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Chaos generalized synchronization of an inertial tachometer with new Mathieu-Van der Pol systems as functional system by GYC partial region stability theory

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ABSTRACT

A new strategy to achieve generalized chaos synchronization by GYC partial region stability theory is proposed. By using the GYC partial region stability theory the Lyapunov function is a simple linear homogeneous function of error states and the controllers are more simple and introduce less simulation error because they are in lower order than that of traditional controllers. In simulation examples, an inertial tachometer system and Mathieu-Van der Pol system are used.

1. Introduction

A large number of studies have shown that chaotic phenomena are observed in many physical nonlinear systems [1,2]. It was also reported that the chaotic occurred in many nonlinear control systems [3,4]. In recent years, synchronization in chaotic dynamic system is a very interesting problem and has been widely studied [4–7]. Among many kinds of synchronizations, the generalized synchronization is investigated in this paper. It means that there exists a given functional relationship between the state vector $\mathbf{x}$ of the master and the state vector $\mathbf{y}$ of the slave $\mathbf{y} = G(\mathbf{x})$. There are many control techniques to synchronize chaotic systems, such as linear error feedback control, adaptive control, active control [6–21].

In this paper, a new chaos generalized synchronization strategy by GYC partial region stability theory is proposed [22,23]. By using the GYC partial region stability theory, the Lyapunov function is a simple linear homogeneous function of states and the controllers are simpler and have less simulation error because they are in lower order than that of traditional controllers.

This paper is organized as follow. In Section 2, chaos generalized synchronization strategy by GCY partial region stability theory is proposed. In Section 3, an inertial tachometer system and Mathieu-Van der Pol system are used as simulated examples. In Section 4, conclusions are given.

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2. Chaos generalized synchronization strategy by GYC partial region stability theory

2.1. GYC partial region stability theory

Consider the differential equations of disturbed motion of a nonautonomous system in the normal form

$$\dot{x}_s = X_s(t, x_1, \ldots, x_n), \quad (s = 1, \ldots, n) \tag{1}$$

where the function X_s is defined on the intersection of the partial region Ω (shown in Fig. 1) and

$$\sum_s x_s^2 \leqslant H \tag{2}$$

and $t > t_0$, where t_0 and H are certain positive constants. X_s which vanishes when the variables x_s are all zero, is a real valued function of $t, x_1, \ldots, x_n$. It is assumed that X_s is smooth enough to ensure the existence, uniqueness of the solution of the initial value problem. When X_s does not contain t explicitly, the system is autonomous.

Obviously, $x_s = 0$ $(s = 1, \ldots, n)$ is a solution of Eq. (1). We are interested to the asymptotical stability of this zero solution on partial region Ω (including the boundary) of the neighborhood of the origin which in general may consist of several subregions (Fig. 1).

Definition 1. For any given number $\varepsilon > 0$, if there exists a $\delta > 0$, such that on the closed given partial region Ω when

$$\sum_s x_{s0}^2 \leqslant \delta, \quad (s = 1, \ldots, n) \tag{3}$$

for all $t \geqslant t_0$, the inequality

$$\sum_s x_s^2 < \varepsilon, \quad (s = 1, \ldots, n) \tag{4}$$

is satisfied for the solutions of Eq. (1) on Ω, then the disturbed motion $x_s = 0$ $(s = 1, \ldots, n)$ is stable on the partial region Ω.

Definition 2. If the undisturbed motion is stable on the partial region Ω, and there exists a $\delta' > 0$, so that on the given partial region Ω when

$$\sum_s x_{s0}^2 \leqslant \delta', \quad (s = 1, \ldots, n) \tag{5}$$

The equality

$$\lim_{t \to \infty} \left(\sum_s x_s^2 \right) = 0 \tag{6}$$

is satisfied for the solutions of Eq. (1) on Ω, then the undisturbed motion $x_s = 0$ $(s = 1, \ldots, n)$ is asymptotically stable on the partial region Ω.

The intersection of Ω and region defined by Eq. (5) is called the region of attraction.

