^*$ and a delta value, Δ_1, such that for each $\Delta < \Delta_1$ (55) defines a *contraction mapping* with respect to $\mathbf{p}^*$ (in an appropriately defined metric), and hence must converge to $\mathbf{p}^*$. We then show that for any neighborhood, $\mathbb{P}_q(\mathbf{p}^*, \epsilon)$, there is also a delta value, Δ_2, sufficiently small to ensure that for each $\Delta < \Delta_2$ and initial starting point, $p_0 \in \mathbb{P}$, the sequence $(\mathbf{p}_\Delta^n)$ generated by (55) must eventually enter $\mathbb{P}_q(\mathbf{p}^*, \epsilon)$. It will then follow that global asymptotic stability of (55) must hold for all $\Delta < \min\{\Delta_1, \Delta_2\}$.

To establish the local contraction property of (55) we begin by recalling that the entire process is restricted to the $(q-1)$-dimensional space defined by the affine hull, $H(\mathbb{P}_q)$, of $\mathbb{P}_q$ in (38). Hence to facilitate the application of standard results, it is convenient to change coordinates so that the process is now defined in $\mathbb{R}^{q-1}$ with a unique steady state at the origin. To do so, let the one-to-one affine transformation, $\mathbf{C}$: $\mathbb{R}^{q-1} \to H(\mathbb{P}_q)$, be defined for all $\mathbf{x} = (x_1, \ldots, x_{q-1}) \in \mathbb{R}^{q-1}$ by

(57) $$\mathbf{C}(\mathbf{x}) = (x_1, \ldots, x_{q-1}, 1 - \sum_{i=1}^{q-1} x_i) \in H(\mathbb{P}_q)$$

and observe that if the linear (projective) transformation, $\mathbf{C}_0$: $\mathbb{R}^q \to \mathbb{R}^{q-1}$, is defined for all $\mathbf{z} = (z_1, \ldots z_{q-1}, z_q) \in \mathbb{R}^q$ by $\mathbf{C}_0(\mathbf{z}) = (z_1, \ldots z_{q-1})$, then the restriction of $\mathbf{C}_0$ to $H(\mathbb{P}_q)$ in the *inverse* of $\mathbf{C}$ [that is, $\mathbf{C}_0(\mathrm{Cz}) = \mathbf{z}$ and $\mathbf{C}(\mathbf{C}_0\mathbf{p}) = \mathbf{p}$ for all $\mathbf{z} \in \mathbb{R}^{q-1}$ and $\mathbf{p} \in H(\mathbb{P}_q)$]. Hence if for each $\mathbf{p} \in H(\mathbb{P}_q)$ we let

(58) $$\mathbf{z} = \mathbf{C}_0(\mathbf{p} - \mathbf{p}^*) = \mathbf{C}_0\mathbf{p} - \mathbf{C}_0\mathbf{p}^*$$

so that by definition, $\mathbf{Cz} = \mathbf{p} - \mathbf{p}^* \Rightarrow \mathbf{p} = \mathbf{Cz} + \mathbf{p}^*$ and $\dot{\mathbf{z}} = \mathbf{C}_0\dot{\mathbf{p}} \Rightarrow \dot{\mathbf{p}} = \mathbf{C}\dot{\mathbf{z}}$, then it follows that (53) can be rewritten as

(59) $$\mathbf{C}\dot{\mathbf{z}}(t) = \lambda\phi[\mathbf{Cz}(t) + \mathbf{p}^*] \Rightarrow \dot{\mathbf{z}}(t) = \lambda\mathbf{C}_0\phi[\mathbf{Cz}(t) + \mathbf{p}^*]$$

Hence, if we now let $\mathbf{Z} = \{\mathbf{C}_0(\mathbf{p} - \mathbf{p}^*): \mathbf{p} \in \mathbb{P}_q\} \subset \mathbb{R}^{q-1}$, and define the map, ψ: $\mathbf{Z} \to \mathbb{R}^{q-1}$, for all $\mathbf{z} \in \mathbf{Z}$ by

