Collective mode contributions to the Meissner effect: Fulde-Ferrell and pair-density wave superfluids

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In this paper we demonstrate the necessity of including the generally omitted collective-mode contributions in calculations of the Meissner effect for nonuniform superconductors. We consider superconducting pairing with nonzero center-of-mass momentum, as is possibly relevant to high transition temperature cuprates, cold atoms, and color superconductors in quantum chromodynamics. For the concrete example of the Fulde-Ferrell phase we present a quantitative calculation of the superfluid density, showing not only that the collective-mode contributions are appreciable but also that they derive from the amplitude mode of the order parameter. This latter mode is generally viewed as being invisible in conventional superconductors. However, our analysis shows that it is extremely important in pair-density-wave-type superconductors, where it destroys stable superfluidity well before the mean-field order parameter vanishes.

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I. INTRODUCTION

There is a large amount of interest in amplitude modes of superconductors in large part stimulated by the excitement surrounding the discovery of the Higgs boson [1]. Nevertheless, there is a widespread belief that observing these modes, directly or indirectly, is particularly challenging [2,3]. As a result they only infrequently appear in condensed-matter physics [4–10]. In this paper we show that in a class of very topical superconductors collective-mode effects associated with the amplitude of the order parameter play an essential role in the most fundamental quantity, the superfluid density tensor n_s^{ij}. The superconductors in question are those which have a "pair-density wave" order parameter. These are a large class of theoretical models (awaiting firm experimental confirmation) associated with pairing of electrons at nonzero center-of-mass momentum $\mathbf{Q}$. Much attention has focused on these systems from the perspective of high-temperature superconductivity (in condensed-matter physics [11,12]) and color superconductors (in particle physics [13]).

For this class of superfluids, the collective-mode contribution to the superfluid density has been largely ignored in previous literature [14,15], with the exception of the original calculation of the electromagnetic current by Larkin and Ovchinnikov [16]. Discussion of this effect can also be found in Ref. [17] for a different situation involving non-s-wave superconductors [18]. In both cases, however, the size of the collective-mode contributions was not accessible.

We provide two different, but related, derivations of the superfluid density for the tractable case of the Fulde-Ferrell (FF) superfluid [19]. Importantly, this enables us to compute numerical values for the sizable collective-mode effects in n_s^{ij}. The first method is based on using the Ward-Takahashi identity (WTI) in the Kubo formalism, while the second method is based on studying the equilibrium current. In both approaches particle number is manifestly conserved and gauge invariance is maintained. Through these approaches we find that amplitude collective modes drive the superfluid density (along the direction parallel to $\mathbf{Q}$) to zero at temperatures lower than those associated with the vanishing of the mean-field order parameter.

Before giving these more complete calculations, here we provide a general argument for the necessity of including collective-mode effects in nonuniform superconductors. The origin of collective-mode contributions to the Meissner effect [16,17] is due to the fact that, in the presence of a vector potential A^μ, the order parameter Δ will depend on A^μ through the gap (saddle-point) equation [9,20]. A series expansion of $\Delta[A]$, in powers of A^μ, is thus

$$\Delta[A] = \Delta^{(0)}[A=0] + \Delta^{(1)}[A] + \mathcal{O}(A^2). \qquad (1.1)$$

Here $\Delta^{(0)}$ is the order parameter in the absence of A^μ and $\Delta^{(1)}$ is a correction linear in A^μ. It is this term which gives rise to the rarely discussed collective-mode contributions to the superfluid density. Since $\Delta^{(1)}$ is a scalar quantity, it can depend on only scalar, linear functionals of A^μ. Therefore, in a uniform superfluid $\Delta^{(1)}$ is a functional of only $\nabla \cdot \mathbf{A}$ [21]. Thus, if one chooses the ("transverse") gauge such that $\nabla \cdot \mathbf{A} = 0$, the collective-mode contribution $\Delta^{(1)}$ vanishes identically [20].