Definition of Functions $V(t, x_1, \ldots, x_n)$:

Let us consider the functions $V(t, x_1, \ldots, x_n)$ given on the intersection Ω_1 of the partial region Ω and the region

$$\sum_s x_s^2 \leqslant h, \quad (s = 1, \ldots, n) \tag{7}$$

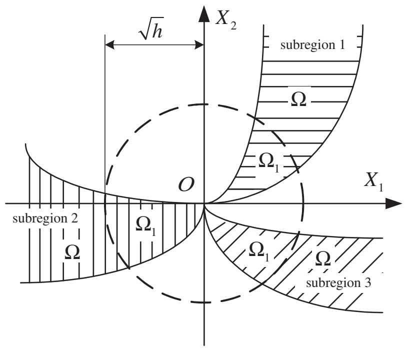

Fig. 1. Partial regions Ω and Ω_1.

for $t \geqslant t_0 > 0$, where t_0 and h are positive constants. We suppose that the functions are single-valued and have continuous partial derivatives and become zero when $x_1 = \cdots = x_n = 0$.

Definition 3. If there exists $t_0 > 0$ and a sufficiently small $h > 0$, so that on partial region Ω_1 and $t \geqslant t_0$, $V \geqslant 0$ (or $\leqslant 0$), then V is a positive (or negative) semi-definite, in general semi-definite, function on the Ω_1 and $t \geqslant t_0$.

Definition 4. If there exists a positive (negative) definitive function $W(x_1,\ldots,x_n)$ on Ω_1, so that on the partial region Ω_1 and $t \geqslant t_0$

$$V - W \geqslant 0 \quad (\text{or } -V - W \geqslant 0) \tag{8}$$

then $V(t,x_1,\ldots,x_n)$ is a positive definite function on the partial region Ω_1 and $t \geqslant t_0$.

Definition 5. If $V(t,x_1,\ldots,x_n)$ is neither definite nor semi-definite on Ω_1 and $t \geqslant t_0$, then $V(t,x_1,\ldots,x_n)$ is an indefinite function on partial region Ω_1 and $t \geqslant t_0$. That is, for any small $h > 0$ and any large $t_0 > 0$, $V(t,x_1,\ldots,x_n)$ can take either positive or negative value on the partial region Ω_1 and $t \geqslant t_0$.

Definition 6. Bounded function V

If there exist $t_0 > 0$, $h > 0$, so that on the partial region Ω_1, we have

$$|V(t,x_1,\ldots,x_n)| < L \tag{9}$$

where L is a positive constant, then V is said to be bounded on Ω_1.

Definition 7. Function with infinitesimal upper bound

If V is bounded, and for any $\lambda > 0$, there exists $\mu > 0$, so that on Ω_1 when $\sum_s x_s^2 \leqslant \mu$, and $t \geqslant t_0$, we have

$$|V(t,x_1,\ldots,x_n)| \leqslant \lambda \tag{10}$$

then V admits an infinitesimal upper bound on Ω_1.

Theorem of stability and of asymptotical stability on partial region

Theorem 1. *If there can be found for the differential equations of the disturbed motion* (Eq. (1) *a definite function $V(t,x_1,\ldots,x_n)$ on the partial region, and for which the derivative with respect to time based on these equations as given by the following:*

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{s=1}^{n} \frac{\partial V}{\partial x_s} X_s \tag{11}$$

is a semi-definite function on the partial region whose sense is opposite to that of V, or if it becomes zero identically, then the undisturbed motion is stable on the partial region.

Proof. Let us assume for the sake of definiteness that V is a positive definite function. Consequently, there exists a sufficiently large number t_0 and a sufficiently small number $h < H$, such that on the intersection Ω_1 of partial region Ω and

$$\sum_s x_s^2 \leqslant h, \quad (s = 1,\ldots,n) \tag{12}$$

and $t \geqslant t_0$, the following inequality is satisfied

$$V(t,x_1,\ldots,x_n) \geqslant W(x_1,\ldots,x_n), \tag{13}$$

where W is a certain positive definite function which does not depend on t. Besides that, Eq. (11) may assume only negative or zero value in this region.

Let ε be an arbitrarily small positive number. We shall suppose that in any case $\varepsilon < h$. Let us consider the aggregation of all possible values of the quantities $x_1,\ldots,x_n$, which are on the intersection Ω_2 of Ω_1 and

$$\sum_s x_s^2 = \varepsilon, \tag{14}$$

and let us designate by $l > 0$ the precise lower limit of the function W under this condition. By virtue of Eq. (8), we shall have

$$V(t,x_1,\ldots,x_n) \geqslant l \quad \text{for} \quad (x_1,\ldots,x_n) \quad \text{on} \quad \omega_2. \tag{15}$$