(60) $$\psi(\mathbf{z}) = \mathbf{C}_0\phi[\mathbf{Cz} + \mathbf{p}^*]$$

then we may replace the differential equation system (53) on $\mathbb{P}_q$ with the equivalent differential equation system

(61) $$\dot{\mathbf{z}}(t) = \lambda\psi[\mathbf{z}(t)], \qquad t \in \mathbb{R}_+$$

on $\mathbf{Z}$. Similarly if we let $\mathbf{z}_\Delta^n = \mathbf{C}_0(\mathbf{p}_\Delta^n - \mathbf{p}^*)$, then (55) can be rewritten as

(62) $$\mathbf{Cz}_\Delta^{n+1} + \mathbf{p}^* = (\mathbf{Cz}_\Delta^n + \mathbf{p}^*) + \alpha(\Delta)\phi[\mathbf{Cz}_\Delta^n + \mathbf{p}^*]$$

$$\Rightarrow \mathbf{z}_\Delta^{n+1} = \mathbf{z}_\Delta^n + \alpha(\Delta)\mathbf{C}_0\phi(\mathbf{Cz}_\Delta^n + \mathbf{p}^*) = \mathbf{z}_\Delta^n + \alpha(\Delta)\psi(\mathbf{z}_\Delta^n)$$

and is hence is seen to yield a 'one-step' method for approximating solutions to (1). To analyze the convergence properties of (62), we start by considering those of (61). First observe from Proposition I.2 together with (59) that the set $\mathbf{Z}$ must be an invariant set for the differential equation system (61). Moreover, since $\psi(0) = \phi(\mathbf{p}^*) = 0$, and since the positivity of $\phi(\cdot)$ implies that $\mathbf{p}^* \in \mathbb{P}_q^+ = \text{int}(\mathbb{P}_q)$, it also follows that (61) must be globally asymptotically stable with respect to the unique steady state, $\mathbf{z}^* = 0 \in \text{int}(\mathbf{Z})$. In addition, since the continuous differentiability of $\phi(\cdot)$ on $\mathbb{P}_q$ implies that that the derivative of $\psi(\cdot)$ is continuous on $\mathbf{Z}$ and is representable by the matrix of partial derivatives, $\nabla\psi(\cdot) = [\partial\psi_i(\cdot)/\partial z_j: i, j = 1, \ldots, q-1]$, it is well known that $\nabla\psi(0)$ can have no eigenvalues with positive real parts (Theorem 9.2.1 in Hirsch and Smale, 1974, p. 187). Hence if the steady state is *hyperbolic,* that is, if the matrix $\nabla\psi(0)$ has no eigenvalues with zero real parts, then it follows that all eigenvalues of $\nabla\psi(0)$ must have *negative real parts.* In this case, (61) exhibits a strong form of local convergence to the steady state, $\mathbf{z}^* = 0$, which can be readily analyzed. Moreover, it is also well known (Theorem 7.3.3 in Hirsch and Smale, 1974, p. 157) that *almost all* matrices have eigenvalues with nonzero real parts.[8] Hence if the admissible parameterizations of ϕ allow an open set of possible values for the matrix $\nabla\psi(0)$, then this 'hyperbolic' property can reasonably be expected to hold for almost parameterizations of ϕ that yield globally asymptotically to hold for almost parameterizations of ϕ that yield globally asymptotically stable steady states. We now assume this to be the case, and designate the steady state, $\mathbf{p}^*$, as *hyperbolic* in $\mathbb{P}_q$ iff all eigenvalues of $\nabla\psi(0)$ have negative real parts.[9] Under this assumption, it can be shown (see Smith and Hsieh, 1996) that:[10]