However, for a nonuniform system there are other scalar, linear functionals of A^μ. In particular, for a pair-density wave superfluid with pairing vector $\mathbf{Q}$, $\Delta^{(1)}$ can depend on other scalar, linear quantities such as $\mathbf{A} \cdot \mathbf{Q}$. Hence, for this nonuniform superfluid, even in the gauge where $\nabla \cdot \mathbf{A} = 0$, $\Delta^{(1)}$ may still be nonzero. In principle, this allows for a collective-mode contribution to the superfluid density. [For future use in the discussion below, we define $\Delta^{(1)} = \int dq\ \Pi^\mu(q) A_\mu(q)$ and $\Pi^\mu(q) = (\delta\Delta[A]/\delta A_\mu(q))|_{A=0}$]. This argument emphasizes that, for any nonuniform superfluid with a pairing vector present, one must consider collective-mode contributions. In particular, it also applies to a system where $\mathbf{Q}$ is *a priori* fixed, such as in a crystalline superconductor, where the rotational symmetry is explicitly, and not spontaneously, broken. Here one would still need to consider the collective-mode contributions to the superfluid density, due to a finite $\mathbf{Q}$.

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For concreteness we will illustrate these collective-mode effects in the FF superfluid, where pairing at finite $\mathbf{Q}$ arises due to a spin (or mass) imbalance causing a Fermi-surface mismatch between up- and down-spins. For simplicity we take the FF pairing vector as $\mathbf{Q} = Q\hat{z}$. In the FF phase specifically, both a continuous rotational and global $U(1)$ symmetry are spontaneously broken. Similarly discrete time-reversal symmetry is also spontaneously broken. However, gauge invariant observables are translationally invariant [22]. Due to the underlying rotational symmetry of the FF state, the superfluid density vanishes along the directions transverse to $\mathbf{Q}$. Hence $n_s^{xx} = n_s^{yy} = 0$, and thus only n_s^{zz} needs to be considered [22].

As has been posited [19], and will be shown in more detail below, the superfluid density for the FF superfluid can be written as

$$\left.\frac{\partial j^z}{\partial Q}\right|_{\mu,h} = \frac{1}{2}\left(\frac{n_s^{zz}}{m}\right), \tag{1.2}$$

where $j^z(Q)$ is the equilibrium current. It is useful to express Eq. (1.2) in terms of the mean-field thermodynamic potential Ω, where $j^z(Q) = 2(\partial\Omega/\partial Q)|_{\mu,h,|\Delta|}$. The saddle-point condition which determines the mean-field value of Q is then $j^z(Q) = 0$. Similarly, the saddle-point condition which determines the mean-field value of Δ is $(\partial\Omega/\partial\Delta)|_{\mu,h,Q} = 0$. In terms of Ω, Eq. (1.2) becomes

$$\left.\frac{\partial j^z}{\partial Q}\right|_{\mu,h} = 2\left[\left.\frac{\partial^2\Omega}{\partial Q^2}\right|_{\mu,h,|\Delta|} - \left(\frac{\partial^2\Omega}{\partial|\Delta|\partial Q}\right)^2 \Big/ \left.\frac{\partial^2\Omega}{\partial|\Delta|^2}\right|_{\mu,h,Q}\right], \tag{1.3}$$

where both saddle-point equations and the symmetry of mixed partial derivatives have been used.

Equation (1.3) indicates that there are two contributions to the superfluid density. The first is the conventional "bubble" term (which is usually assumed to be sufficient) and the second represents the collective-mode contribution required for gauge invariance. Importantly, a stability inequality for the FF superfluid based on the thermodynamic potential curvature [23–25] is equivalent to requiring that both n_s^{zz}, as derived above, and $(\partial^2\Omega/\partial|\Delta|^2)|_{\mu,h,Q}$ are positive. It follows from the latter condition that for a stable FF superfluid the collective-mode contribution always acts to reduce the overall size of the superfluid density. These arguments, however, still do not indicate how large the magnitude of this effect is.

In this paper the collective-mode contribution will be found to be appreciable; this underlines the inadequacy of including only the so-called bubble term [14,15]. Equally important is the nature of these collective-mode corrections. For the FF superfluid we will show that they derive from the *amplitude* mode of the order parameter. This mode is thought to be rather invisible in conventional superconductors [2]. Nevertheless we demonstrate how it arises to ensure the electromagnetic (EM) response is manifestly gauge invariant. Readers who wish to quickly take away the main message of this paper can proceed directly to the numerical results.