We shall now consider the quantities x_s as functions of time, which satisfy the differential equations of disturbed motion. We shall assume that the initial values x_{s0} of these functions for $t = t_0$ lie on the intersection Ω_2 of Ω_1 and the region

$$\sum_s x_s^2 \leqslant \delta, \tag{16}$$

where δ is so small that

$$V(t_0, x_{10}, \ldots, x_{n0}) < l \tag{17}$$

By virtue of the fact that $V(t_0, 0, \ldots, 0) = 0$, such a selection of the number δ is obviously possible. We shall suppose that in any case the number δ is smaller than ε. Then the inequality

$$\sum_s x_s^2 < \varepsilon, \tag{18}$$

being satisfied at the initial instant will be satisfied, in the very least, for a sufficiently small $t - t_0$, since the functions $x_s(t)$ very continuously with time. We shall show that these inequalities will be satisfied for all values $t > t_0$. Indeed, if these inequalities were not satisfied at some time, there would have to exist such an instant $t = T$ for which this inequality would become an equality. In other words, we would have

$$\sum_s x_s^2(T) = \varepsilon, \tag{19}$$

and consequently, on the basis of Eq. (15)

$$V(T, x_1(T), \ldots, x_n(T)) \geqslant l \tag{20}$$

On the other hand, since $\varepsilon < h$, the inequality Eq. (7) is satisfied in the entire interval of time $[t_0, T]$, and consequently, in this entire time interval $\dot{V} \leqslant 0$. This yields

$$V(T, x_1(T), \ldots, x_n(T)) \leqslant V(t_0, x_{10}, \ldots, x_{n0}), \tag{21}$$

which contradicts Eq. (18) on the basis of Eq. (17). Thus, the inequality Eq. (4) must be satisfied for all values of $t > t_0$, hence follows that the motion is stable.

Finally, we must point out that from the viewpoint of mathematics, the stability on partial region in general does not be related logically to the stability on whole region. If an undisturbed solution is stable on a partial region, it may be either stable or unstable on the whole region and vice versa. From the viewpoint of dynamics, we were not interesting to the solution starting from Ω_2 and going out of Ω.

Theorem 2. *If in satisfying the conditions of Theorem 1, the derivative $\dot{V}$ is a definite function on the partial region with opposite sign to that of V and the function V itself permits an infinitesimal upper limit, then the undisturbed motion is asymptotically stable on the partial region.*

Proof. Let us suppose that V is a positive definite function on the partial region and that consequently, $\dot{V}$ is negative definite. Thus on the intersection Ω_1 of Ω and the region defined by Eq. (7) and $t \geqslant t_0$ there will be satisfied not only the inequality Eq. (8), but the following inequality as will:

$$\dot{V} \leqslant -W_1(x_1, \ldots, x_n), \tag{22}$$

where W_1 is a positive definite function on the partial region independent of t.

Let us consider the quantities x_s as functions of time, which satisfy the differential equations of disturbed motion assuming that the initial values $x_{s0} = x_s(t_0)$ of these quantities satisfy the inequalities Eq. (16). Since the undisturbed motion is stable in any case, the magnitude δ may be selected so small that for all values of $t \geqslant t_0$ the quantities x_s remain within Ω_1. Then, on the basis of Eq. (22) the derivative of function $V(t, x_1(t), \ldots, x_n(t))$ will be negative at all times and, consequently, this function will approach a certain limit, as t increases without limit, remaining larger than this limit at all times. We shall show that this limit is equal to some positive quantity different from zero. Then for all values of $t \geqslant t_0$ the following inequality will be satisfied:

$$V(t, x_1(t), \ldots, x_n(t)) > \alpha \tag{23}$$

where $\alpha > 0$.

Since V permits an infinitesimal upper limit, it follows from this inequality that

$$\sum_s x_s^2(t) \geqslant \lambda, \quad (s = 1, \ldots, n), \tag{24}$$

where λ is a certain sufficiently small positive number. Indeed, if such a number λ did not exist, that is, if the quantity $\sum_s x_s(t)$ were smaller than any preassigned number no matter how small, then the magnitude $V(t, x_1(t), \ldots, x_n(t))$, as follows from the definition of an infinitesimal upper limit, would also be arbitrarily small, which contradicts (13).

