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Optimal production run time for two-stage production system with imperfect processes and allowable shortages

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Abstract This paper considers a two-stage production system with imperfect processes. Shortages are allowed, and the unsatisfied demand is completely backlogged. In addition, the capital investment in process quality improvement is adopted. Under these assumptions, we first formulate the proposed problem as a cost minimization model where the production run time and process quality are decision variables. Then we develop the criterion for judging whether the optimal solution not only exists but also is unique. If the criterion is not satisfied, the production system should not be opened. An algorithm for the computations of the optimal solutions is also provided. Finally, a numerical example and sensitivity analysis are carried out to illustrate the model.

Keywords Inventory · Imperfect production process · Quality improvement · Shortages · Two-stage system

1 Introduction

In the classical economics manufacturing quantity (EMQ) models, one often assumed that the production facility is perfect and all the finished products are good quality. However, in the real world cases, the product quality is not always perfect and usually

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depends on the state of the production process. Recently, some studies have pointed out that the unreliable production facility is subject to random deterioration from the in-control state to the out-of-control state (Rosenblatt and Lee 1986; Porteus 1986) or random failure (Groenevelt et al. 1992; Hong et al. 1992; Das and Sarkar 1999; Liu and Cao 1999). Moreover, extending this unreliable production facility based on various considerations. Kim and Hong (1999) extended Rosenblatt and Lee (1986) to incorporate a more generalized assumption that an elapsed time until process shift is arbitrarily distributed. Lin and Hou (2005) took the restoration cost into account when the system is an out-of-control at the end of a production run. Chung and Hou (2003) and Chen and Lo (2006) determined the optimal production run time with imperfect production process and allowable shortages. Lin and Lin (2007) assumed that the defective products can not be repaired and reworked and must be scraped with additional cost. Chiu (2007) examined the production run time problem with random machine breakdowns under abort/resume (AR) policy and reworking of defective products produced. Furthermore, the effect of investment on quality improvement is often investigated. Keller and Noori (1988), Hong and Hayya (1995), Hariga and Ben-Daya (1998) studied the economic benefits of reducing setup cost and improving process quality by simultaneously investing in new technology. Ouyang et al. (2002) investigate the joint effects of quality improvement and setup cost reduction in which the lot size, process quality, setup cost, and lead time are decision variables. Next, Chang and Ouyang (2002) extended Porteus (1986) model with investment in quality improvement in the fuzzy sense.

In the other hand, the present industrial settings, products are processed through multi-stage production system. Several authors have developed various multi-stags models in the literatures and pointed out that the two-stage models can be also used to approximate more complex multi-stage systems. Szendrovits (1983) proposed several two-stage production/inventory models in which smaller lots are produced at one stage and one larger lot is produced at the other stage. Kim's (1999) considered two-stage lot sizing problems with various lot sizing depending on batch transfer and production rates between stages. Hill (2000) extended Kim's (1999) model provide an alternative way of performing the analysis which is easier to understand. Darwish and Ben-Daya (2007) investigated the effect of imperfect production processes involving variable the frequency of preventive maintenance. Sarker et al. (2008) considered a multi-stage serial production problem in an unreliable production environment for two different operational policies. Besides, they also argued that the defective products are produced during the production time. The defective products are then corrected during the rework period.

The main purpose of this paper is to consider that the production system with imperfect processes is separated into two stages. The former stage (Stage 1; raw material to semi-finished products), is an automatic process (forming, polish or cutting), and this process is treated by machines. The latter stage (Stage 2; semi-finished products to finished products) is a manual process (painting, assembly or packing), and this process is treated by labors. It is well known that the production rate of Stage 1 is always higher than Stage 2. Under this situation, the semi-finished products are going to be accumulated between the Stage 1 and Stage 2. In addition, the capital investment in process quality improvement is adopted. Then, we formulate the proposed problem as a cost minimization model where the production run time and process quality are decision variables. We also prove that the optimal solution not only exists but also is unique. Finally, a numerical example is presented to demonstrate the developed model and the solution procedure, then the sensitivity analysis of the optimal solution with respect to major parameters is carried out.