II. THEORETICAL FORMALISM

A. Mean-field results

The FF mean-field Hamiltonian, in the $\psi_{\mathbf{k}}^{\mathsf{T}} = (c_{\mathbf{k},\uparrow}, c^{\dagger}_{-\mathbf{k}+\mathbf{Q},\downarrow})$ basis, is $H_{\mathrm{FF}} = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^{\dagger}\mathcal{H}_{\mathrm{FF}}\psi_{\mathbf{k}}$, where [26]

$$\mathcal{H}_{\mathrm{FF}} = \begin{pmatrix} \xi_{\mathbf{k},\uparrow} & -\Delta \\ -\Delta^* & -\xi_{\mathbf{k}-\mathbf{Q},\downarrow} \end{pmatrix}. \tag{2.1}$$

Here an irrelevant constant $-\sum_{\mathbf{k}} \xi_{\mathbf{k}-\mathbf{Q},\downarrow}$ has been ignored. The notation is as follows: the dispersion relation is defined by $\xi_{\mathbf{k},\sigma} = \mathbf{k}^2/2m - \mu_\sigma$, where μ_σ is the fermionic chemical potential for a species with spin $\sigma = \uparrow, \downarrow$, m is the fermion mass, and Δ denotes an s-wave pairing gap. It is useful to define $\mu = \frac{1}{2}(\mu_\uparrow + \mu_\downarrow)$ and $h = \frac{1}{2}(\mu_\uparrow - \mu_\downarrow)$. The dispersion relations are then written compactly as $\xi_{\mathbf{kQ}} = (1/2m)[\mathbf{k}^2 + (\mathbf{Q}/2)^2] - \mu$, $E_{\mathbf{kQ}}^2 = \xi_{\mathbf{kQ}}^2 + |\Delta|^2$, $h_{\mathbf{kQ}} = h - \mathbf{k}\cdot\mathbf{Q}/2m$. Throughout the paper $\hbar = k_B = c = 1$.

The inverse Nambu Green's function is then $\mathcal{G}^{-1} = i\omega_n - \mathcal{H}_{\mathrm{FF}}$, where $i\omega_n$ is a fermionic Matsubara frequency. The inverse bare Green's function is defined by $G_{0,\sigma}^{-1}(k) = i\omega_n - \xi_{\mathbf{k}}$. Thus, the off-diagonal Gorkov function is $\mathcal{G}_{12}(k) = \Delta G_{0,\downarrow}(-k+Q)G_\uparrow(k)$, where the (spin-up) Green's function is $G_\uparrow(k) = \mathcal{G}_{11}(k)$. Note that Greek indices denote space-time coordinates: $x^\mu = (t,x,y,z)$, whereas Roman indices denote spatial coordinates: $x^i = (x,y,z)$. Here $Q^\mu = (0,\mathbf{Q})$. Explicit calculation then gives the full Green's function which has appeared in the literature [14,26]. Indeed, from Dyson's equation, $G_\sigma^{-1}(k) = G_{0,\sigma}^{-1}(k) - \Sigma_\sigma(k)$, the self-energy is $\Sigma_\sigma(k) = -|\Delta|^2 G_{0,\bar{\sigma}}(-k+Q)$. [For convenience later, we define $\widetilde{k}_\pm^\mu \equiv k^\mu \pm Q^\mu/2$].

B. Electromagnetic response and Ward-Takahashi identity

We now study the EM response of the FF superfluid and consider the superfluid density for the case of a neutral superfluid. Our observation that collective modes are necessary to ensure a gauge invariant superfluid density calculation should apply to a charged system as well, but we avoid here the complexities associated with the Coulomb interaction. We apply linear-response theory, where a fictitious vector potential A^μ is applied to the system. The resulting EM current is $j^\mu(q) = K^{\mu\nu}(q)A_\nu(q)$, where $K^{\mu\nu}(q)$ is the EM response kernel. The response kernel can also be expressed as $K^{\mu\nu}(q) = P^{\mu\nu}(q) + (n/m)\delta^{\mu\nu}(1-\delta_{\mu,0})$ (with μ and ν not summed over) where the EM response functions are denoted by $P^{\mu\nu}(q)$. In the Kubo formalism the EM response functions for a superfluid are

$$P^{\mu\nu}(q) = \sum_\sigma \sum_k G_\sigma(k_+)\Gamma_\sigma^\mu(k_+,k_-)G_\sigma(k_-)\gamma_\sigma^\nu(k_-,k_+), \tag{2.2}$$

where $q^\mu = (i\Omega_m, \mathbf{q})$, with $i\Omega_m$ a bosonic Matsubara frequency.