LEMMA 1: *If* $(\mathbf{M}, \lambda)$ *is a positive interactive Markov process with globally asymptotically stable hyperbolic steady state,* $\mathbf{p}^*$, *then there exists some* $\epsilon_1 > 0$, $\Delta_1 > 0$, *and* $\mu \in (0, 1)$ *such that for each* $\Delta \in (0, \Delta_1)$, $\epsilon \in (0, \epsilon_1)$, *and* $n \in \mathbb{Z}_+$

$$(63) \qquad \|\mathbf{p}_\Delta^n - \mathbf{p}^*\| < \epsilon_1 \Rightarrow \lim_{m\to\infty} \mathbf{p}_\Delta^m = \mathbf{p}^* \text{ and}$$

$$(64) \qquad \|\mathbf{p}_\Delta^n - \mathbf{p}^*\| < \mu\epsilon \Rightarrow \|\mathbf{p}_\Delta^m - \mathbf{p}^*\| < \epsilon \text{ for all } m \geq n$$

This result covers behavior in a small neighborhood $\mathbf{p}^*$. The next result shows that such a neighborhood can always be reached. In particular, it can be verified that (see Smith and Hsieh, 1996):

LEMMA 2: *If* $(\mathbf{M}, \lambda)$ *is a positive interactive Markov process with globally asymptotically stable steady state,* $\mathbf{p}^*$, *then for each* $\epsilon > 0$ *there exists a* $\Delta_\epsilon > 0$ *sufficiently small to ensure that for each* $\Delta \in (0, \Delta_\epsilon)$ *there is some* $n(\Delta) \in \mathbb{Z}_+$ *such*

[8]More precisely, the set of n-square matrices not exhibiting this property has measure zero in $\mathbb{R}^{n\times n}$ for each n.

[9]It should be noted that the above transformation, $\mathbf{C}$, is essential for this purpose. In particular, if the matrix of partial derivatives of ϕ is denoted by $\nabla\phi(\cdot) = [\partial\phi_i(\cdot)/\partial p_j: ij = 1, \ldots, q]$, then the identity $\Sigma_i\, \phi_i(\cdot) = 1$ implies that $\Sigma_i\, \partial\phi_i(\cdot)/\partial p_j = 0$, $j = 1, \ldots, q$, which can be written in matrix form as $\nabla\phi(\cdot)\mathbf{1} = 0$. Thus $\nabla\phi(\mathbf{p}^*)$ must always have a zero eigenvalue (with unit eigenvector), so that the equilibrium, $\mathbf{p}^*$, can *never be hyperbolic* in $\mathbb{R}^q$.

[10]This result can also be established by an application of Corollary 2.3.9 in Stetter (1973) together with standard properties of linear difference equations.

that for every starting point, $\mathbf{p}^0 \in \mathbb{P}_q$, *the sequence* $(\mathbf{p}_\Delta^n)$ *defined by* (55) *satisfies*

$$\|\mathbf{p}_\Delta^{n(\Delta)} - \mathbf{p}^*\| < \epsilon \tag{65}$$

By combining these results, we can now establish the global asymptotic stability of interactive Markov chains that are 'sufficiently close' to their continuous (globally asymptotically stable) counterparts. To do so, we now say (as a parallel to the continuous case) that an interactive Markov chain $(\mathbf{M}, \alpha, \Delta)$ is *stable* with respect to a steady state, $\mathbf{p}^* \in \mathbb{P}_q^+$, iff for each $\epsilon > 0$ there is some $\delta(\epsilon) \in (0, \epsilon)$ sufficiently small to ensure that $\|\mathbf{p}_\Delta^n - \mathbf{p}^*\| < \epsilon$ holds for all $n \in \mathbb{Z}_+$ on every adjustment path $(\mathbf{p}_\Delta^n)$ with $\|\mathbf{p}_\Delta^0 - \mathbf{p}^*\| < \delta(\epsilon)$. In addition, $(\mathbf{M}, \alpha, \Delta)$ is said to be *globally asymptotically stable* with respect to $\mathbf{p}^*$ iff for every starting point, $\mathbf{p}_\Delta^0 \in \mathbb{P}_q$ it is also true that $\lim_{n\to\infty} \|\mathbf{p}_\Delta^n - \mathbf{p}^*\| = 0$. With these definitions, we now have:

THEOREM 6: Global Stability of Interactive Markov Chains. *If* $(\mathbf{M}, \lambda)$ *is a positive interactive Markov process with globally asymptotically stable hyperbolic steady state,* $\mathbf{p}^*$, *then there exists a* $\Delta_0 > 0$ *sufficiently small to ensure that each (positive) interactive Markov chain* $(\mathbf{M}, \alpha, \Delta)$ *with* $\Delta < \Delta_0$ *is also globally asymptotically stable with respect to* $\mathbf{p}^*$.