2 Notation and assumptions

2.1 Notation

To develop the mathematical model of inventory system, the notation adopted in this paper is as below:

$p_1 =$ production rate of Stage 1 in units per unit time.
$p_2 =$ production rate of Stage 2 in units per unit time.
$D =$ demand rate in units per unit time.
$k =$ setup cost per cycle.
$c_p =$ purchasing and labor cost per unit.
$c_s =$ shortage cost per unit per unit time.
$c_{h1} =$ holding cost for a semi-finished product per unit time.
$c_{h2} =$ holding cost for a finished product per unit time.
$c_{r1} =$ rework cost for a defective semi-finished product.
$c_{r2} =$ rework cost for a defective finished product.
$N_1 =$ number of defective semi-finished products.
$N_2 =$ number of defective finished products.
$\theta_1 =$ percentage of defective semi-finished products produced. (decision variable)
$\theta_2 =$ percentage of defective finished products produced.
$t_1 =$ time period when there is no production and shortage occurs. (decision variable)
$t_2 =$ production run time when backorder is replenished.
$t_3 =$ production run time of Stage 1 in a production cycle. (decision variable)
$t_4 =$ time period when inventory of semi-finished product depletes.
$t_5 =$ time period when inventory of finished product depletes.
$T =$ cycle time, $T = t_1 + t_3 + t_4 + t_5$.
$Z_0 =$ maximum backorder level.
$Z_1 =$ maximum inventory level of semi-finished product.
$Z_2 =$ maximum inventory level of finished product.

2.2 Assumptions

In addition, the following assumptions are used throughout this paper:

(1) The production cycle repeats infinitely.
(2) The production system is imperfect, and the inspection time and rework time of defective products are very short, which can be neglected.

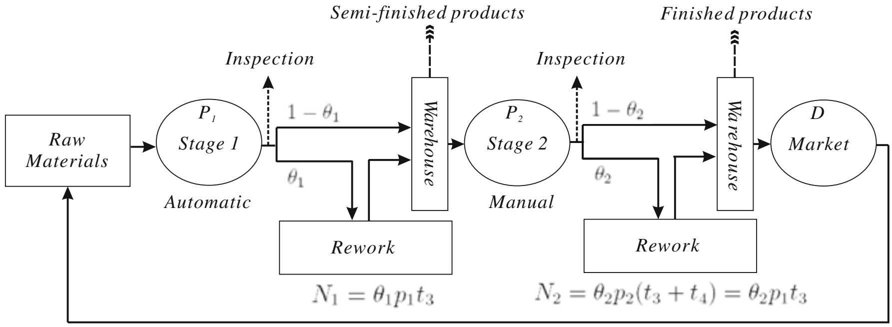

Fig. 1 Two-stage production system with imperfect processes

(3) The production system is separated into two stages. The former stage (Stage 1; raw material to semi-finished products) is an automatic process; the latter stage (Stage 2; semi-finished products to finished products) is a manual process, and the production rates of the two stages satisfy the condition $p_1 > p_2 > D$. This system is depicted in Fig. 1.

(4) Shortages are allowed and completely backlogged.

(5) As stated in the literature Porteus (1986): a logarithmic investment function is assumed for investment in quality improvement. In this situation, the investment, $IV_i(\theta_i)$, to reduce the defective probability θ_i is described by

$$IV_i(\theta_i) = a_i \ln\left(\frac{\theta_{0i}}{\theta_i}\right) \quad \text{for } 0 < \theta_i \le \theta_{0i} \text{ and } i = 1, 2,$$

where θ_{0i} is original percentage of defective products produced and $a_i = \frac{1}{\delta_i}$, with δ_i denoting the percentage decrease in θ_i per dollar increase in $IV_i(\theta_i)$.

(6) In our case, it is difficult to avoid the human negligence (i.e., $\delta_2 \to 0$). Therefore, the capital investment in process quality improvement should be only implemented in Stage 1. Then we set $\theta_2 = \theta_{02}$.

(7) The process quality of two stages are independent, i.e., θ_1 is independent of θ_2.

3 Model formulation

Under the notation and assumptions in the previous section, the graphic representation of the two-stage production system with allowable shortages can be shown as Fig. 2. Then, we formulate the objective function as a cost minimization problem. The overall cost for our problem includes setup cost, inventory holding cost, shortage cost, rework cost, production cost and investment cost. The formulations of these six costs are described in detail as follows. Before building the objective function, we first clarify the relationship between t_1, t_2, t_3, t_4, t_5 and T. Referring to Fig. 2, we have the following results:

$$t_2 = \frac{Z_0}{p_2 - D} = \frac{Dt_1}{p_2 - D}, \tag{1}$$

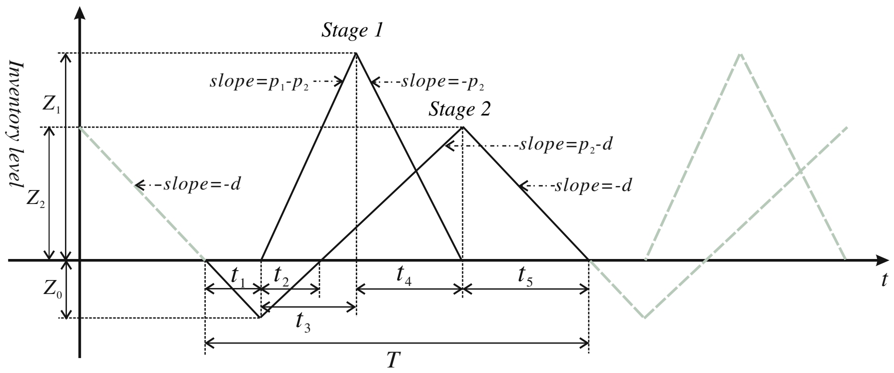

Fig. 2 The graph of inventory level during time period $[0, T]$

$$t_4 = \frac{Z_1}{p_2} = \frac{(p_1 - p_2)t_3}{p_2}, \tag{2}$$

$$t_5 = \frac{Z_2}{D} = \frac{(p_2 - D)(-t_2 + t_3 + t_4)}{D} = \frac{(p_2 - D)\left(\frac{p_1 t_3}{p_2} - \frac{D t_1}{p_2 - D}\right)}{D}. \tag{3}$$

From the Eqs. (1)–(3), the time duration of a cycle can be calculated as

$$T = t_1 + t_3 + t_4 + t_5 = \frac{p_1 t_3}{D}.$$

Based on t_1, t_2, t_3, t_4 and t_5, the total cost per cycle consists of the following elements:

1. Setup cost per cycle $= k$.
2. Holding cost per cycle for semi-finished products $= \frac{c_{h1} Z_1 (t_3 + t_4)}{2} = \frac{c_{h1}(p_1 - p_2) p_1 t_3^2}{2 p_2}$.
3. Holding cost per cycle for finished products $= \frac{c_{h2} Z_2 (-t_2 + t_3 + t_4 + t_5)}{2} = \frac{c_{h2}(p_2 - D)\left(\frac{p_1}{\sqrt{D p_2}} t_3 - \frac{\sqrt{D p_2}}{p_2 - D} t_1\right)^2}{2}$.
4. Shortage cost per cycle $= \frac{c_s (t_1 + t_2) Z_0}{2} = \frac{c_s p_2 D t_1^2}{2(p_2 - D)}$.
5. Rework cost per cycle $= c_{r1} N_1 + c_{r2} N_2 = (c_{r1}\theta_1 + c_{r2}\theta_{02}) p_1 t_3$.
6. Production cost per cycle $= c_p p_1 t_3$.
7. Investment cost per cycle $= a_1 \ln\left(\frac{\theta_{01}}{\theta_1}\right)$.

Then, our problem is to minimize the total cost per unit time by simultaneously optimizing t_1, t_3 and θ_1, constrained on $0 < \theta_1 \le \theta_{01}$. That is:

$$\text{Min } AC(t_1,t_3,\theta_1) = \frac{1}{T}\{\text{Setup cost} + \text{Holding cost} + \text{Shortage cost} + \text{Rework cost} + \text{Production cost} + \text{Investment cost}\}$$
$$= \frac{D}{p_1t_3}\left\{k + \frac{c_s p_2 D t_1^2}{2(p_2-D)} + (c_{r1}\theta_1 + c_{r2}\theta_{02} + c_p)p_1t_3 + a_1\ln\left(\frac{\theta_{01}}{\theta_1}\right) + \frac{c_{h1}(p_1-p_2)p_1t_3^2}{2p_2} + \frac{c_{h2}(p_2-D)\left(\frac{p_1t_3}{\sqrt{Dp_2}} - \frac{\sqrt{Dp_2}t_1}{p_2-D}\right)^2}{2}\right\},$$
$$\text{Subject to } 0 < \theta_1 \le \theta_{01}. \tag{4}$$

In order to solve this nonlinear programming problem, we first ignore the restriction $0 < \theta_1 \le \theta_{01}$, and take the first-order derivative of $AC(t_1, t_3, \theta_1)$ with respect to t_1, t_3 and θ_1, respectively. We obtain

$$\frac{\partial AC(t_1,t_3,\theta_1)}{\partial t_1} = \frac{D}{p_1t_3(p_2-D)}\{(c_{h2}+c_s)Dp_2t_1 - c_{h2}(p_2-D)p_1t_3\}, \tag{5}$$

$$\frac{\partial AC(t_1,t_3,\theta_1)}{\partial t_3} = \frac{D}{p_1t_3^2}\left\{-k - \frac{c_s p_2 D t_1^2}{2(p_2-D)} - a_1\ln\left(\frac{\theta_{01}}{\theta_1}\right) + \frac{c_{h1}(p_1-p_2)p_1t_3^2}{2p_2} + \frac{c_{h2}(p_2-D)}{2}\left(\frac{p_1^2t_3^2}{Dp_2} - \frac{Dp_2t_1^2}{(p_2-D)^2}\right)\right\}, \tag{6}$$

and

$$\frac{\partial AC(t_1,t_3,\theta_1)}{\partial \theta_1} = D\left\{-\frac{a_1}{\theta_1p_1t_3} + c_{r1}\right\}. \tag{7}$$

