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Relation-Based Variations of the Discrete Radon Transform

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Abstract—The finite Radon transform was introduced by Bolker around 1976. Since then, many variations of the discrete Radon transform have been proposed. In this paper, we first propose a variation of the discrete Radon transform which is based on a binary relation. Then, we generalize this variation to weighted Radon transformation based on a weighted relation. Under such generalization, we show that discrete convolution is a special case of weighted Radon transformation. To further generalize Radon transformation to be defined on lattice-valued functions, we propose two nonlinear variations of Radon transformation. These two nonlinear variations have very close relations with morphological operations. Finally, we generalize Matheron's representation theorem to represent translation-invariant operations on functions from an abelian group to a complete lattice.

Keywords—Discrete Radon transform, Discrete convolution, Nonlinear Radon transforms, Galois connection, Mathematical morphology.

1. INTRODUCTION

The Radon transformation is a very powerful mathematical tool in computed tomography (CT). The problem in CT is the determination of some property of the internal structure of an object without having to cut the object. The object is acted on by a probe, such as X-rays, nuclear magnetic resonance signals, or ultrasound waves, which can be detected to produce a projected distribution or profile.

Let $f(x,y)$ represent the internal property of an object on $\mathbb{R}^2$. For any line L, the *Radon transform* of f, written as $\mathcal{R}f$, is given by

$$\mathcal{R}\, f(L) = \int_L F(x,y)\, ds,$$

where ds is an increment of length along L. If the unit normal vector of and the distance from the origin to L is p, then the Radon transform of f can also be written as

$$\mathcal{R}\, f(p,\phi) = \int_{-\infty}^{\infty} f(p\cos\,\phi - s\,\sin\,\phi,\ p\,\sin + s\,\cos\,\phi)\, ds.$$

For each (x,y), let

$$F(x,y,q) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{R}\, f(x\,\cos\,\phi + y\,\sin\,\phi + q,\ \phi)\, d\phi.$$

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If $f(x,y)$ satisfies some "regularity conditions," Radon [3], then $f(x,y)$ can be reconstructed by

$$f(x,y) = -\frac{1}{\pi}\int_0^\infty \frac{dF(x,y,q)}{q}.$$

In Radon's original paper [1] (see [2] for an English translation), the Radon transformation is defined on functions from $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}$. It is generalized to functions from $\mathbb{R}^n$ to $\mathbb{R}$ by Ludwig [3] and Deans [4]. Further generalized Radon transformation can be found in [5].

A very interesting variation of Radon transformation is its discrete version. Let X be a finite set and $\mathcal{C}$ be a family of subsets of X. If f is a function from X to $\mathbb{C}$, then the *finite Radon transform* of f, also written as $\mathcal{R}\, f$, is a function from $\mathcal{C}$ to $\mathbb{C}$ defined by

$$\mathcal{R}\, f(Y) = \sum_{x\in Y} f(x), \qquad \forall Y \in \mathcal{C}.$$

For each $x \in X$, denote $G_x = \{Y \in \mathcal{C} \mid x \in Y\}$. Then the *dual Radon transform* of a function F from $\mathcal{C}$ to $\mathbb{C}$, written as $\mathcal{R}^d F$, is defined by

$$\mathcal{R}^d F(x) = \sum_{Y\in G_x} F(Y).$$

The finite Radon transformation is introduced by Bolker [6] and has been employed by Kung [7] to find matchings and to prove rank and covering inequalities in finite lattices.

Another variation of Radon transformation is to define it as an operation. For instance, in [8], the Radon transformation on a finite abelian group $(A,+,-,0)$ is defined as a linear operation on the space of functions from A to $\mathbb{C}$. Let B be a subset of A. For every function f from A to $\mathbb{C}$, the Radon transform of f with respect to B, written as $\mathcal{R}_B\, f$, is a function from A to $\mathbb{C}$ given by

$$\mathcal{R}_B\, f(a) = \sum_{b\in B} f(a+b).$$

In this paper, we will consider a more general variation of the Radon transformation that will unify most variations of the finite Radon transformation. The basic idea of our approach is to define transformations based on binary relations between two sets. If a given binary relation is weighted, a weighted Radon transformation can be defined. We will show that weighted Radon transformations can be reduced to convolutions if particularly specified weighted binary relations are provided. The details will be given in Section 2. A characterization theorem for weighted Radon transformation to be translation-invariant will be given in Section 3.

Convolutions play essential roles in linear signal and image processing. An *n-dimensional signal* is usually defined as a function from $\mathbb{R}^n$ to $\mathbb{R}$, while an *n-dimensional image* is usually defined as a function from $\mathbb{R}^n$ to $\mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geq 0\}$. To be suitable for computer manipulation, a signal or an image must be digitized. For instance, an *n-dimensional digital* image is considered as a function from $\mathbb{Z}^n$ to the set $\{0,1,\dots,l-1\}$, where l is a positive integer. Most linear transformations on digital images map digital images to real- or complex-valued functions. In other words, most linear transformations on digital images are not operations on digital images. For instance, the output of a mean filter on digital images is a function with values from the interval $[0, l-1]$; the Fourier transform of a digital image is a complex-valued function. In Section 4, we will propose two nonlinear variations of Radon transformation on lattice-valued functions by employing the two lattice operations $\vee$ (join) and $\wedge$ (meet). When applied to digital images, these two nonlinear transformations can be used as operations on digital images. In Section 5, we will show that all morphological operations [9–11] on $\mathcal{P}(\mathbb{R}^n)$, the power set of $\mathbb{R}^n$, can be derived from these two nonlinear transformations if suitable binary relations are provided.

A very fundamental theorem in mathematical morphology related to translation-invariant transformations is Matheron's representation theorem, which represents any isotone translation-invariant operation on $\mathcal{P}(\mathbb{R}^n)$ in terms of erosions or, dually, dilations. Matheron's theorem has been recently generalized for translation-invariant operations, which are not necessarily isotone, by Banon and Barrera [12]. In Section 6, we will further generalize Banon and Barrera's result for translation-invariant operations on functions from an abelian group to a complete lattice. Conclusions and some remarks will be given in Section 7.

2. TRANSFORMS BASED ON BINARY RELATIONS

Let X and Y be two arbitrary sets. A *binary relation* ϱ between X and Y is a subset of the Cartesian product $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$. For any $x \in X$ and $y \in Y$, a conventional notation for $(x,y) \in \varrho$ is $x\varrho y$. Then, consider the following variation of finite Radon transformation.