The important quantity $\Gamma^\mu(k_+,k_-)$ denotes the full EM vertex, where the incoming (outgoing) momentum is k_+ (k_-), with $k_\pm^\mu \equiv k^\mu \pm q^\mu/2$. To determine the full vertex $\Gamma^\mu(k_+,k_-)$, we apply the WTI [27]:

$$\begin{aligned} q_\mu\Gamma_\sigma^\mu(k_+,k_-) &= G_\sigma^{-1}(k_+) - G_\sigma^{-1}(k_-) \\ &= q_\mu\gamma_\sigma^\mu(k_+,k_-) + \Sigma_\sigma(k_-) - \Sigma_\sigma(k_+). \end{aligned} \tag{2.3}$$

This is an exact relation in quantum field theory which relates the single-particle Green's function to the full vertex. It is a gauge invariant statement and here it reflects the underlying global $U(1)$ symmetry. Similarly, the bare WTI, $q_\mu \gamma^\mu_\sigma(k_+,k_-) = G^{-1}_{0,\sigma}(k_+) - G^{-1}_{0,\sigma}(k_-)$, is satisfied by the bare vertex $\gamma^\mu_\sigma(k_+,k_-) = (1,\mathbf{k}/m)$.

Satisfying the WTI ensures conservation of particle number (or charge in the charged superfluid case). The particle number can be written in terms of the single-particle Green's function as $n = \sum_\sigma \sum_k G_\sigma(k)$, where $\sum_k \equiv \beta^{-1}\sum_{i\omega_n}\sum_{\mathbf{k}}$ with β being inverse temperature. In the limit $q^\mu \to 0$, the WTI reduces to the Ward identity: $\Gamma^\mu_\sigma(k,k) = \gamma^\mu_\sigma(k,k) - [\partial\Sigma_\sigma(k)/\partial k_\mu]$. The second term in this equation diagrammatically represents a vertex insertion in the self-energy. This relation then importantly shows that the full vertex can be obtained by performing all possible vertex insertions in the full Green's function [27,28].

For the FF self-energy given in Sec. II A, there are three possible positions where the bare vertex can be inserted: the bare Green's function and the two gaps Δ and Δ^*. Inserting the bare vertex into the position of the two gaps leads to the collective-mode vertices, discussed in more detail in the next section, which are of crucial importance to ensure gauge invariance.

C. Collective-mode vertices

This section discusses the properties of the collective-mode vertices and how they contribute to the superfluid density. By inserting the bare vertex in the two gaps (Δ,Δ^*) one obtains the collective-mode vertices $\Pi^\mu(q)$ and $\bar{\Pi}^\mu(q)$, respectively. Reference [29] presents details showing how $\Pi^\mu(q)$ and $\bar{\Pi}^\mu(q)$ are obtained by performing these vertex insertions in the gap equation, which can be written as $\Delta/g = \Delta\sum_\sigma\sum_k G_{0,\downarrow}(-k+Q)G_\uparrow(k) = \sum_\sigma\sum_k \mathcal{G}_{12}(\widetilde{k})$ [26].

Due to the spontaneously broken global $U(1)$ symmetry, the gaps Δ,Δ^* are themselves not gauge invariant. There are, however, two natural gauge invariant combinations of the collective-mode vertices; these appear as $(\Delta^*\Pi^\mu - \Delta\bar{\Pi}^\mu)$ and $(\Delta^*\Pi^\mu + \Delta\bar{\Pi}^\mu)$. In order to associate these combinations with the appropriate phase or amplitude modes of the order parameter, we contract these quantities with q_μ. In Ref. [29] it is proved that, for $q_\mu \neq 0$, the collective-mode vertices obey $q_\mu\Pi^\mu(q) = 2\Delta, q_\mu\bar{\Pi}^\mu(q) = -2\Delta^*$. These relations then lead to

$$q_\mu(\Delta^*\Pi^\mu - \Delta\bar{\Pi}^\mu) = 4|\Delta|^2, \tag{2.4}$$

$$q_\mu(\Delta^*\Pi^\mu + \Delta\bar{\Pi}^\mu) = 0. \tag{2.5}$$

The right-hand sides of these expressions are gauge invariant quantities, and thus so too are the expressions in parentheses, as claimed.