It is well known that the necessary condition for (t_1, t_3, θ_1) to be optimal must satisfy the equations $\frac{\partial AC(t_1,t_3,\theta_1)}{\partial t_1} = 0$, $\frac{\partial AC(t_1,t_3,\theta_1)}{\partial t_3} = 0$ and $\frac{\partial AC(t_1,t_3,\theta_1)}{\partial \theta_1} = 0$, simultaneously, which implies

$$t_1 = \frac{c_{h2}(p_2-D)p_1}{(c_s+c_{h2})\,p_2D}t_3, \tag{8}$$

$$k + \frac{c_s p_2 D t_1^2}{2(p_2-D)} + a_1\ln\left(\frac{\theta_{01}}{\theta_1}\right) = \frac{c_{h1}(p_1-p_2)p_1t_3^2}{2p_2} + \frac{c_{h2}(p_2-D)}{2}\left(\frac{p_1^2t_3^2}{Dp_2} - \frac{Dp_2t_1^2}{(p_2-D)^2}\right), \tag{9}$$

and

$$\theta_1 = \frac{a_1}{c_{r1} p_1} t_3^{-1}. \tag{10}$$

From Eqs. (8) and (10), it is clear that t_1 and θ_1 can be uniquely determined as functions of t_3. For finding the optimal solution of (t_1, t_3, θ_1), we substitute t_1 and θ_1 given by Eqs. (8) and (10) into Eq. (9), and then obtain

$$\frac{G}{2} t_3^2 - a_1 \ln \left(\frac{c_{r1}\theta_{01} p_1}{a_1} t_3 \right) - k = 0, \tag{11}$$

where

$$G = \frac{c_{h1}(p_1 - p_2)p_1}{p_2} + \frac{c_s c_{h2}(p_2 - D)p_1^2}{D p_2 (c_s + c_{h2})} > 0.$$

Therefore, the optimal solution t_3^* can be obtain by solving the Eq. (11). Now, we let $f(t_3)$ denote the left hand side of Eq. (11). Taking the first-order derivative of $f(t_3)$ with respect to t_3, it yields $G t_3 - a_1 t_3^{-1}$. Hence $f(t_3)$ is a continuous function which decreases strictly in $t_3 \in [0, \tilde{t_3}]$ and increases strictly in $t_3 \in [\tilde{t_3}, \infty)$, respectively, where $\tilde{t_3} = \sqrt{a_1/G}$. As a result, $f(t_3)$ has a minimum at the point $t_3 = \tilde{t_3}$, and is

$$f(\tilde{t_3}) = \frac{a_1}{2} - a_1 \ln \left(\frac{c_{r1}\theta_{01} p_1}{a_1} \sqrt{\frac{a_1}{G}} \right) - k. \tag{12}$$

Then we have the following result.

Theorem 1 *For any $t_3 \geq 0$, we have*

(a) If $f(\tilde{t_3}) < 0$, then the solution $(t_1^*, t_3^*, \theta_1^*)$ which minimizes $AC(t_1, t_3, \theta_1)$ not only exists but also is unique, and $t_3^* \in (\tilde{t_3}, \infty)$.
(b) If $f(\tilde{t_3}) \geq 0$, then the optimal value of t_3 is $t_3^* \to 0$. The production system should not be opened.

Proof (a) First, we consider the interval $t_3 \in [\tilde{t_3}, \infty)$. Because $\lim_{t_3 \to \infty} f(t_3) = \infty$ and $f(t_3)$ is strictly increasing in the interval $t_3 \in [\tilde{t_3}, \infty)$, and on condition that $f(\tilde{t_3}) < 0$, from the Intermediate Value Theorem, we can find a unique solution $t_3^* \in (\tilde{t_3}, \infty)$ such that $f(t_3^*) = 0$. Substituting t_3^* into Eqs. (8) and (10), the corresponding t_1^* and θ_1^* can be determined. Furthermore, in order to examine the second-order sufficient conditions (SOSC) for a minimum value, we first obtain the Hessian matrix $\mathbf{H}$ as follows:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 AC(t_1,t_3,\theta_1)}{\partial t_1^2} & \frac{\partial^2 AC(t_1,t_3,\theta_1)}{\partial t_1 \partial t_3} & \frac{\partial^2 AC(t_1,t_3,\theta_1)}{\partial t_1 \partial \theta_1} \\ \frac{\partial^2 AC(t_1,t_3,\theta_1)}{\partial t_3 \partial t_1} & \frac{\partial^2 AC(t_1,t_3,\theta_1)}{\partial t_3^2} & \frac{\partial^2 AC(t_1,t_3,\theta_1)}{\partial t_3 \partial \theta_1} \\ \frac{\partial^2 AC(t_1,t_3,\theta_1)}{\partial \theta_1 \partial t_1} & \frac{\partial^2 AC(t_1,t_3,\theta_1)}{\partial \theta_1 \partial t_3} & \frac{\partial^2 AC(t_1,t_3,\theta_1)}{\partial \theta_1^2} \end{bmatrix}.$$