DEFINITION 2.1. *Let ϱ be a binary relation between two sets X and Y. For any function f from X to $\mathbb{R}$ or $\mathbb{C}$, the Radon transform of f with respect to ϱ, written as $\mathcal{R}_\varrho f$, if it exists, is a function from Y to $\mathbb{R}$ or $\mathbb{C}$ defined by*

$$\mathcal{R}_\varrho f(y) = \sum_{x\varrho y} f(x).$$

If $Y = \mathcal{C}$ is a class of subsets of X, and $x\varrho S$ means (for every $x \in X$ and $S \in \mathcal{C}$, then Definition 2.1 reduces to Bolker's definition for Radon transforms. If $X = Y = A, (A,+)$ is a finite abelian group, and $a\varrho b$ means $a + b \in B$ for a given subset B, then Definition 2.1 reduces to the definition of Radon transforms on abelian group given by Frankl and Graham. Moreover, if $X = Y = \mathbb{Z}_p^n, S$ is a subset of $\mathbb{Z}_p^n$, and $\vec{x}\varrho\vec{y}$ means $\vec{x} \in S_{\vec{y}} = \{\vec{z} + \vec{y} \mid \vec{z} \in S\}$ for all $\vec{x}$, $\vec{y} \in \mathbb{Z}_p^n$, then the transform $\mathcal{R}_\varrho f$ of a function f from $\mathbb{Z}_p^n$ to $\mathbb{C}$ is called the Radon transform of f based on *translates* of S by Fill [13].

Next, we consider transforms based on weighted binary relations.

DEFINITION 2.2. *A weighted relation w between two sets X and Y is a function from $X \times Y$ to $\mathbb{R}$ or $\mathbb{C}$.*

Note that an unweighted binary relation ϱ between X and Y can be considered as a weighted relation w defined by

$$w(x,y) = \begin{cases} 1, & \text{if } x\varrho y, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the Radon transform $\mathcal{R}_\varrho f$ can be written as

$$\mathcal{R}_\varrho f(y) = \sum_{x\varrho y} f(x) = \sum_{x\in X} f(x)\, w(x,y).$$

This leads us to the following definition.

DEFINITION 2.3. *Let w be a weighted relation between X and Y. Then the weighted Radon transform of a function f from X to $\mathbb{R}$ or $\mathbb{C}$, written as $\mathcal{R}_w f$, if it exists, is defined by*

$$\mathcal{R}_w f(y) = \sum_{x\in X} f(x)\, w(x,y).$$

EXAMPLE 2.4. Let $X = Y = \mathbb{Z}^n$. Let $F(\mathbb{Z}^n \to \mathbb{R})$ denote the set of all real-valued functions on $\mathbb{Z}^n$. For a neighborhood N of the origin $\vec{0}$, define a weighted relation w on $\mathbb{Z}^n$ by

$$w(\vec{x},\vec{y}) = \begin{cases} \dfrac{1}{|N|}, & \text{if } \vec{x} \in N_{\vec{y}}, \\ 0, & \text{otherwise,} \end{cases}$$

where $|N|$ denotes the cardinality of N and $N_{\vec{y}}$ is the translate of N by $\vec{y}$. Then, the weighted Radon transform $\mathcal{R}_w f$ of a function $f \in F(\mathbb{Z}^n \to \mathbb{R})$ is given by

$$\mathcal{R}_w f(y) = \sum_{x \in \mathbb{Z}^n} f(x)\, w(x,y) = \frac{1}{|N|} \sum_{x \in N_y} f(x).$$

This is known as the *local averaging* of f with respect to the "window" N.

EXAMPLE 2.5. Let $\mathcal{F}(\mathbb{Z}^n \to C)$ denote the set of all complex-valued functions on $\mathbb{Z}^n$. For any function $g \in \mathcal{F}(\mathbb{Z}^n \to C)$, define a weighted relation w on $\mathbb{Z}^n$ by $w(\vec{x},\vec{y}) = g(\vec{y}-\vec{x})$. Then, the weighted Radon transform $\mathcal{R}_w f$, given by

$$\mathcal{R}_w f(\vec{y}) = \sum_{\vec{x}} f(\vec{x})\, g(\vec{y}-\vec{x}),$$

is known as the *discrete convolution* of f and g.

EXAMPLE 2.6. Let $X = Y = \{0,1,\dots,m-1\} \times \{0,1,\dots,n-1\}$. Define a weighted relation w on X by

$$w((x,y),(u,v)) = \frac{1}{mn} e^{-j2\pi(ux/m+vy/n)},$$

where $j = \sqrt{-1}$. Then the weighted Radon transform $\mathcal{R}_w f$ of any $f \in \mathcal{F}(X \to C)$, given by

$$\mathcal{R}_w f(u,v) = \frac{1}{mn} \sum_{x=0}^{m-1} \sum_{y=0}^{n-1} f(x,y) e^{-j2\pi(ux/m+vy/n)},$$

is indeed the *discrete Fourier transform* of f.

3. TRANSLATION INVARIANT TRANSFORMATIONS

In this section, we will consider the case when $X = Y = A$, where $(A,+,-,0)$ is an abelian group. For any $a, b \in A$, the element $a+b$ is sometimes called the *translation* of a by b.

DEFINITION 3.1. *Let w be a weighted relation on A. Then w is called translation-invariant if $w(x+a,\, y+a) = w(x,y)$ for all $a,x,y \in A$. In particular, a binary relation ϱ on A is called translation-invariant if $x\varrho y$ implies $(x+a)\varrho(y+a)$ for all $a,x,y \in A$.*

EXAMPLE 3.2. Consider the weighted relation w defined in Example 2.5. Since

$$\begin{aligned} w(\vec{x}+\vec{a},\vec{y}+\vec{a}) &= g(\vec{y}+\vec{a}-(\vec{x}+\vec{a})) \\ &= g(\vec{y}-\vec{x}) = w(\vec{x},\vec{y}), \end{aligned}$$

the weighted relation w is translation-invariant.

Now, consider the group action of A on the set $\mathcal{F}(a \to K)$ of functions from A to K, where $K = \mathbb{R}$ or $\mathbb{C}$. For each $f \in \mathcal{F}(a \to K)$ and $a \in A$, there is a function $f_a \in \mathcal{F}(A \to K)$ defined by

$$f_a(x) = f(x-a), \qquad \forall x \in A.$$

The function f_a is usually called the *translation* of f by a. Obviously, we have $f_0 = f$ and $(f_a)_b = f_{a+b}$ for all $a,b \in A$.

DEFINITION 3.3. *Let Ψ be a unary operation on $\mathcal{F}(A \to K)$. Then, Ψ is called translation-invariant if $\Psi(f_a) = (\Psi f)_a$ for all $f \in \mathcal{F}(A \to K)$ and for all $a \in A$.*

The following result gives a necessary and sufficient condition for weighted Radon transformations to be translation-invariant.