Proof. For Δ_1 in Lemma 1 and for Δ_{ϵ_1} defined in Lemma 2 with respect to ϵ_1 in Lemma, let $\Delta_0 = \min\{\Delta_1, \Delta_{\epsilon_1}\} > 0$. Then to establish stability of $(\mathbf{p}_\Delta^n)$ for any $\Delta < \Delta_0 \le \Delta_1$, it suffices from the argument in the proof of Theorem 4 (*i*) to show that for each $\epsilon < \epsilon_1$ there is some $\delta(\epsilon) \in (0, \epsilon)$ small enough to ensure that $\|\mathbf{p}_\Delta^n - \mathbf{p}^*\| < \epsilon$ holds for all $n \in \mathbb{Z}_+$ whenever $\|\mathbf{p}_\Delta^0 - \mathbf{p}^*\| < \delta(\epsilon)$. However, by setting $\delta(\epsilon) = \mu\epsilon$ in Lemma 1, it then follows from (64) with $n = 0$ that $(\mathbf{p}_\Delta^n)$ must be stable with respect to $\mathbf{p}^*$. Finally, to establish that for each $\Delta < \Delta_0$ and choice of starting points, $\mathbf{p}_\Delta^0 \in \mathbb{P}_q$ it is true that $\lim_{n\to\infty} \|\mathbf{p}_\Delta^n - \mathbf{p}^*\| = 0$, observe from (63) that we need only show that $\|\mathbf{p}_\Delta^n - \mathbf{p}^*\| < \epsilon_1$ for some n. But since $\Delta < \Delta_0 \le \Delta_{\epsilon_1}$ implies from Lemma 2 that $\|\mathbf{p}_\Delta^n - \mathbf{p}^*\| < \epsilon_1$ for $n = n(\Delta)$, we may thus conclude that $(\mathbf{p}_\Delta^n)$ is globally asymptotically stable with respect to $\mathbf{p}^*$ for all $\Delta < \Delta_0$. ■

5. EXTENSIONS AND DIRECTIONS FOR FURTHER RESEARCH

In this two-part paper we have shown that the uniqueness and stability properties of steady states for a large class of gravity-type interactive Markov chains can be fully analyzed in terms of an associated programming problem. This implies that the dynamical behavior of such processes must be intimately related to the structure of the objective function Z in (21). In particular, it is clear from Theorem 5 that asymptotically stable steady states for these processes are implicitly associated with *minimal* values of Z. This minimizing property can be given an interesting behavioral interpretation. In particular, when attraction functions are given a cost interpretation, such as in the 'logit' model of Example I.1 in Part I, the resulting objective function in (21) can be shown to represent 'cumulative costs' analogous to those employed in stochastic network equilibrium models (as for example in Smith, 1988). Hence, by employing similar methods, such migration dynamics can be shown to be consistent with a 'cost efficiency' theory of migration behavior in which individuals exhibit an overall cost-minimizing tendency. This interpretation will be developed further in a subsequent paper.