The limit $q_\mu \to 0$ of these contractions is of particular interest. For Eq. (2.4), the right-hand side is finite, nonzero, and independent of q_μ; this applies to the left-hand side as well. As $q_\mu \to 0$, it follows that the quantity in parentheses must become singular in this limit. This indicates that it has a zero-momentum pole; we can conclude that this is to be associated with the Nambu-Goldstone boson that restores the global $U(1)$ symmetry. Since the phase mode of the order parameter is responsible for restoring gauge invariance, it follows that $(\Delta^*\Pi^\mu - \Delta\bar{\Pi}^\mu)$ corresponds to the phase mode of the order parameter.

On the other hand, for Eq. (2.5), the right-hand side is zero and independent of q_μ; this applies to the left-hand side as well. As $q_\mu \to 0$, it follows that the quantity in parentheses must be nonsingular in this limit. This indicates the quantity in parentheses does not have a zero-momentum pole. It follows that $(\Delta^*\Pi^\mu + \Delta\bar{\Pi}^\mu)$ corresponds to the amplitude mode of the order parameter.

The next section studies the superfluid density where we find that $(\Delta^*\Pi^z + \Delta\bar{\Pi}^z)$ and not $(\Delta^*\Pi^z - \Delta\bar{\Pi}^z)$ contributes. Thus the phase mode, while contained within the individual Π^μ and $\bar{\Pi}^\mu$ expressions, does not directly contribute to the superfluid density [30].

We end by noting that Δ is a function of the FF pairing vector Q, and by differentiating the gap equation with respect to Q (at fixed μ and h) one can obtain $(\partial|\Delta|^2/\partial Q)|_{\mu,h}$. An explicit calculation then gives the following important identity, for $\Delta \neq 0$, which relates in a more transparent way to the amplitude mode:

$$\Delta^*\Pi^z(0) + \Delta\bar{\Pi}^z(0) = P^z_0/M_0 = 2(\partial|\Delta|^2/\partial Q)|_{\mu,h}. \tag{2.6}$$

The order of limits in which frequency and momentum are taken to zero is important; frequency $i\Omega_m$ and q^z are set to zero, and then $q^x,q^y \to 0$. In the following section this will be clarified. The quantities P^z_0 and M_0 are generalized three-particle and four-particle Green's functions, respectively, which are defined in the next section. The generalized Green's functions in Eq. (2.6) also appear in a similar form in the work of Larkin and Ovchinnikov [16] and Millis [17]. Finally, note that, when $Q = 0$, $P^z_0 = 0$, and thus this collective-mode term does not contribute for a uniform superfluid. The size of the amplitude mode contribution is thus set (in part) by Q/k_F. In principle this allows for a significant collective-mode contribution, as will be discussed further in Sec. IV.

III. SUPERFLUID DENSITY

A. Superfluid density derivation via Kubo formula

In this section we use the Kubo formula and Eq. (2.2) to derive the superfluid density tensor:

$$(n^{ij}_s/m) = (n/m)\delta^{ij} + P^{ij}(\omega = 0,\mathbf{q} \to 0). \tag{3.1}$$

Note that the order of limits in the above expression is crucial. To compute n^{ij}_s, first set $\omega = q^i = q^j = 0$, then take $q^k \to 0$, where $k \neq i,j$. The collective modes are contained within the second term.

Evaluating this expression we find

$$\left(\frac{n^{ij}_s}{m}\right) = \sum_{\mathbf{k}} \frac{|\Delta|^2}{E^2_{\mathbf{kQ}}}\left(\frac{X_{\mathbf{k}}}{E_{\mathbf{kQ}}} - \beta Y_{\mathbf{k}}\right)(\widetilde{k}^i_-/m)(\widetilde{k}^j_+/m) - \delta^{iz}\delta^{jz}\left(P^z_0\right)^2/M_0, \tag{3.2}$$

where we define $X_{\mathbf{k}} \equiv D^{-1}\sinh(\beta E_{\mathbf{kQ}})$, and $Y_{\mathbf{k}} \equiv D^{-2}[1 + \cosh(\beta E_{\mathbf{kQ}})\cosh(\beta h_{\mathbf{kQ}})]$ with $D \equiv \cosh(\beta E_{\mathbf{kQ}}) + \cosh(\beta h_{\mathbf{kQ}})$.