If for all values of $t \geqslant t_0$ the inequality Eq. (24) is satisfied, then Eq. (22) shows that the following inequality will be satisfied at all times:

$$\dot{V} \leqslant -l_1, \tag{25}$$

where l_1 is positive number different from zero which constitutes the precise lower limit of the function $W_1(t, x_1(t), \ldots, x_n(t))$ under condition Eq. (24). Consequently, for all values of $t \geqslant t_0$ we shall have:

$$V(t, x_1(t), \ldots, x_n(t)) = V(t_0, x_{10}, \ldots, x_{n0}) + \int_{t_0}^{t} \frac{dV}{dt} dt \leqslant V(t_0, x_{10}, \ldots, x_{n0}) - l_1(t - t_0), \tag{26}$$

which is, obviously, in contradiction with Eq. (23). The contradiction thus obtained shows that the function $V(t, x_1(t), \ldots, x_n(t))$ approached zero as t increase without limit. Consequently, the same will be true for the function $W(x_1(t), \ldots, x_n(t))$ as well, from which it follows directly that

$$\lim_{t \to \infty} x_s(t) = 0, \quad (s = 1, \ldots, n) \tag{27}$$

which proves the theorem.

2.2. Generalized chaos synchronization strategy

Consider the following unidirectional coupled chaotic systems

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}) \\ \dot{\mathbf{y}} &= \mathbf{h}(t, \mathbf{y}) + \mathbf{u} \end{aligned} \tag{28}$$

Where $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T \in R^n$, $\mathbf{y} = [y_1, y_2, \ldots, y_n]^T \in R^n$ denote the master state vector and slave state vector respectively, $\mathbf{f}$ and $\mathbf{h}$ are nonlinear vector functions, and $\mathbf{u} = [u_1, u_2, \ldots, u_n]^T \in R^n$ is a control input vector.

The generalized synchronization can be accomplished when $t \to \infty$, the limit of the error vector $\mathbf{e} = [e_1, e_2, \ldots, e_n]^T$ approaches zero:

$$\lim_{t \to \infty} \mathbf{e} = 0 \tag{29}$$

where

$$\mathbf{e} = \mathbf{G}(\mathbf{x}) - \mathbf{y} \tag{30}$$

$G(x)$ is a given function of x.

By using the partial region stability theory, the Lyapunov function is linear homogeneous function of error states. The controllers can be designed in lower order.

3. Chaos of an inertial tachometer system and new Mathieu-Van der Pol system

This section introduces an inertial tachometer system and Mathieu-Van der Pol system, respectively.

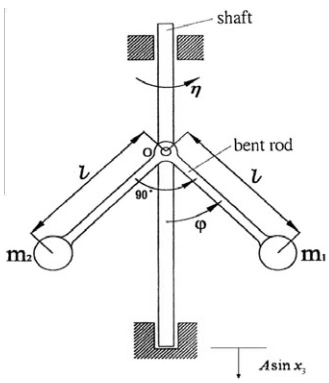

Fig. 2. Sketch of an inertial tachometer.

3.1. An inertial tachometer system

The physical model of the inertial tachometer system is shown in Fig. 2. The mass of bent rod is neglected and the balls m_1 and m_2 are considered as two particles.

We can write the kinetic and potential energies of the system as follow:

$$T = \frac{1}{2}m_1(l^2\dot{\varphi}^2 + l^2\dot{\eta}^2\sin^2\varphi) + \frac{1}{2}m_2(l^2\dot{\varphi}^2 + l^2\dot{\eta}^2\cos^2\varphi) + \frac{1}{2}J\dot{\eta}^2$$
$$\Pi = -gl(m_1\cos\varphi + m_2\sin\varphi) \tag{31}$$

wherem_1 and m_2 the mass of balls and $m_1 > m_2$
J the moment of inertia of the shaft about vertical center axis
l the length of rod
φ the angle between the shaft and the rod
$\dot{\eta}$ constant angular velocity of the tachometer
g gravity acceleration

The Lagrangian is $L = T - \Pi$, the corresponding Lagrange equations are

$$(m_1 + m_2)l^2\ddot{\varphi} - (m_1 - m_2)l^2\dot{\eta}^2\sin\varphi\cos\varphi + gl(m_1\cos\varphi - m_2\cos\varphi) = -k\dot{\varphi} \tag{32}$$

where k is damping coefficient in bent rod bearing.

We assume that the inertial tachometer is subjected to an external vertical vibration basement $A\sin x_3$ where x_3 is state variable, A is the amplitude of vibration. The vertical axis rotates with constant speed $\dot{\eta}$. The Lagrange equation now are given in a noninertial vibrating reference frame, which is fixed with the basement. Due to the inertial force appearing in the noninertial frame, the gravity acceleration in the noninertial frame becomes $g + A\Omega^2\sin\Omega t$. Let $x_1 = \varphi,\ x_2 = \dot{\varphi},\ x_3 = \Omega t,\ x_4 = \dot{x}_3,\ \Omega = \eta$, where η is a constant.