THEOREM 3.4. *A weighted Radon transformation $\mathcal{R}_w$ on $\mathcal{F}(A \to K)$ is translation invariant if and only if the weighted relation w is translation-invariant.*

PROOF. Suppose $\mathcal{R}_w$ is translation-invariant. That is, suppose $\mathcal{R}_w f_a = (\mathcal{R}_w f)_a$, for all $f \in \mathcal{F}(A \to K)$ and for all $a \in A$. Let δ be the unit sample function defined by

$$\delta(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then observe that

$$\mathcal{R}_w \delta_x(y) = \sum_{z \in A} \delta(z - x)\, w(z, y) = w(x, y)$$

and

$$(\mathcal{R}_w \delta)_x (y) = (\mathcal{R}_w \delta)_{x+a-a} (y) = (\mathcal{R}_w \delta)_{x+a} (y + a) = w(x + a, y + a).$$

Since $\mathcal{R}_w$ is translation-invariant, we have $w(x + a, y + a) = w(x, y)$. Thus, w is translation-invariant.

Conversely, suppose w is translation-invariant. Then

$$\begin{aligned} (\mathcal{R}_w f)_a (y) &= \sum_{x \in A} f(x)\, w(x, y - a) \\ &= \sum_{z - a \in A} f(z - a)\, w(z - a, y - a) = \sum_{z \in A} f(z - a)\, w(z, y) = \mathcal{R}_w f_a(y). \end{aligned}$$

Thus, $\mathcal{R}_w$ is translation-invariant. ∎

EXAMPLE 3.5. Consider the local averaging operation with respect to a window N discussed in Example 2.4. Since $\vec{x} \in N_{\vec{y}}$ is equivalent to $\vec{x} + \vec{a} \in N_{\vec{y}+\vec{a}}$ for all $\vec{a} \in \mathbb{Z}^n$, the weighted relation w is translation-invariant. By Theorem 3.4, the local averaging operation with respect to N is translation-invariant.

EXAMPLE 3.6. Consider the discrete convolution discussed in Example 2.5. As shown in Example 3.2, the weighted relation w defined by $w(x, y) = g(y - x)$ is translation-invariant. Thus, the discrete convolution $\mathcal{R}_w$ is also translation-invariant.

The unit sample function δ employed in the proof of Theorem 3.4 has significant importance in representation theorem for linear translation-invariant operations. Let $\mathcal{O}$ be an operation on $\mathcal{F}(A \to K)$, where $(A, +, -, 0)$ is a finite abelian group. The function $h = \mathcal{O}\delta$ is usually called the *impulse response* of $\mathcal{O}$. If $\mathcal{O}$ is linear and translation-invariant, then a very fundamental theorem in digital signal and image processing says that $\mathcal{O}f$ can be represented as the discrete convolution of f with the impulse response h, for any function f in $\mathcal{F}(A \to K)$. In terms of weighted Radon transforms, we can rephrase the above result as a theorem.

THEOREM 3.7. *Let $\mathcal{O}$ be a linear translation-invariant operation on $\mathcal{F}(A \to K)$, where $(A, +, -, 0)$ is a finite abelian group and $K = \mathbb{R}$ or $\mathbb{C}$. Then,*

$$\mathcal{O}f = \mathcal{R}_w f,$$

for all $f \in \mathcal{F}(A \to K)$, where the weighted relation w is given by $w(x, y) = \mathcal{O}\delta(y - x)$.

4. NONLINEAR RADON TRANSFORMATIONS

Now consider the generalization of Radon transformation to lattice-valued functions such as digital images. As mentioned before, a digital image is usually considered as a function from $\mathbb{Z}^n$ to the set $\{0, 1, \ldots, l - 1\}$, where a typical value for l is 256. Since addition is an inadequate operation on digital images, a reasonable substitution for summation used in discrete Radon transforms might be one of the two lattice operations $\vee$ (join) or $\wedge$ (meet). This motivates us to the following definition.

DEFINITION 4.1. *Let ϱ be a binary relation between two sets X and Y, and $(L, \vee, \wedge)$ be a complete lattice. For each function $f : X \to L$, the supremum Radon transform of f with respect to ϱ, written as $\check{\mathcal{R}}_\varrho f$, is a function from Y to L defined by*

$$\check{\mathcal{R}}_\varrho f(y) = \bigvee_{x\varrho y} f(x).$$

The infimum Radon transform of f with respect to ϱ, written as $\hat{\mathcal{R}}_\varrho f$, is a function from Y to L defined by

$$\hat{\mathcal{R}}_\varrho f(y) = \bigwedge_{x\varrho y} f(x).$$

In the following, we will discuss the properties of these two nonlinear transformations from lattice theoretical point of view.

Let $(P, \leq)$ and $(P', \leq')$ be two posets. A pair (f, g) of functions, $f : P \to P', g : P' \to P$ is called a *Galois connection* [14,15] between P and P' if

(GC1) f and g are isotone, i.e., $f(x) \leq' f(y)$ if $x \leq y$ and $g(u) \leq g(v)$ if $u \leq' v$;
(GC2) $x \leq g(f(x))$ for all $x \in P$ and $f(g(y)) \leq' y$ for all $y \in P'$.

Given two sets X, Y and a complete lattice $(L, \vee, \wedge)$, denote the set of functions from X to L as $\mathcal{F}(X \to L)$ and the set of functions from Y to L as $\mathcal{F}(Y \to L)$. It is well-known that $(\mathcal{F}(X \to L), \vee, \wedge)$ and $(\mathcal{F}(Y \to L), \vee, \wedge)$ are complete lattices, where $f_1 \leq f_2$ in $\mathcal{F}(X \to L)$ means $f_1(x) \leq f_2(x)$ for all x in X. In the following, we will establish Galois connections between $\mathcal{F}(X \to L)$ and $\mathcal{F}(Y \to L)$.

Note that for any binary relation ϱ between X and Y the inverse relation ϱ^{-1} is a binary relation between Y and X. Thus, the transformations $\check{\mathcal{R}}_{\varrho^{-1}}$ and $\hat{\mathcal{R}}_{\varrho^{-1}}$ are functions from $\mathcal{F}(Y \to L)$ to $\mathcal{F}(X \to L)$.