Then we proceed by evaluating the principal minor determinants of $\mathbf{H}$ at stationary point $\left(t_1^*, t_3^*, \theta_1^*\right)$. The first, second and third principal minor determinant of $\mathbf{H}$ are calculated as follows, respectively.

$$|\mathbf{H}_{11}| = \frac{(c_{h2} + c_s)\, D^2 p_2}{p_1 t_3^* \,(p_2 - D)} > 0,$$
$$|\mathbf{H}_{22}| = \frac{D^2}{t_3^{*2}} \left\{ \frac{c_{h1}(c_{h2} + c_s)(p_1 - p_2)D}{p_1} + c_{h2}c_s \right\} > 0,$$

and

$$|\mathbf{H}_{33}| = \frac{a_1 D^4 (c_{h2} + c_s) p_2 G}{p_1^2 (p_2 - D) t_3^{*2}} \left(t_3^{*2} - \tilde{t_3}^2 \right) > 0.$$

According to the above results, it is clear that the Hessian matrix $\mathbf{H}$ is positive definite. Next, we consider the interval $t_3 \in [0, \tilde{t_3}]$. Since $f(0) = \infty$ and $f(t_3)$ is strictly decreasing in the interval $t_3 \in [0, \tilde{t_3}]$, and on condition that $f(\tilde{t_3}) < 0$, there also exists a unique solution $t_3^{**} \in (0, \tilde{t_3})$ such that $f(t_3^{**}) = 0$, then the corresponding t_1^{**} and θ_1^{**} can be obtained. However, the third principal minor determinant of $\mathbf{H}$ at the stationary point $(t_1^{**}, t_3^{**}, \theta_1^{**})$ is negative ($|\mathbf{H}_{33}| < 0$), the Hessian matrix $\mathbf{H}$ is not positive definite. Therefore, we conclude that the solution $(t_1^*, t_3^*, \theta_1^*)$ which minimizes $AC(t_1, t_3, \theta_1)$ not only exists but also is unique, and $t_3^* \in (\tilde{t_3}, \infty)$. This completes the proof.

(b) Since $f(t_3)$ has a global minimum at $\tilde{t_3}$, if $f(\tilde{t_3}) > 0$, then $f(t_3) > f(\tilde{t_3}) > 0$ for all $t_3 \neq \tilde{t_3}$. From Eq. (6) and $f(t_3)$, we obtain that $\frac{\partial AC(t_1,t_3,\theta_1)}{\partial t_3} = \frac{D}{p_1 t_3^2} f(t_3) > 0$, which implies that a small value of t_3 causes a lower value of $AC(t_1, t_3, \theta_1)$. Hence the minimum value of $AC(t_1, t_3, \theta_1)$ occurs at the point $t_3^* \to 0$. Consequently, the production system should not be opened. For the another case $f(\tilde{t_3}) = 0$, since $\frac{\partial AC(t_1,t_3,\theta_1)}{\partial t_3} = 0$ and the $AC(t_1, t_3, \theta_1)$ is strictly increasing in $(0, \tilde{t_3})$ and $(\tilde{t_3}, \infty)$, respectively. As a result, $t_3 = \tilde{t_3}$ is an inflection point and the minimum value of $AC(t_1, t_3, \theta_1)$ occurs at the point $t_3^* \to 0$. This completes the proof. □

We now consider the constraint $0 < \theta_1 \leq \theta_{01}$. If $\theta_1^* < \theta_{01}$ then $(t_1^*, t_3^*, \theta_1^*)$ is an interior optimal solution. However, if $\theta_1^* \geq \theta_{01}$, then it is unrealistic to invest in improving process quality; in this case, the optimal $\theta_1^* = \theta_{01}$. Summarize the above results, we establish the following algorithm to obtain the optimal solution of our problem.

Algorithm

Step 1 Calculate $\tilde{t_3} = \sqrt{a_1/G}$, and then from Eq. (12) to obtain $f(\tilde{t_3})$.