And the state equations can be written as:

$$\begin{cases} \dot{x}_1 = x_2; \quad \dot{x}_2 = \frac{1}{m_1+m_2}\left[(m_1 - m_2)\eta^2\cos x_1\sin x_1 - \frac{1}{l}(g + A\eta^2\sin x_3)(m_1\sin x_1 - m_2\cos x_1) - \frac{k}{l^2}x_2\right] \\ \dot{x}_3 = x_4; \quad \dot{x}_4 = -A\sin x_3 \end{cases} \tag{33}$$

where x_1, x_2, x_3, x_4 are state variables and $m_1, m_2, A, \eta, l, k, g$ are parameters. This system exhibits chaos when the parameters of system are $m_1 = 9$, $m_2 = 1$, $A = 10.5$, $\eta = 1$, $l = 0.3$, $k = 0.5$, $g = 9.81$ and the initial condition is $(x_1,x_2,x_3,x_4) = (0,0,2,2)$. Its phase portraits and time histories as shown in Figs. 3 and 4.

3.2. New Mathieu-Van der Pol system

Mathieu equation and van der Pol equation are two typical nonlinear non-autonomous systems:

$$\dot{z}_1 = z_2; \quad \dot{z}_2 = -(a_1 + b_1\sin wt)z_1 - (a_1 + b_1\sin wt)z_1^3 - c_1z_2 + d_1\sin wt \tag{34}$$

$$\dot{z}_3 = z_4; \quad \dot{z}_4 = -e_1z_3 + f_1(1 - z_3^2)z_4 + g_1\sin wt \tag{35}$$

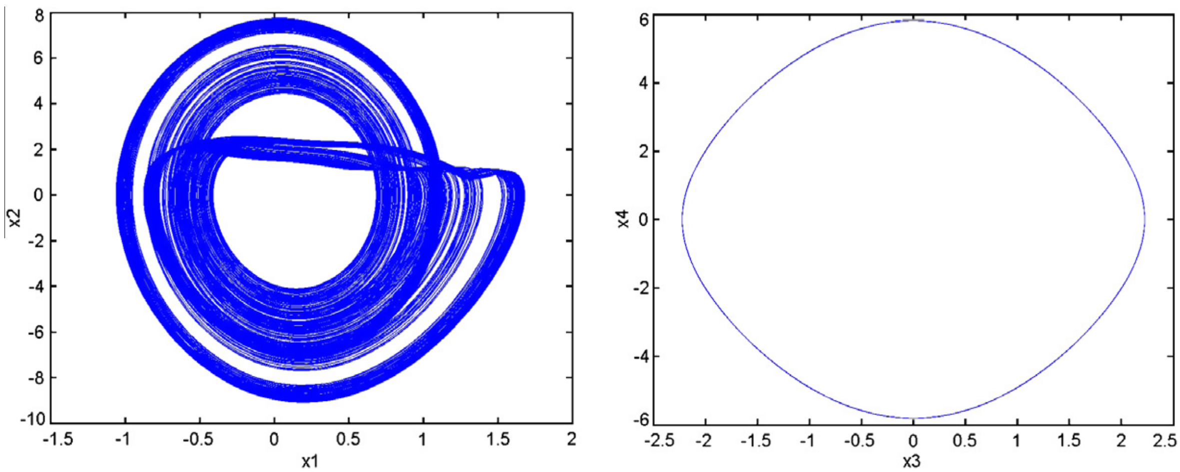

Fig. 3. Phase portraits of an inertial tachometer.