THEOREM 4.2. *Suppose ϱ is a binary relation between X and Y. Then, the pair $(\check{\mathcal{R}}_\varrho, \hat{\mathcal{R}}_{\varrho^{-1}})$ is a Galois connection between $\mathcal{F}(X \to L)$ and $\mathcal{F}(Y \to L)$. Similarly, the pair $(\check{\mathcal{R}}_{\varrho^{-1}}, \hat{\mathcal{R}}_\varrho)$ is a Galois connection between $\mathcal{F}(Y \to L)$ and $\mathcal{F}(X \to L)$.*

PROOF. It is easy to see that all Radon transformations are isotone. Thus, to show $(\check{\mathcal{R}}_\varrho, \hat{\mathcal{R}}_{\varrho^{-1}})$ is a Galois connection, it remains for us to show that $\hat{\mathcal{R}}_{\varrho^{-1}}\check{\mathcal{R}}_\varrho f \geq f$ and $\check{\mathcal{R}}_\varrho \hat{\mathcal{R}}_{\varrho^{-1}} g \leq g$, for all $f \in \mathcal{F}(Y \to L)$ and $\mathcal{F}(X \to L)$. For all $x \in X$, since

$$\hat{\mathcal{R}}_{\varrho^{-1}}\check{\mathcal{R}}_\varrho f(x) = \bigwedge_{y\varrho^{-1}x} \bigvee_{z\varrho y} f(z) \geq \bigwedge_{y\varrho^{-1}x} f(x) = f(x),$$

we have $\hat{\mathcal{R}}_{\varrho^{-1}}\check{\mathcal{R}}_\varrho f \geq f$, for all $f \in \mathcal{F}(X \to L)$. Similarly, we have $\check{\mathcal{R}}_\varrho \hat{\mathcal{R}}_{\varrho^{-1}} g \leq g$, for all $g \in \mathcal{F}(Y \to L)$. Thus, $(\check{\mathcal{R}}_\varrho, \hat{\mathcal{R}}_{\varrho^{-1}})$ is a Galois connection between $\mathcal{F}(X \to L)$ and $\mathcal{F}(Y \to L)$. That the pair $(\check{\mathcal{R}}_{\varrho^{-1}}, \hat{\mathcal{R}}_\varrho)$ is a Galois connection between $\mathcal{F}(Y \to L)$ and $\mathcal{F}(X \to L)$ follows easily by interchanging the roles of X and Y. ∎

COROLLARY 4.3. *For any binary relation ϱ between X and Y, we have $\check{\mathcal{R}}_\varrho \hat{\mathcal{R}}_{\varrho^{-1}} \check{\mathcal{R}}_\varrho f = \check{\mathcal{R}}_\varrho f$, for all $f \in \mathcal{F}(X \to L)$ and $\check{\mathcal{R}}_{\varrho^{-1}} \hat{\mathcal{R}}_\varrho \check{\mathcal{R}}_{\varrho^{-1}} g = \check{\mathcal{R}}_{\varrho^{-1}} g$, for all $g \in \mathcal{F}(Y \to L)$. Dually, $\hat{\mathcal{R}}_\varrho \check{\mathcal{R}}_{\varrho^{-1}} \hat{\mathcal{R}}_\varrho f = \hat{\mathcal{R}}_\varrho f$ for all $\mathcal{F}(Y \to L)$ and $\hat{\mathcal{R}}_{\varrho^{-1}} \check{\mathcal{R}}_\varrho \hat{\mathcal{R}}_{\varrho^{-1}} g = \hat{\mathcal{R}}_{\varrho^{-1}} g$, for all $g \in \mathcal{F}(Y \to L)$.*

PROOF. Since $\hat{\mathcal{R}}_{\varrho^{-1}} \check{\mathcal{R}}_\varrho f \geq f$ and $\check{\mathcal{R}}_\varrho$ is isotone, we have $\check{\mathcal{R}}_\varrho \hat{\mathcal{R}}_{\varrho^{-1}} \check{\mathcal{R}}_\varrho f \geq \check{\mathcal{R}}_\varrho f$, for all $f \in \mathcal{F}(X \to L)$. On the other hand, since $\check{\mathcal{R}}_\varrho \hat{\mathcal{R}}_{\varrho^{-1}} g \leq g$, for all $g \in \mathcal{F}(Y \to L)$, we have $\check{\mathcal{R}}_\varrho \hat{\mathcal{R}}_{\varrho^{-1}} \check{\mathcal{R}}_\varrho f \leq \check{\mathcal{R}}_\varrho f$, for all $f \in \mathcal{F}(X \to L)$. Thus, $\check{\mathcal{R}}_\varrho \hat{\mathcal{R}}_{\varrho^{-1}} \check{\mathcal{R}}_\varrho f = \check{\mathcal{R}}_\varrho f$, for all $f \in \mathcal{F}(Y \to L)$. Other equalities follow similarly. ∎

Let $(P, \leq)$ be a poset. An operation $^{-} : x \mapsto \bar{x}$ on P is called a *closure operation* [14,15] on P if

(CL1) $x \leq \bar{x}$ (Extensive);
(CL2) $x \leq y \Rightarrow \bar{x} \leq \bar{y}$ (Isotone);
(CL3) $\bar{\bar{x}} = \bar{x}$ (Idempotent).

THEOREM 4.4. *Suppose ϱ is a binary relation between X and Y. Then, the operation $\hat{\mathcal{R}}_{\varrho^{-1}}\check{\mathcal{R}}_\varrho$ is a closure operation on $\mathcal{F}(X \to L)$ and the operation $\hat{\mathcal{R}}_\varrho\check{\mathcal{R}}_{\varrho^{-1}}$ is a closure operation on $\mathcal{F}(Y \to L)$.*

PROOF. It suffices to show that the two operations $\hat{\mathcal{R}}_{\varrho^{-1}}\check{\mathcal{R}}_\varrho$ and $\hat{\mathcal{R}}_\varrho\check{\mathcal{R}}_{\varrho^{-1}}$ are idempotent. By Corollary 4.3, we have $\check{\mathcal{R}}_\varrho\hat{\mathcal{R}}_{\varrho^{-1}}\check{\mathcal{R}}_\varrho = \check{\mathcal{R}}_\varrho$ and, hence, $\hat{\mathcal{R}}_{\varrho^{-1}}\check{\mathcal{R}}_\varrho\hat{\mathcal{R}}_{\varrho^{-1}}\check{\mathcal{R}}_\varrho = \hat{\mathcal{R}}_{\varrho^{-1}}\check{\mathcal{R}}_\varrho$. That is, $\hat{\mathcal{R}}_{\varrho^{-1}}\check{\mathcal{R}}_\varrho$ is idempotent. By interchanging X and Y, it is obvious that the operation is also idempotent. ∎

The dual notion of closure operation is coclosure operation [14] (or kernel operation [15]). Let $(P, \leq)$ be a poset. An operation $^{-} : x \mapsto \bar{x}$ on P is called a *coclosure operation* on P if

(CCL1) $\bar{x} \leq x$ (Extensive);
(CL2) $x \leq y \Rightarrow \bar{x} \leq \bar{y}$ (Isotone);
(CL3) $\bar{\bar{x}} = \bar{x}$ (Idempotent).