Step 2 If $f(\tilde{t_3}) < 0$, go to Step 3. Otherwise, set the optimal solutions $t_1^* = 0$, $t_3^* = 0$ and $\theta_1^* = \theta_{01}$ (i.e., the production system should not be opened), then stop the Algorithm.

Step 3 Find the optimal value t_3' such that $f(t_3') = 0$.

Step 4 Set $t_3'' = t_3' > \tilde{t_3}$, and then put t_3'' into Eqs. (8) and (10) to obtain the corresponding value of t_1 and θ_1, i.e., t_1'' and θ_1''.

Step 5 Compare θ_1'' and θ_{01}. If $\theta_1'' < \theta_{01}$, let $t_1^* = t_1''$, $t_3^* = t_3''$ and $\theta_1^* = \theta_1''$, go to Step 7. Otherwise, go to Step 6.

Step 6 Set $\theta_1^* = \theta_{01}$ (no capital investment for process quality improvement is made), and utilize Eqs. (8) and (9) (replace θ_1 by θ_{01}) to solve the optimal solution (t_1^*, t_3^*). Then go to Step 7.

Step 7 The minimum total cost per unit time $AC(t_1^*, t_3^*, \theta_1^*)$ can be obtain by substituting t_1^*, t_3^* and θ_1^* into Eq. (4).

4 Numerical example and sensitivity analysis

To illustrate the results, we consider an inventory situation proposed by Chen and Lo (2006): k = \$100/cycle, p_1 = 600/unit time, D = 400/unit time, c_{h1} = \$0.1/unit/unit time, c_s = \$0.5/unit/unit time, c_p = \$10/unit and c_{r1} = \$0.1/unit. Besides, we take p_2 = 500/unit time, c_{h2} = \$0.2/unit/unit time, c_{r2} = \$0.2/unit, $a_1 = 20$ (i.e., $\delta_1 = 0.05$), $\theta_{01} = 0.25$ and $\theta_{02} = 0.2$. Applying the proposed algorithm, we find $t_1^* = 0.20896$, $t_3^* = 2.43783$, $T^* = 3.65675$, $\theta_1^* = 0.13673$, the variation of original and treated process quality $\Delta = \theta_{01} - \theta_1^* = 0.11327$, the optimal production lot size for each cycle $Q = p_1 t_3^* = 1462.70$ and $AC(t_1^*, t_3^*, \theta_1^*) = 4082.76$.

Now, this numerical example is considered to study the effects of changes in the system parameters $a_1, \theta_{01}, \theta_{02}, k, c_p, c_s, c_{h1}, c_{h2}, c_{r1}$ and c_{r2} on the optimal values of $t_1^*, t_3^*, T^*, \theta_1^*, \Delta, Q, AC(t_1^*, t_3^*, \theta_1^*)$. The sensitivity analysis is performed by changing each of the parameters by +50, +25, −25 and −50%; taking one parameter at a time and keeping the remaining parameters unchanged. The results are shown in Table 1.

On the basis of the results of Table 1, the following observations can be made:

(1) When the values of parameters θ_{02}, c_p and c_{r2} increase, $t_1^*, t_3^*, T^*, \theta_1^*, \Delta$ and Q are still fixed but the minimum total cost per unit time $AC(t_1^*, t_3^*, \theta_1^*)$ increases. It implies that if these costs and the defective rate of finished products could be reduced effectively, the total cost per unit time could be improved.

(2) With increase in the value of parameter a_1, $AC(t_1^*, t_3^*, \theta_1^*)$ and θ_1^* increase, but Δ decreases. Therefore, in order to decrease the minimum total cost per unit time and the defective rate of semi-finished products, simultaneously, one should select an effective investment strategy in process quality improvement (i.e., the lower value of a_1). In addition, the investment is not implemented when the value of a_1 exceeds some limit value. Table 2 displays the upper limit value of a_1 for $\theta_{01} = 0.1(0.05)0.4$ and $c_{r1} = 0.1(0.05)0.4$.