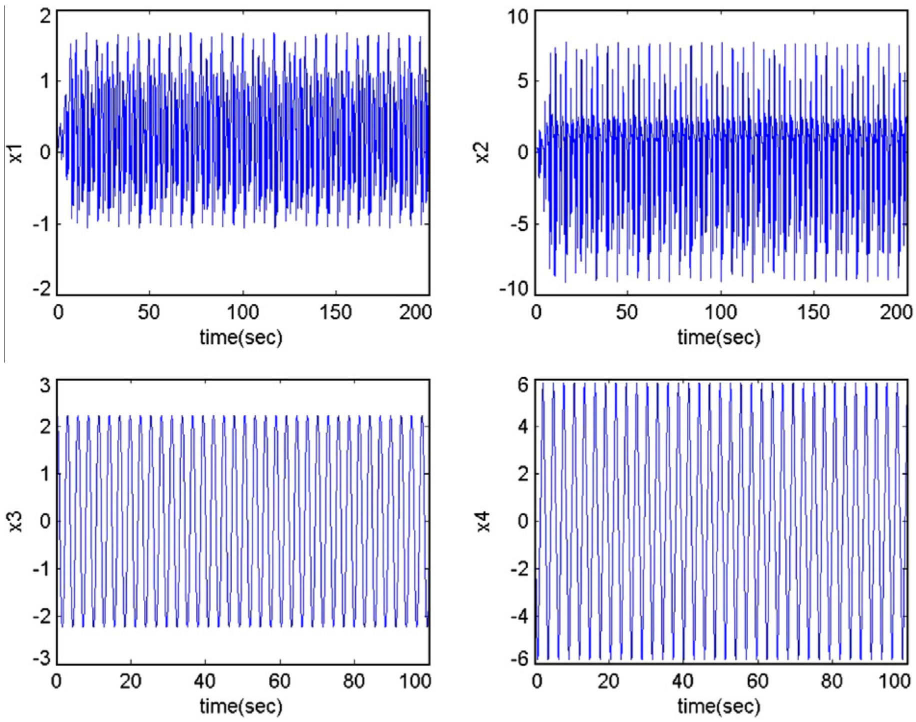

Fig. 4. Time histories of the four states of an inertial tachometer.

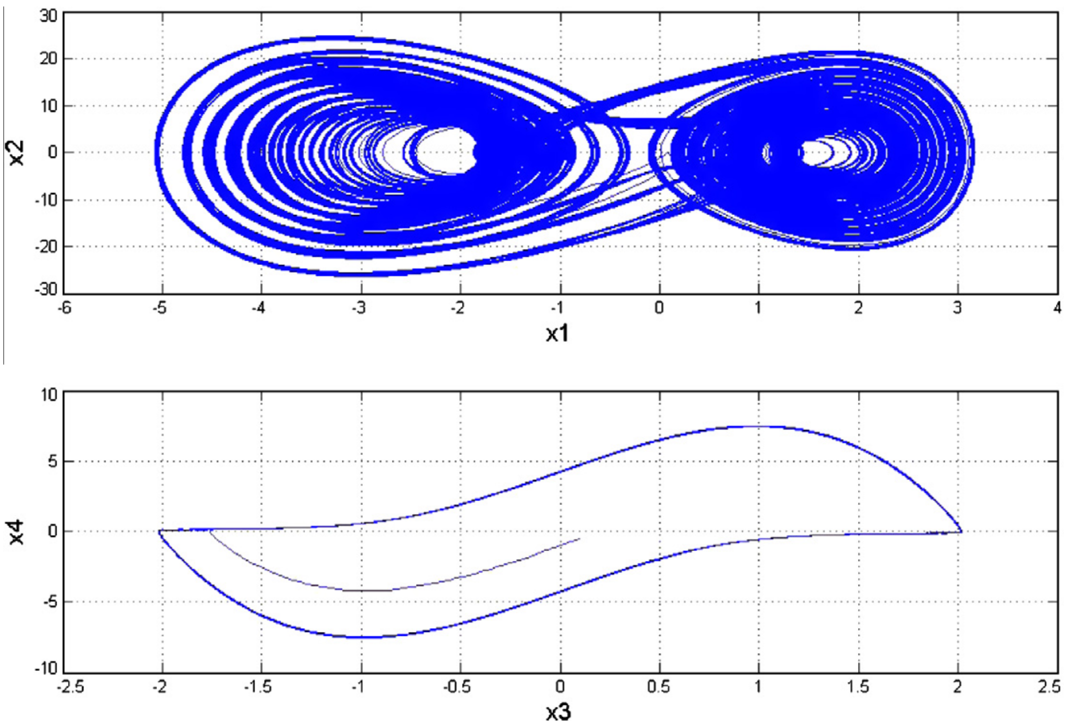

Fig. 5. Phase portraits of new Mathieu-Van der Pol system.