The following result is the dual of Theorem 4.4. We omit its proof.

THEOREM 4.5. *Suppose ϱ is a binary relation between X and Y. Then, the operation $\check{\mathcal{R}}_{\varrho^{-1}}\hat{\mathcal{R}}_\varrho$ is a coclosure operation on $\mathcal{F}(X \to L)$ and the operation $\check{\mathcal{R}}_\varrho\hat{\mathcal{R}}_{\varrho^{-1}}$ is a coclosure operation on $\mathcal{F}(Y \to L)$.*

DEFINITION 4.6. *Let ϱ be a binary relation between X and Y, and $(L, \vee, \wedge)$ a complete lattice. A function $f \in \mathcal{F}(Y \to L)$ is called ϱ-closed if $\hat{\mathcal{R}}_{\varrho^{-1}}\check{\mathcal{R}}_\varrho f = f$. Dually, f is called ϱ-open if $\check{\mathcal{R}}_{\varrho^{-1}}\hat{\mathcal{R}}_\varrho f = f$.*

The following theorem is essentially due to Serra [10].

THEOREM 4.7. *Let $f \in \mathcal{F}(X \to L)$. Then f is ϱ-closed if and only if $f = \hat{\mathcal{R}}_{\varrho^{-1}}g$, for some $g \in \mathcal{F}(Y \to L)$. Similarly, f is ϱ-open if and only if $f = \check{\mathcal{R}}_{\varrho^{-1}}g$, for some $g \in \mathcal{F}(Y \to L)$.*

PROOF. Suppose f is ϱ-closed. Let $g = \check{\mathcal{R}}_\varrho f$. Since $f = \hat{\mathcal{R}}_{\varrho^{-1}}\check{\mathcal{R}}_\varrho f$, we have $f = \hat{\mathcal{R}}_{\varrho^{-1}}g$. Conversely, suppose $f = \hat{\mathcal{R}}_{\varrho^{-1}}g$, for some $g \in \mathcal{F}(Y \to L)$. Then, $\hat{\mathcal{R}}_{\varrho^{-1}}\check{\mathcal{R}}_\varrho f = \hat{\mathcal{R}}_{\varrho^{-1}}\check{\mathcal{R}}_\varrho\hat{\mathcal{R}}_{\varrho^{-1}}g = \hat{\mathcal{R}}_{\varrho^{-1}}g = f$, by Corollary 4.3. Thus, f is ϱ-closed. The dual statement can be proved similarly. ∎

5. RELATION TO MORPHOLOGICAL OPERATIONS

Consider the special case when $L = \{0, 1\}$. It is well-known that every function $f \in \mathcal{F}(A \to \{0,1\})$ can be characterized by its support $\text{Supp}(f) = \{a \in A \mid f(a) = 1\}$, which is a subset of A. Conversely, given any subset B of A, the characteristic function χ_B of B is a function in $\mathcal{F}(A \to \{0,1\})$ with support B. Thus, we can make the following definition.

DEFINITION 5.1. *Let ϱ be a binary relation on A. For any subset B of A, the supremum and infimum Radon transforms of B, respectively written as $\check{\mathcal{R}}_\varrho B$ and $\hat{\mathcal{R}}_\varrho B$, are defined by*

$$\hat{\mathcal{R}}_\varrho B = \text{Supp}\left(\check{\mathcal{R}}_\varrho \chi_B\right) \quad \textit{and} \quad \hat{\mathcal{R}}_\varrho B = \text{Supp}\left(\hat{\mathcal{R}}_\varrho \chi_B\right).$$

REMARK 5.2. Note that $\check{\mathcal{R}}_\varrho$ and $\hat{\mathcal{R}}_\varrho$ are operations on $\mathcal{P}(A)$, the power set of A.

Let A be an abelian group and S a fixed subset of A. Define a binary relation ϱ on A by $a\varrho b$ if and only if $a \in S_b = \{s + b \mid s \in S\}$. Then it is easy to see that ϱ is a translation-invariant relation on A. Write $\check{\mathcal{R}}_S$ for $\check{\mathcal{R}}_\varrho$ and $\hat{\mathcal{R}}_S$ for $\hat{\mathcal{R}}_\varrho$ if $a\varrho b$ means $a \in S_b$. By Theorem 3.4, the two operations $\check{\mathcal{R}}_S$ and $\hat{\mathcal{R}}_S$ are translation-invariant. Observe that

$$\check{\mathcal{R}}_S\chi_B(b) = \bigvee_{a \in S_b} \chi_B(a) = \begin{cases} 1, & \text{if } S_b \cap B \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

In other words,

$$b \in \check{\mathcal{R}}_S B, \qquad \text{if and only if } S_b \cap B \neq \emptyset.$$

Thus, we have

$$\check{\mathcal{R}}_S B = \{b \in A \mid S_b \cap B \neq \emptyset\}.$$

Similarly, by observing that

$$\hat{\mathcal{R}}_S \chi_B(b) = \bigwedge_{a \in S_b} \chi_B(a) = \begin{cases} 1, & \text{if } S_b \subset B, \\ 0, & \text{otherwise,} \end{cases}$$

we then have

$$\hat{\mathcal{R}}_S B = \{b \in A \mid S_b \subset B\}.$$

In the literature of mathematical morphology, $\check{\mathcal{R}}_S B$ is called the *dilation* of B by S, $\hat{\mathcal{R}}_S B$ is called the *erosion* of B by S, and S is called a *structuring element*. Moreover, the closure operation $\hat{\mathcal{R}}_{-S}\check{\mathcal{R}}_S$, where $-S = \{-s \mid s \in S\}$, is called a *closing* and the coclosure operation $\check{\mathcal{R}}_{-S}\hat{\mathcal{R}}_S$ is called an *opening*. Details on the theory and applications of mathematical morphology can be found in [9–11,16–24].