The first term in Eq. (3.2) represents the usual [14,15] "bubble" contribution, due to bubble terms in both $(n/m)\delta^{ij}$

and $P^{ij}(0)$. The second term represents the collective-mode contribution arising solely from $P^{ij}(0)$. As an important check, we note that Eq. (3.2) is identical to the superfluid density obtained from Eqs. (1.2) and (1.3). Explicit calculation shows that the bubble term is $4(\partial^2\Omega/\partial Q^2)|_{\mu,h,|\Delta|}$ and $(\partial^2\Omega/\partial|\Delta|\partial Q) = -|\Delta|P_0^z, (\partial^2\Omega/\partial|\Delta|^2)|_{\mu,h,Q} = 4|\Delta|^2 M_0$, where the mean-field thermodynamic potential [14,25] is $\Omega = |\Delta|^2/g - \beta^{-1}\sum_{\mathbf{k}}\{\log[2\cosh(\beta E_{\mathbf{kQ}}) + 2\cosh(\beta h_{\mathbf{kQ}})] - \beta\xi_{\mathbf{kQ}}\}$.

Note that the collective-mode contribution is only along the direction of the FF pairing vector, in agreement with the general arguments presented earlier. Direct calculation shows that n_s^{ij} is diagonal, with $n_s^{xx} = n_s^{yy} = 0$, as required by symmetry.

B. Derivation via equilibrium current

A verification of this Kubo analysis and the collective mode contributions can be made in a slightly simpler fashion. Here we derive the superfluid density in the direction along the FF pairing vector using only the equilibrium current and its partial derivative with respect to Q. The equilibrium current in the z direction is $j^z(Q) = \sum_\sigma \sum_k (\tilde{k}_+^z/m)G_\sigma(\tilde{k})$. This expression follows from $j^z = 2(\partial\Omega/\partial Q)|_{\mu,h,|\Delta|}$. By symmetry the mean-field currents in the other directions vanish: $j^x = j^y = 0$.

In what follows it will be important to fix μ and h to their mean-field values, and to consider the Q dependence of only the gap: $\Delta(Q)$. The following lemma, the proof of which is given in Ref. [29], will also be required: $(\partial G_\sigma^{-1}(\tilde{k}_+)/\partial Q)|_{\mu,h} = -(1/2)\Gamma_\sigma^z(\tilde{k}_+,\tilde{k}_+)$. The partial derivative of j^z can now be computed. Using the number equation $n = \sum_\sigma\sum_k G_\sigma(k)$, along with the aforementioned lemma, the partial derivative of j^z is then $(\partial j^z/\partial Q)|_{\mu,h} = (n/2m) - \sum_\sigma\sum_k(\tilde{k}_+^z/m)G_\sigma^2(\tilde{k})(\partial G_\sigma^{-1}(\tilde{k}_+)/\partial Q)|_{\mu,h} = (n_s^{zz}/2m)$. Note that the above expression, which reproduces Eqs. (1.2) and (3.2), includes collective-mode contributions arising through $\Gamma^z(\tilde{k}_+,\tilde{k}_+)$.

IV. NUMERICAL RESULTS

We now illustrate numerically the regime of stability of the FF phase. We require first that the superfluid density n_s^{zz} as computed in the theory outlined above is positive and second that the state of interest is a minimum of the thermodynamic potential. These conditions correspond to $n_s^{zz} > 0$ and $(\partial^2\Omega/\partial|\Delta|^2)|_{\mu,h,Q} > 0$. Although derived differently, these criteria coincide with results in the recent literature [25]. Importantly, they are a useful way to characterize the various temperature regimes in mean-field FF superfluid systems. We associate the critical temperature T_c^{FF} with the temperature at which either one of these stability conditions fails. Additionally, we associate the temperature T_Q as the temperature at which the FF pairing vector Q vanishes. Finally, T_Δ represents the temperature at which the mean-field pairing gap vanishes.

For the specific region of the phase diagram studied, our numerical calculations show that there are three temperature regimes of interest:

(i) $0 \leqslant T < T_c^{\mathrm{FF}}$ is the regime where a stable FF phase exists: both $n_s^{zz} > 0$ and $(\partial^2\Omega/\partial|\Delta|^2)|_{\mu,h,Q} > 0$.