(3) When the value of parameters θ_{01} increases, $AC(t_1^*, t_3^*, \theta_1^*)$ and Δ increase. Therefore, one should select a process with lower defective rate (i.e., the lower value of θ_{01}) to decrease the minimum total cost per unit time. In addition, the

Table 1 Effect of changes in various parameters of the inventory model

Parameter	% Change	Optimal solutions						
		t_1^*	t_3^*	T^*	θ_1^*	Δ	Q	$AC(t_1^*, t_3^*, \theta_1^*)$
a_1	+50%	0.2022	2.3592	3.5388	0.2119	0.0381	1415.52	4083.79
	+25%	0.2062	2.4062	3.6093	0.1732	0.0768	1443.73	4083.43
	−25%	0.2103	2.4529	3.6794	0.1019	0.1481	1471.15	4081.75
	−50%	0.2098	2.4480	3.6720	0.0681	0.1819	1468.81	4080.27
θ_{01}	+50%	0.2171	2.5325	3.7987	0.1316	0.2434	1519.49	4084.94
	+25%	0.2135	2.4905	3.7357	0.1338	0.1787	1494.27	4083.97
	−25%	0.2030	2.3679	3.5519	0.1408	0.0467	1420.75	4081.17
	−50%	0.1974	2.3028	3.4543	0.1250	0.0000	1381.70	4078.90
θ_{02}	+50%	0.2090	2.4378	3.6568	0.1367	0.1133	1462.70	4090.76
	+25%	0.2090	2.4378	3.6568	0.1367	0.1133	1462.70	4086.76
	−25%	0.2090	2.4378	3.6568	0.1367	0.1133	1462.70	4078.76
	−50%	0.2090	2.4378	3.6568	0.1367	0.1133	1462.70	4074.76
k	+50%	0.2543	2.9670	4.4505	0.1123	0.1377	1780.18	4095.09
	+25%	0.2329	2.7173	4.0760	0.1227	0.1273	1630.40	4089.23
	−25%	0.1811	2.1132	3.1698	0.1577	0.0923	1267.93	4075.44
	−50%	0.1463	1.7069	2.5603	0.1953	0.0547	1024.13	4066.73
c_p	+50%	0.2090	2.4378	3.6568	0.1367	0.1133	1462.70	6082.76
	+25%	0.2090	2.4378	3.6568	0.1367	0.1133	1462.70	5082.76
	−25%	0.2090	2.4378	3.6568	0.1367	0.1133	1462.70	3082.76
	−50%	0.2090	2.4378	3.6568	0.1367	0.1133	1462.70	2082.76
c_s	+50%	0.1482	2.3468	3.5202	0.1420	0.1080	1408.07	4084.92
	+25%	0.1734	2.3842	3.5763	0.1398	0.1102	1430.54	4084.02
	−25%	0.2631	2.5210	3.7815	0.1322	0.1178	1512.60	4080.92
	−50%	0.3557	2.6678	4.0016	0.1249	0.1251	1600.66	4077.91
c_{h1}	+50%	0.1927	2.2479	3.3719	0.1483	0.1017	1348.75	4087.44
	+25%	0.2004	2.3375	3.5062	0.1426	0.1074	1402.49	4085.15
	−25%	0.2187	2.5513	3.8269	0.1307	0.1193	1530.77	4080.27
	−50%	0.2298	2.6809	4.0214	0.1243	0.1257	1608.54	4077.66
c_{h2}	+50%	0.2466	2.1923	3.2885	0.1520	0.0980	1315.41	4088.95
	+25%	0.2298	2.2979	3.4468	0.1451	0.1049	1378.73	4086.14
	−25%	0.1823	2.6333	3.9499	0.1266	0.1234	1579.95	4078.59
	−50%	0.1464	2.9279	4.3919	0.1138	0.1362	1756.76	4073.26
c_{r1}	+50%	0.2171	2.5325	3.7987	0.0877	0.1623	1519.49	4084.94
	+25%	0.2135	2.4905	3.7357	0.1071	0.1429	1494.27	4083.97
	−25%	0.2030	2.3679	3.5519	0.1887	0.0623	1420.75	4081.17
	−50%	0.1974	2.3028	3.4543	0.2500	0.0000	1381.70	4078.90
c_{r2}	+50%	0.2090	2.4378	3.6568	0.1367	0.1133	1462.70	4090.76
	+25%	0.2090	2.4378	3.6568	0.1367	0.1133	1462.70	4086.76
	−25%	0.2090	2.4378	3.6568	0.1367	0.1133	1462.70	4078.76
	−50%	0.2090	2.4378	3.6568	0.1367	0.1133	1462.70	4074.76

Table 2 The criterion of a_1 for judging whether the investment is implemented

Upper limit value of a_1	θ_{01}						
c_{r1}	0.10	0.15	0.20	0.25	0.30	0.35	0.40
0.10	13.817	20.725	27.634	34.542	41.451	48.359	55.268
0.15	20.725	31.088	41.451	51.814	62.176	72.539	82.902
0.20	27.634	41.451	55.268	69.085	82.902	96.719	104.41
0.25	34.542	51.814	69.085	86.356	100.72	109.87	118.79
0.30	41.451	62.176	82.902	100.72	111.67	122.31	132.70
0.35	48.359	72.539	96.719	109.87	122.31	134.42	146.28
0.40	55.268	82.902	104.41	118.79	132.70	146.28	159.61