As mentioned in the first section, a very interesting result in mathematical morphology related to translation-invariant operations is Matheron's [9]'s representation theorem. Let Ψ be a unary operation on $\mathcal{P}(A)$. The *kernel* of Ψ, written as $\mathcal{K}(\Psi)$, is given by

$$\mathcal{K}(\Psi) = \{B \in \mathcal{P}(A) \mid 0 \in \Psi(B)\}.$$

THEOREM 5.3. *[9] Let Ψ be an isotone translation-invariant operation on $\mathcal{P}(A)$. Then*

$$\Psi B = \bigcup_{S \in \mathcal{K}(\Psi)} \hat{\mathcal{R}}_S B, \qquad \textit{for all } B \in \mathcal{P}(A).$$

PROOF. Let $b \in \Psi B$. Since Ψ is translation-invariant, we have $0 \in (\Psi B)_{-b} = \Psi(B_{-b})$. Thus, $B_{-b} \in \mathcal{K}(\Psi)$. Then

$$b \in \hat{\mathcal{R}}_{B_{-b}} B \subset \bigcup_{S \in \mathcal{K}(\Psi)} \hat{\mathcal{R}}_S B.$$

Thus, $\Psi B \subset \cup_{S \in \mathcal{K}(\Psi)} \hat{\mathcal{R}}_S B$. Conversely, let $b \in \cup_{S \in \mathcal{K}(\Psi)} \hat{\mathcal{R}}_S B$. Then $b \in \hat{\mathcal{R}}_S B$, for some $S \in \mathcal{K}(\Psi)$. This means that $S_b \subset B$ or, equivalently, $S \subset B_{-b}$. Since Ψ is isotone and $S \in \mathcal{K}(\Psi)$, we have

$$0 \in \Psi S \subset \Psi(B_{-b}) = (\Psi B)_{-b}.$$

Thus, $b \in \Psi B$. This shows that $\cup_{S \in \mathcal{K}(\Psi)} \hat{\mathcal{R}}_S B \subset \Psi B$. We then conclude that $\Psi B = \cup_{S \in \mathcal{K}(\Psi)} \times \hat{\mathcal{R}}_S B$. ∎

For any operation Ψ on $\mathcal{P}(A)$, the *dual* operation of Ψ, written as Ψ^d, is defined by

$$\Psi^d B = (\Psi B^c)^c, \qquad \text{for all } B \subset A,$$

where B^c denotes the complement of B in A. Observe that for any $B \subset A$,

$$\begin{aligned} \left(\hat{\mathcal{R}}_S\right)^d B = \left(\hat{\mathcal{R}}_S B^c\right)^c &= \{b \mid S_b \subset B^c\}^c, \\ &= \{b \mid S_b \cap B \neq \emptyset\} = \check{\mathcal{R}}_S B. \end{aligned}$$

Thus, $(\hat{\mathcal{R}}_S)^d = \check{\mathcal{R}}_S$ for any $S \subset A$.

THEOREM 5.4. *[9] Let Ψ be an isotone translation-invariant operation on $\mathcal{P}(A)$. Then*

$$\Psi B = \bigcap_{S \in \mathcal{K}(\Psi^d)} \check{\mathcal{R}}_S B, \qquad \text{for all } B \in \mathcal{P}(A).$$

PROOF. Since $\Psi^d B^c = \cup_{S \in \mathcal{K}(\Psi^d)} \hat{\mathcal{R}}_S B^c$ by Theorem 5.3, we have

$$\Psi B = \left(\Psi^d B^c\right)^c = \left(\bigcup_{S \in \mathcal{K}(\Psi^d)} \hat{\mathcal{R}}_S B^c\right)^c = \bigcap_{S \in \mathcal{K}(\Psi^d)} \check{\mathcal{R}}_S B. \qquad \blacksquare$$

Matheron's representation theorem has been recently extended to represent all translation-invariant operations on $\mathcal{P}(A)$ by Banon and Barrera [12]. For any $B, C \subset A$, denote $[B, C] = \{S \subset A \mid B \subset S \subset C\}$. Then consider the ternary operation $T(S, B, C)$ on $\mathcal{P}(A)$ defined by

$$T(S, B, C) = \{a \mid S \in [B_a, C_a]\}.$$

Since "$S \in [B_a, C_a]$" is equivalent to "$B_a \subset S$ and $C_a^c \subset S^c$", we have

$$T(S, B, C) = \hat{\mathcal{R}}_B S \cap \hat{\mathcal{R}}_{C^c} S^c.$$

THEOREM 5.5. *[12] Let Ψ be a translation-invariant operation on $\mathcal{P}(A)$. Then for all $S \subset A$,*

$$\Psi S = \bigcup_{[B,C] \subset \mathcal{K}(\Psi)} \left(\hat{\mathcal{R}}_B S \cap \hat{\mathcal{R}}_{C^c} S^c\right).$$

PROOF. Let $a \in \Psi S$. Then, $0 \in (\Psi S)_{-a} = \Psi(s_{-a})$, $S_{-a} \in \mathcal{K}(\Psi)$, and $[S_{-a}, S_{-a}] \subset \mathcal{K}(\Psi)$. Obviously, $S \in [(S_{-a})_a, (S_{-a})_a]$ and, hence, $a \in \hat{\mathcal{R}}_{S_{-a}} S \cap \hat{\mathcal{R}}_{S^c_{-a}} S^c \subset \bigcup_{[B,C] \subset \mathcal{K}(\Psi)} (\hat{\mathcal{R}}_B S \cap \hat{\mathcal{R}}_{C^c} S^c)$. Conversely, suppose $a \in \bigcup_{[B,C] \subset \mathcal{K}(\Psi)} (\hat{\mathcal{R}}_B S \cap \hat{\mathcal{R}}_{C^c} S^c)$. Then $a \in \hat{\mathcal{R}}_b S \cap \hat{\mathcal{R}}_{C^c} S^c$ for some $[B, C] \subset \mathcal{K}(\Psi)$. This means that $B_a \subset S \subset C_a$ for some $[B, C] \subset \mathcal{K}(\Psi)$. Thus, $S_{-a} \in \mathcal{K}(\Psi)$ and, hence, $a \in \Psi S$. $\blacksquare$

REMARK 5.6. If Ψ is also isotone, then Theorem 5.5 reduces to Theorem 5.3. Indeed, then it is easy to show that $B \in \mathcal{K}(\Psi)$ if and only if $[B, C] \subset \mathcal{K}(\Psi)$ for all C with $C \supset B$, and thus Theorem 5.5 asserts

$$\begin{aligned}
\Psi S &= \bigcup_{B \in \mathcal{K}(\Psi)} \bigcup_{C \supset B} \left(\hat{\mathcal{R}}_b S \cap \hat{\mathcal{R}}_{C^c}\right) \\
&= \bigcup_{B \in \mathcal{K}(\Psi)} \left(\hat{\mathcal{R}}_B S \cap \bigcup_{C \supset B} \hat{\mathcal{R}}_{C^c} S^c\right) \\
&= \bigcup_{B \in \mathcal{K}(\Psi)} \left(\hat{\mathcal{R}}_B S \cap \hat{\mathcal{R}}_{A^c} S^c\right) \\
&= \bigcup_{B \in \mathcal{K}(\Psi)} \left(\hat{\mathcal{R}}_B S \cap \hat{\mathcal{R}}_\emptyset S^c\right) \\
&= \bigcup_{B \in \mathcal{K}(\Psi)} \left(\hat{\mathcal{R}}_B S \cap A\right) \\
&= \bigcup_{B \in \mathcal{K}(\Psi)} \hat{\mathcal{R}}_B S,
\end{aligned}$$

which is the assertion of Theorem 5.3.