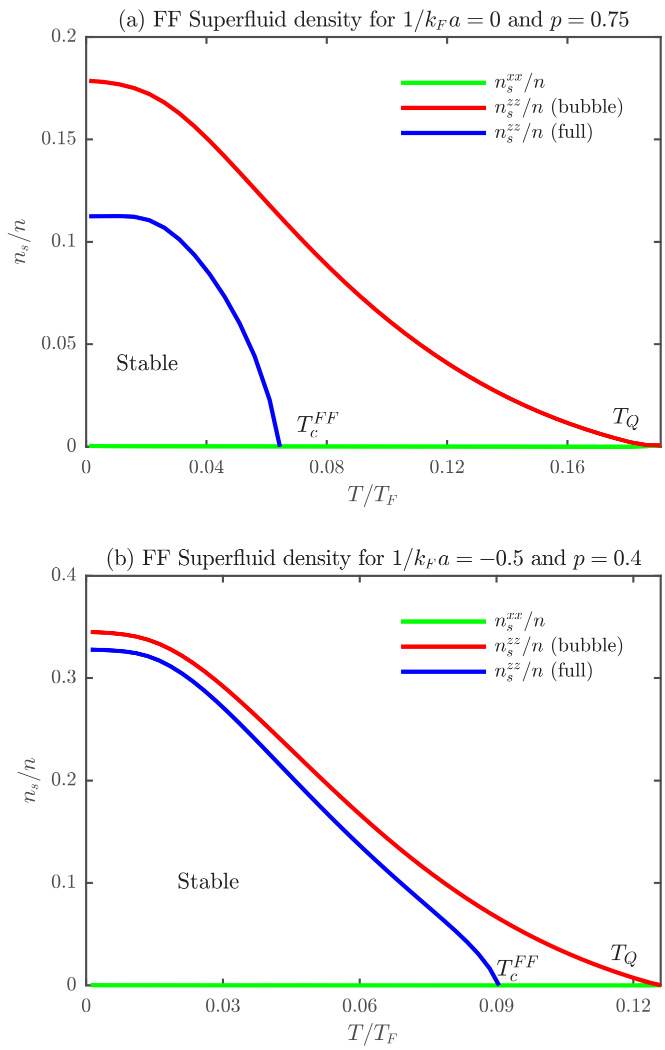

FIG. 1. Superfluid density as a function of temperature for the FF phase (a) at unitarity ($1/k_Fa = 0$) and with polarization $p = (n_\uparrow - n_\downarrow)/n$ set to $p = 0.75$ and (b) in the near-BCS regime ($1/k_Fa = -0.5$) with polarization $p = 0.4$. The blue curves denote the full expressions for n_s^{zz}/n given in Eq. (3.2), while the red curves denote the bubble contribution alone, given in the first line of Eq. (3.2). The green curves denote n_s^{xx}/n; in this case there are no collective modes and so the bubble and full expressions are equivalent.

(ii) $T_c^{\mathrm{FF}} \leqslant T < T_Q$ is the regime where an unstable FF phase exists. Either one or the other (or both) of the stability conditions fails; that is, $n_s^{zz} < 0$ or $(\partial^2\Omega/\partial|\Delta|^2)|_{\mu,h,Q} < 0$.

(iii) $T_Q < T < T_\Delta$ is the regime where $\Delta \neq 0$, but $Q = 0$. This corresponds to the Sarma phase. Since $Q = 0$ in this regime, there is no collective-mode contribution to the superfluid density, and moreover $n_s^{zz} = n_s^{xx} = n_s^{yy} > 0$.

Figure 1 encapsulates the important point that collective modes of the order parameter will substantially reduce the region where there is a stable FF phase. In Fig. 1(a) we plot the superfluid density with collective-mode effects (blue curve) as a function of temperature for the case of polarization $p = 0.75$ and interaction strength (in terms of the scattering amplitude) $1/k_Fa = 0$. Similarly, Fig. 1(b) plots the superfluid density with collective-mode effects (blue curve) for a more BCS-like case $p = 0.4$ and $1/k_F a = -0.5$. The red curves in Fig. 1 denote the bubble contribution to the superfluid density which is historically [14,15] all that is considered. The green curves plot the transverse superfluid density. As required by symmetry, $n_s^{xx} = 0$ for all $T < T_Q$ for which the FF pairing vector Q persists.