However, it should be emphasized that the present programming formulation depends critically on the assumptions of both *symmetry* and *constancy* of accessibility measures. Although these two assumptions are quite reasonable in most empirical applications of gravity models, it is nonetheless of interest to ask whether the present uniqueness and stability properties continue to hold under more general conditions. Turning first to symmetry, it will be shown in a subsequent paper that this assumption can often be relaxed for constant accessibility measures. In particular, it will be shown that the uniqueness and stability properties for certain generalization of the 'pure congestion' case continue to hold in the asymmetric case. However, there appears to be no simple global 'Lyapunov' characterization of flow dynamics in this more general setting. Turning next to constancy, it is not surprising that relaxations of this assumption are considerably more difficult. However, one possible approach is suggested by the mathematical parallel between the structure of this problem and logit-type stochastic user equilibrium models of traffic flows. Here certain uniqueness and stability results have been obtained which may be extendable to the present setting (see for example Daganzo, 1982). Such possibilities will be explored in subsequent research.

Our remaining discussion focuses on the general nature of interactive Markov chains. We begin by recalling from the introductory discussion in Part I that for the case of unique steady states, the global asymptotic stability results in Section 4 above show that (for sufficiently small adjustment periods) deterministic interactive Markov chains are indeed valid approximations to their (higher dimensional) probabilistic counterparts. However, even when steady states are not unique, it can still be argued that interactive Markov chains may constitute viable models of population adjustment behavior in their own right. In particular, the 'large number' arguments used to characterize interactive Markov chains as probability limits of large-population Markov chains typically assume *conditionally independent* decision-making behavior by individuals. Hence, when such independence assumptions are violated, it is possible that interactive Markov chains may be better approximations to behavior than their higher dimensional counterparts. This is well illustrated by the case of 'pure agglomeration' behavior in Example I.2 of Part I, where strong interactions between individuals (such as 'band wagon' effects) may violate the conditional independence assumption. Here the multiple steady states of the interactive Markov chain model may indeed have more behavioral relevance than the (unique) steady state of its higher dimensional probabilistic Markov chain version. In particular, it can be shown that in this probabilistic steady state, the system will 'visit' the three locally stable vertex states equally often. However, in reality the first vertex state visited may act more like an 'absorbing state' which persists for the entire future of the system. If so, then the local convergence property of the interactive Markov chain model may in fact be a more accurate description of behavior.

These observations suggest that the dynamical properties of interactive Markov chain models (and their continuous approximations) can be of genuine behavioral interest in their own right. In particular, one may ask whether similar hereditary relations exist between interactive Markov chains and their continuous approximations when steady states are not unique. As one result along these lines, it can be shown that in typical cases consisting of finitely many locally stable hyperbolic steady states, with domains of attraction including all but a set of measure zero in $\mathbb{P}_q$, there are step sizes Δ sufficiently small to ensure that all but an arbitrarily small set of adjustment paths $(\mathbf{p}_\Delta^n)$ must converge to some locally stable steady state in $\mathbb{P}_q$. Results of this type for multiple steady states will be developed in a subsequent paper.

Finally, one may consider a number of possible extensions of the present class of interactive Markov chains, $(\mathbf{M}, \alpha, \Delta)$. First observe that fraction, $\alpha(\Delta)$, of the population making migration decisions may in fact depend on the current state of the system, $\mathbf{p}^t = (p_1^t, \ldots, p_q^t)$, as well as the time interval Δ (as in De Palma and Lefevre, 1983), and may also vary from region to region, $\alpha_i(\Delta, \mathbf{p}^t)$, $i = 1, \ldots, q$. For example in states, $\mathbf{p}^t = (p_1^t, \ldots, p_q^t)$, where gross inequalities exist among the current attraction levels, $a_t(p_i^t)$, of various locations, it may be postulated that more individuals consider migrating from 'unattractive' regions than from 'attractive' regions during the given adjustment period Δ. Such variations in participation levels can be incorporated by employing convergence results for more general classes of discrete dynamical adjustment processes (as for example in Derevitskii and Fradkov, 1974). Moreover, by employing more recent results in stochastic approximation theory, it is possible to introduce probabilistic variations directly into the interactive Markov chain model itself. For example, the 'large deviation' results of Kushner and Kuang (1981) allow certain 'weak convergence' results to be obtained for cases in which random terms, ϵ_t, are added to $\phi(\mathbf{p}^t)$ in (54). Such extensions will be considered in subsequent papers.

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