6. REPRESENTATION THEOREM FOR TRANSLATION-INVARIANT OPERATIONS

Representation theorems discussed in previous section are developed primarily for translation-invariant operations on $\mathcal{P}(A)$. In the following, we will generalize the alluded representation theorems to represent translation-invariant operations on $\mathcal{F}(A \to L)$ for arbitrary complete lattice $(L, \vee, \wedge)$.

Before generalizing Theorem 5.3 and Theorem 5.5, we first note that any function in $\mathcal{F}(A \to \{0,1\})$ is of the form $f = \chi_S$, where $S = \mathrm{Supp}(f) \subset A$. Thus, the ternary operation T on $\mathcal{P}(A)$ can be easily redefined as a ternary operation on $\mathcal{F}(A \to \{0,1\})$ by

$$T(\chi_S, \chi_B, \chi_C)(x) = \begin{cases} 1, & \text{if } B_X \subset S \subset C_x, \\ 0, & \text{otherwise,} \end{cases}$$

for all $\chi_S, \chi_B, \chi_C \in \mathcal{F}(A \to \{0,1\})$ and for all $x \in A$. An analogous ternary operation on $\mathcal{F}(a \to L)$, also written as T, can then be defined by

$$T(f,g,h)(x) = \begin{cases} \top, & \text{if } g_x \leq f \leq h_x, \\ \perp, & \text{otherwise,} \end{cases}$$

for all $x \in A$, where $\top$ and $\perp$ are the greatest and least elements of L, respectively. Next, we introduce the threshold operations on L. For any $a \in L$, let Θ_a be the operation on L defined by

$$\Theta_a \beta = \begin{cases} \top, & \text{if } a \leq \beta, \\ \perp, & \text{otherwise.} \end{cases}$$

The operation Θ_a is known as the *a-level threshold operation*. Complementarily, we can define Θ_a^c to be an operation on L by

$$\Theta_a^c \beta = \begin{cases} \top, & \text{if } \beta \leq a, \\ \perp, & \text{otherwise.} \end{cases}$$

Then for any $g \in \mathcal{F}(A \to L), a \in L$, define a relation ϱ on A by $x \varrho y$, if and only if $g_y(x) \geq a$, for all $x, y \in A$ and write $\check{\mathcal{R}}_{g \geq a}$ and $\hat{\mathcal{R}}_{g \geq a}$, for $\check{\mathcal{R}}_\varrho$ and $\hat{\mathcal{R}}_\varrho$, respectively. Similarly, we will write $\check{\mathcal{R}}_{g \leq a}$ and $\hat{\mathcal{R}}_{g \leq a}$ for $\check{\mathcal{R}}_\varrho$ and $\hat{\mathcal{R}}_\varrho$, respectively, if $x \varrho y$ means $g_y(x) \leq a$, for all $x, y \in A$.

REMARK 6.1. If $L = \{0,1\}$ with $0 < 1$ and $f \in \mathcal{F}(A \to L)$ with $\mathrm{Supp}(g) = S$. Then, it is easy to see that the definition $g_y(x) \geq 1$ of $x \varrho y$ is the same as that for S in Section 5 and, hence, that

$$\check{\mathcal{R}}_{g \geq 1} = \check{\mathcal{R}}_S \quad \text{and} \quad \hat{\mathcal{R}}_{g \geq 1} = \hat{\mathcal{R}}_S.$$

Likewise,

$$\check{\mathcal{R}}_{g \leq 0} = \check{\mathcal{R}}_{S^c} \quad \text{and} \quad \hat{\mathcal{R}}_{g \leq 0} = \hat{\mathcal{R}}_{S^c}.$$

Moreover,

$$\check{\mathcal{R}}_{g \geq 0} = \check{\mathcal{R}}_{g \leq 1} = \check{\mathcal{R}}_A \qquad \text{(which maps } \emptyset \text{ to itself and everything else to } A\text{)},$$

and

$$\hat{\mathcal{R}}_{g \geq 0} = \hat{\mathcal{R}}_{g \leq 1} = \hat{\mathcal{R}}_A \qquad \text{(which maps } A \text{ to itself and everything else to } \emptyset\text{)}.$$

LEMMA 6.2. *For any $f, g, h \in \mathcal{F}(A \to L)$, we have*

$$T(f,g,h) = \bigwedge_{a \in L} \left(\hat{\mathcal{R}}_{g \geq a} \Theta_a f \right) \wedge \left(\hat{\mathcal{R}}_{h \leq a} \Theta_a^c f \right).$$

PROOF. Suppose $T(f,g,h)(y) = \top$. Then $g_y \leq f \leq h_y$. Thus, for all $a \in L$, if $g_y(x) \geq a$ then $f(x) \geq a$ and $\Theta_a f(x) = \top$. Similarly, if $h_y(x) \leq a$ then $f(x) \leq a$ and $\Theta_a^c f(x) = \top$. Therefore,

$$\bigwedge_{a \in L} \left(\hat{\mathcal{R}}_{g \geq a} \Theta_a f(y)\right) \wedge \left(\hat{\mathcal{R}}_{h \leq a} \Theta_a^c f(y)\right) = \top.$$

Conversely, if $\bigwedge_{a \in L} (\hat{\mathcal{R}}_{g \geq a} \Theta_a f(y)) \wedge (\hat{\mathcal{R}}_{h \leq a} \Theta_a^c f(y)) = \top$, then for all $a \in L$, we have

$$\hat{\mathcal{R}}_{g \geq a} \Theta_a f(y) = \top \quad \text{and} \quad \hat{\mathcal{R}}_{h \leq a} \Theta_a^c f(y) = \top.$$

Thus, $f(x) \geq a$, for all x with $g_y(x) \geq a$ and $f(x) \leq a$, for all x with $h_y(x) \leq a$. In other words, we have $g_y \leq f \leq h_y$. Thus, $T(f,g,h)(y) = \top$. ∎

REMARK 6.3. If $L = \{0,1\}$, then Lemma 6.2 asserts

$$T(\chi_S, \chi_B, \chi_C) = \hat{\mathcal{R}}_B \chi_S \wedge \hat{\mathcal{R}}_{C^c} \chi_{S^c},$$

which is equivalent to the one right above Theorem 5.5.