In Fig. 1(a) we identify $T_c^{\mathrm{FF}}/T_F \sim 0.06$–$0.065$, in rough agreement with Ref. [25]. Additionally, $T_Q \sim 0.2T_F$, and $T_\Delta \sim 0.24T_F$ (not shown in Fig. 1). Even though this plot is in the strong interaction regime, for quantitative purposes, strict mean-field parameters are used in this plot. It should be noted that the effects of the collective modes are quite appreciable in this plot. This follows because the bubble term is proportional to $(\Delta/E_F)^2$, whereas the collective-mode term is proportional to $(Q/k_F)^2$. (Note though the integrands in both expressions are somewhat different.) Near zero temperature, with $p = 0.75$ and $1/k_F a = 0$, $\Delta/E_F \sim 0.16$ whereas $Q/k_F \sim 0.71$. Thus qualitatively the collective-mode contribution is expected to be an important contribution in this regime.

One can determine from Fig. 1(b) that the highest temperature for which the FF superfluid is stable is given by $T_c^{\mathrm{FF}}/T_F \sim 0.09$–$0.095$ (in rough agreement with Ref. [24]). The other temperature scales of interest are found to be roughly $T_Q \sim 0.13T_F$, and $T_\Delta \sim 0.17T_F$. In this plot the effects of the collective modes are not as appreciable. Near zero temperature, with $p = 0.4$ and $1/k_F a = -0.5$, we find $\Delta/E_F \sim 0.13$ whereas $Q/k_F \sim 0.38$, so that the collective-mode contribution is still expected to be appreciable, albeit not as large as exhibited in Fig. 1(a).

For numerical checks on our results we have verified that, in the stable regime, our mean-field solutions are global minima of the thermodynamic potential [31] and that the blue curve computed via Eq. (3.2) is numerically equivalent to that computed via the equilibrium current using Eq. (1.2).

V. CONCLUSIONS

In this paper we have computed the superfluid density tensor n_s^{ij} for the FF superfluid phase. Importantly, we have shown (using multiple, distinct theoretical frameworks) that widely neglected collective (amplitude) mode contributions cannot be ignored. In general they will affect n_s^{ij} for the broad class of $\mathbf{Q} \neq 0$ pair-density wave superconductors. Indeed, while Fig. 1 was obtained using the specific microscopic approach of Fulde and Ferrell, we believe its qualitative features (except for the behavior of the transverse superfluid density) are more generic. It is useful to note that in the original paper [16] by Larkin and Ovchinnikov the authors addressed the superfluid density of pair-density wave phases; however, they used a small Δ expansion, necessarily valid near T_Δ. Note that our numerical results show this may be well removed from the stable FF regime.

Given the intense interest in condensed-matter observations of a Higgs mode, one can inquire as to what is the relation between the amplitude mode evident in pair-density wave superconductors and the Higgs mode in condensed matter [4–6,8–10,32]. The Higgs mechanism is associated with a charged system, where the Nambu-Goldstone mode is gapped out due to the Englert-Brout-Higgs-Guralnik-Hagen-Kibble mechanism. In the present theory we argue that the general observation that amplitude modes affect the superfluid density applies, even though we have implemented the calculations for the neutral case. Moreover, in the present theory we incorporate the effects of the amplitude mode only at zero frequency and zero wave number so that the amplitude mode is not observed as a collective resonance. Nevertheless, we have ascertained in this paper that the existence of an amplitude mode has important consequences for readily accessible physical quantities in pair-density wave superconductors.

Note added. After this paper was completed a paper addressing the "Higgs" mode in FF and other pair-density wave superfluids appeared [32]. However, as in the present paper, the authors focused on neutral systems. Their work addresses the finite frequency behavior of the amplitude mode in these systems. In our formalism, this mode can be obtained from the finite frequency branch cut in $\Pi^\mu, \bar{\Pi}^\mu$, which can be found from the analytic expressions given for these quantities in Ref. [29].

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[30] It is important to note that while phase fluctuations alone are sufficient to restore gauge invariance, they are not necessary [9]. Amplitude fluctuations must also be considered to derive the full gauge invariant response. The $\Pi^\mu,\bar{\Pi}^\mu$ collective-mode vertices contain both phase and amplitude fluctuations of the order parameters Δ,Δ^*; however, it is found that for the FF superfluid density only the gauge invariant combination $\Delta^*\Pi^z(0)+\Delta\bar{\Pi}^z(0)$ contributes. The superfluid density is a zero-momentum response and thus the gapped nature of the amplitude modes is not of crucial importance. In particular, for the FF superfluid, Q/k_F is an important scale for the size of the contribution of the amplitude modes to the superfluid density. In Sec. IV it is found that this effect can be appreciable.
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