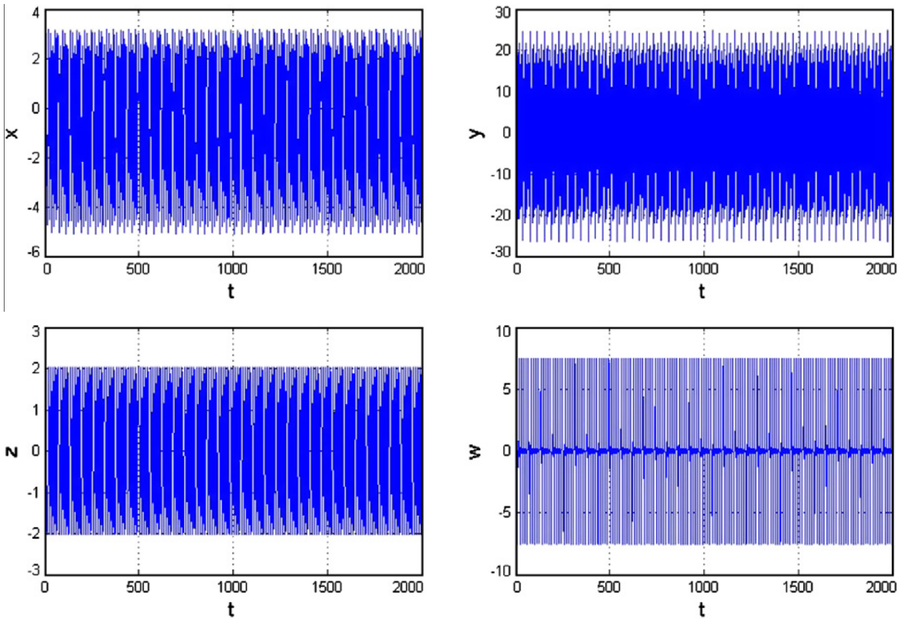

Fig. 6. Time histories of the four states of new Mathieu-van der Pol system.

Exchanging *sinwt* in Eq. (32) with z_3 and *sinwt* in Eq. (33) with z_1, we obtain the autonomous new Mathieu-Van der Pol system:

$$\begin{cases} \dot{z}_1 = z_2; \quad \dot{z}_2 = -(a_1 + b_1 z_3) z_1 - (a_1 + b_1 z_3) z_1^3 - c_1 z_2 + d_1 z_3 \\ \dot{z}_3 = z_4; \quad \dot{z}_4 = -e_1 z_3 + f_1 (1 - z_3^2) z_4 + g_1 z_1 \end{cases} \tag{36}$$

where $a_1, b_1, c_1, d_1, e_1, f_1, g_1$, are uncertain parameter. This system exhibits chaos when the parameters of system are $a_1 = 10$, $b_1 = 3$, $c_1 = 0.4$, $d_1 = 70$, $e_1 = 1$, $f_1 = 5$, $g_1 = 0.1$ and the initial states of system are $(z_1, z_{,2}, z_3, z_4) = (0.1, -0.5, 0.1, -0.5)$, its phase portraits and time histories are shown in Figs. 5 and 6.

4. Numerical simulations

The following example is two inertial tachometer systems with unidirectional coupling:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{m_1+m_2}\left[(m_1 - m_2)\eta^2 \cos x_1 \sin x_1 - \frac{1}{l}(g + A\eta^2 \sin x_3)(m_1 \sin x_1 - m_2 \cos x_1) - \frac{k}{l^2} x_2\right] \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -A \sin x_3 \end{cases} \tag{37}$$

$$\begin{cases} \dot{y}_1 = y_2 + u_1 \\ \dot{y}_2 = \frac{1}{m_1+m_2}\left[(m_1 - m_2)\eta^2 \cos y_1 \sin y_1 - \frac{1}{l}(g + A\eta^2 \sin y_3)(m_1 \sin y_1 - m_2 \cos y_1) - \frac{k}{l^2} y_2\right] + u_2 \\ \dot{y}_3 = y_4 + u_3 \\ \dot{y}_4 = -A \sin y_3 + u_4 \end{cases} \tag{38}$$

CASE I. The generalized synchronization error function is $e_i = x_i - y_i + 20$ $(i = 1,2,3,4)$

$$\text{Our goal is } y_i = x_i + 20, \quad \text{i.e. } \lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + 20) = 0, \quad (i = 1,2,3,4) \tag{39}$$

The addition of 20, makes that error dynamics always happens in first quadrant.