DEFINITION 6.4. *Let Ψ be a unary operation on $\mathcal{F}(A \to L)$, where $(A, +, -, 0)$ is an abelian group and $(L, \vee, \wedge)$ is a complete lattice. Then for all $a \in L$, the a-level kernel of Ψ, written as $\mathcal{K}_a(\Psi)$, is defined by*

$$\mathcal{K}_a(\Psi) = \{f \in \mathcal{F}(A \to L) \mid \Psi f(0) \geq a\}.$$

REMARK 6.5. If $L = \{0,1\}$ and we identify sets with their characteristic functions, then $\mathcal{K}_1 = \mathcal{K}$ as defined in Section 5 and $\mathcal{K}_0 = \mathcal{P}(A)$.

THEOREM 6.6. *Let Ψ be a translation-invariant operation on $\mathcal{F}(A \to L)$. Then*

$$\Psi f = \bigvee_{a \in L} \left[a \wedge \bigvee_{[g,h] \subset \mathcal{K}_a(\Psi)} \left(\bigwedge_{\beta \in L} \left(\hat{\mathcal{R}}_{g \geq \beta} \, \Theta_\beta \, f\right) \wedge \left(\hat{\mathcal{R}}_{h \leq \beta} \, \Theta_\beta^c \, f\right) \right) \right].$$

PROOF. Let $\Psi f(y) = \gamma$. Then $\Psi f_{-y}(0) = \gamma$ and then $f_{-y} \in \mathcal{K}_\gamma(\Psi)$. Since $[f_{-y}, f_{-y}]$ and $f_{-y})_y \leq f \leq (f_{-y})_y$, we have

$$\bigvee_{[g,h] \subset \mathcal{K}_\gamma(\Psi)} \left(\bigwedge_{\beta \in L} \left(\hat{\mathcal{R}}_{g \geq \beta} \, \Theta_\beta \, f(y)\right) \wedge \left(\hat{\mathcal{R}}_{h \leq \beta} \, \Theta_\beta^c \, f(y)\right) \right) = \top$$

and

$$\bigvee_{a \in L} \left[a \wedge \bigvee_{[g,h] \subset \mathcal{K}_a(\Psi)} \left(\bigwedge_{\beta \in L} \left(\hat{\mathcal{R}}_{g \geq \beta} \, \Theta_\beta \, f(y)\right) \wedge \left(\hat{\mathcal{R}}_{h \leq \beta} \, \Theta_\beta^c \, f(y)\right) \right) \right] \geq \gamma.$$

Conversely, if

$$\bigvee_{a \in L} \left[a \wedge \bigvee_{[g,h] \subset \mathcal{K}_a(\Psi)} \left(\bigwedge_{\beta \in L} \left(\hat{\mathcal{R}}_{g \geq \beta} \, \Theta_\beta \, f(y)\right) \wedge \left(\hat{\mathcal{R}}_{h \leq \beta} \, \Theta_\beta^c \, f(y)\right) \right) \right] = \gamma,$$

then there exists a subset M of L such that

$$a \wedge \bigvee_{[g,h] \subset \mathcal{K}_a(\Psi)} \left(\bigwedge_{\beta \in L} \left(\hat{\mathcal{R}}_{g \geq \beta} \, \Theta_\beta \, f(y)\right) \wedge \left(\hat{\mathcal{R}}_{h \leq \beta} \, \Theta_\beta^c \, f(y)\right) \right) = a,$$

for all $a \in M$. Note that $\vee_{a\in M} a = \gamma$. This means that for each $a \in M$, there exists g and h such that $[g,h] \subset \mathcal{K}_a(\Psi)$ and $\wedge_{\beta\in L}(\hat{\mathcal{R}}_{g\geq a}\Theta_a f(y)) \wedge (\hat{\mathcal{R}}_{h\leq a}\Theta_a^c f(y)) = \top$. By Lemma 6.2, we have $g_y \leq f \leq h_y$, i.e., $g \leq f_{-y} \leq h$. Thus, $f_{-y} \in \mathcal{K}_a(\Psi)$, and hence, $\Psi f_{-y}(0) \geq a$ for all $a \in M$. Since Ψ is translation-invariant, we then have $\Psi f(y) \geq a$ for all $a \in M$. Therefore, $\Psi f(y) \geq \vee_{a\in M} a = \gamma$. ∎

REMARK 6.7. If $L = \{0,1\}$, Theorem 6.6 reduces easily to Theorem 5.5.

In particular, if Ψ is isotone, then $g \in \mathcal{K}_a[\Psi]$ implies $f \in \mathcal{K}[\Psi]$ for all $g \leq f$. Thus, the following corollary is an easy consequence of Theorem 6.6.

COROLLARY 6.8. *If Ψ is an isotone translation-invariant operation on $\mathcal{F}(A \to L)$, then*

$$\Psi = \bigvee_{a\in L} \left(a \wedge \bigvee_{g\in\mathcal{K}_a(\Psi)} \left(\bigwedge_{\beta\in L} \left(\hat{\mathcal{R}}_{g\geq\beta}\,\Theta_\beta\, f \right) \right) \right).$$

REMARK 6.9. In the used fashion, Corollary 6.8 generalizes Theorem 5.3.

7. CONCLUSION

The proposed weighted Radon transformations based on weighted relations unify many linear transformations commonly used in signal and image processing. In [25], many applications of discrete Radon transforms to signal processing have been discussed. Moreover, Theorem 3.4 is an interesting result on characterizing translation-invariant weighted Radon transformations.

Mathematical morphology is recently very vital in the area of image processing. Most new published textbooks on digital image processing [26–29] have included mathematical morphology as one of their topics. The proposed nonlinear Radon transformations can be considered as generalizations of morphological operations.

It should be noted that if $X = Y = V(G)$, where $V(G)$ is the vertex set of a simple graph G, and if ϱ is a binary relation on $V(G)$, then the Radon transforms $\check{\mathcal{R}}_\varrho B$ and $\hat{\mathcal{R}}_\varrho B$, where B is any subset of $V(G)$, are generalizations of dilation and erosion of B in graph morphology. Interested readers can be referred to [30,31] for details.

It should also be noted that Theorem 6.6 is developed for representing translation-invariant operations on $\mathcal{F}(A \to L)$, where L is the underlying set of a complete lattice. Since neither $\mathbb{R}$ nor $\mathbb{C}$ form a complete lattice, Theorem 6.6 cannot be applied directly to translation-invariant operations discussed in Section 3.

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