Table 3 The criterion of θ_{01} for judging whether the investment is implemented

Lower limit value of θ_{01}	a_1						
c_{r1}	10	15	20	25	30	35	40
0.10	0.0724	0.1086	0.1448	0.1810	0.2172	0.2534	0.2895
0.15	0.0482	0.0724	0.0965	0.1206	0.1447	0.1689	0.1930
0.20	0.0362	0.0543	0.0724	0.0905	0.1086	0.1267	0.1447
0.25	0.0289	0.0434	0.0579	0.0724	0.0868	0.1013	0.1158
0.30	0.0241	0.0362	0.0482	0.0604	0.0724	0.0844	0.0965
0.35	0.0207	0.0310	0.0414	0.0517	0.0620	0.0724	0.0827
0.40	0.0181	0.0271	0.0362	0.0452	0.0543	0.0633	0.0724

capital investment in process quality improvement is unnecessarily implemented when the value of θ_{01} is below some limit value. Table 3 displays the lower limit value of θ_{01} for $a_1 = 10(5)40$ and $c_{r1} = 0.1(0.05)0.4$. Besides, t_1^*, t_3^*, T^* and Q increase as θ_{01} increases. It implies that the cycle time should be lengthened to retard the growth of the investment cost per unit.

(4) The minimum total cost per unit time $AC(t_1^*, t_3^*, \theta_1^*)$ increases as k increases. If the setup cost per cycle could be reduced effectively, the total cost per unit time could be improved. Besides, t_1^*, t_3^*, T^* and Q increase as k increases. It implies that the cycle time is lengthened to retard the growth of the setup cost per unit.

(5) t_1^* and Δ decrease while $AC(t_1^*, t_3^*, \theta_1^*)$ and θ_1^* increase with increase in the value of parameter c_s. It implies that if the shortage cost per unit per unit time increases, one should decrease the capital investment in process quality improvement, and focus on the length of the period during which shortages are allowed for reducing the shortage quantity.

(6) t_3^*, Q and Δ decrease while $AC(t_1^*, t_3^*, \theta_1^*)$ and θ_1^* increase with increase in the value of parameter c_{h1} (or c_{h2}). It implies that if the holding cost per unit per unit time increases, one should decrease the capital investment in process quality improvement, and focus on the production run time for reducing the production (inventory) quantity.

Table 4 The criterion of c_{r1} for judging whether the investment is implemented

Lower limit value of c_{r1}	θ_{01}						
a_1	0.10	0.15	0.20	0.25	0.30	0.35	0.40
10	0.0724	0.0482	0.0362	0.0289	0.0241	0.0207	0.0181
15	0.1086	0.0724	0.0543	0.0434	0.0362	0.0310	0.0271
20	0.1447	0.0965	0.0724	0.0579	0.0482	0.0414	0.0362
25	0.1809	0.1206	0.0905	0.0724	0.0603	0.0517	0.0452
30	0.2171	0.1447	0.1086	0.0868	0.0724	0.0620	0.0543
35	0.2533	0.1689	0.1267	0.1013	0.0844	0.0724	0.0633
40	0.2895	0.1930	0.1447	0.1158	0.0965	0.0827	0.0724

(7) With increase in the value of parameter c_{r1}, θ_1^* decreases but t_1^*, t_3^*, T^*, Q, Δ and $AC(t_1^*, t_3^*, \theta_1^*)$ increase. It implies that if the rework cost of a defective semi-finished product increases, one should focus on process quantity improvement for reducing the rework quantity, and increase the cycle time to retard the growth of the investment cost per unit. In addition, the capital investment in process quality improvement is unnecessarily implemented when the value of c_{r1} is below some limit value. Table 4 displays the lower limit value of c_{r1} for $\theta_{01} = 0.1(0.05)0.4$ and $a_1 = 10(5)40$.

5 Conclusion

In this paper, we develop a two-stage production system with imperfect processes and allowable shortages. The necessary and sufficient conditions of the existence and uniqueness of the optimal solution are shown in Theorems 1. Next, we provide a simple algorithm to find the optimal solution of (t_1, t_3, θ_1) for minimizing the total cost per unit time. The proposed model can be used in inventory control of two-stage (automatic stage and manual stage) production system such as Automotive industry, Glass industry, Food industry, and others. The results of the present study suggest three dimensions that might profitably be addressed by future researchers in the area. One is to investigate the effects of variable deterioration rate and stochastic nature of demand. The Second is to investigate the imprecise production by combining the statistical techniques and fuzzy set concepts. The last is to relax the Assumption (6), and determine what stage is profitable to implement the capital investment in process quality improvement.

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