The error dynamics becomes

$$\begin{aligned}
\dot{e}_1 &= \dot{x}_1 - \dot{y}_1 = x_2 - y_2 - u_1 \\
\dot{e}_2 &= \dot{x}_2 - \dot{y}_2 = \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos x_1\sin x_1 - \frac{1}{l}(g+A\eta^2\sin x_3)(m_1\sin x_1 - m_2\cos x_1) - \frac{k}{l^2}x_2\right] \\
&\quad - \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos y_1\sin y_1 - \frac{1}{l}(g+A\eta^2\sin y_3)(m_1\sin y_1 - m_2\cos y_1) - \frac{k}{l^2}y_2\right] - u_2 \\
\dot{e}_3 &= \dot{x}_3 - \dot{y}_3 = x_4 - y_4 - u_3 \\
\dot{e}_4 &= \dot{x}_4 - \dot{y}_4 = -A\sin x_3 + A\sin y_3 - u_4
\end{aligned} \tag{40}$$

Let initial states be $(x_1,x_2,x_3,x_4) = (0,0,2,2)$, $(y_1,y_2,y_3,y_4) = (2,2,0,0)$, we find the error dynamic always exist in first quadrant showed in Fig. 7. By GYC partial region asymptotical stability theorem, one can choose a Lyapunov function in the form of a positive definite function in first quadrant:

$$V = e_1 + e_2 + e_3 + e_4 \tag{41}$$

Its time derivative is

$$\begin{aligned}
\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 = (x_2 - y_2 - u_1) + \left\{\frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos x_1\sin x_1\right.\right. \\
&\quad \left. - \frac{k}{l^2}x_2 - \frac{1}{l}(g+A\eta^2\sin x_3)(m_1\sin x_1 - m_2\cos x_1)\right] - \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos y_1\sin y_1\right. \\
&\quad \left.\left. - \frac{1}{l}(g+A\eta^2\sin y_3)(m_1\sin y_1 - m_2\cos y_1) - \frac{k}{l^2}y_2\right] - u_2\right\} + (x_4 - y_4 - u_3) + (-A\sin x_3 + A\sin y_3 - u_4)
\end{aligned} \tag{42}$$

Choose

$$\begin{aligned}
u_1 &= x_2 - y_2 + e_1 \\
u_2 &= \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos x_1\sin x_1 - \frac{1}{l}(g+A\eta^2\sin x_3)(m_1\sin x_1 - m_2\cos x_1) - \frac{k}{l^2}x_2\right] \\
&\quad - \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos y_1\sin y_1 - \frac{1}{l}(g+A\eta^2\sin y_3)(m_1\sin y_1 - m_2\cos y_1) - \frac{k}{l^2}y_2\right] + e_2 \\
u_3 &= x_4 - y_4 + e_3 \\
u_4 &= -A\sin x_3 + A\sin y_3 + e_4
\end{aligned} \tag{43}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0 \tag{44}$$

which is negative definite function in first quadrant. Error states versus time and time histories of states are shown in Figs. 8 and 9.

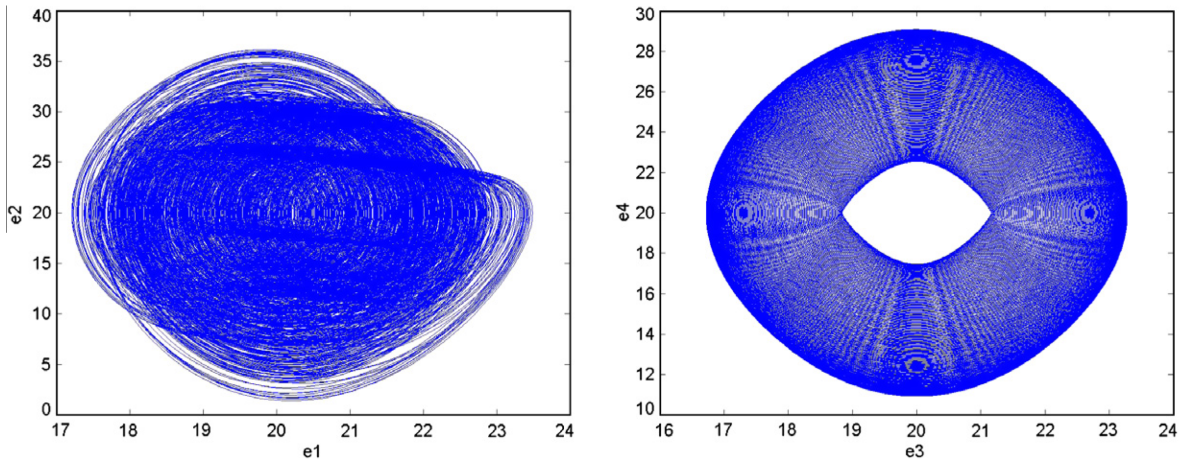

Fig. 7. Phase portraits of four errors dynamics for Case I.

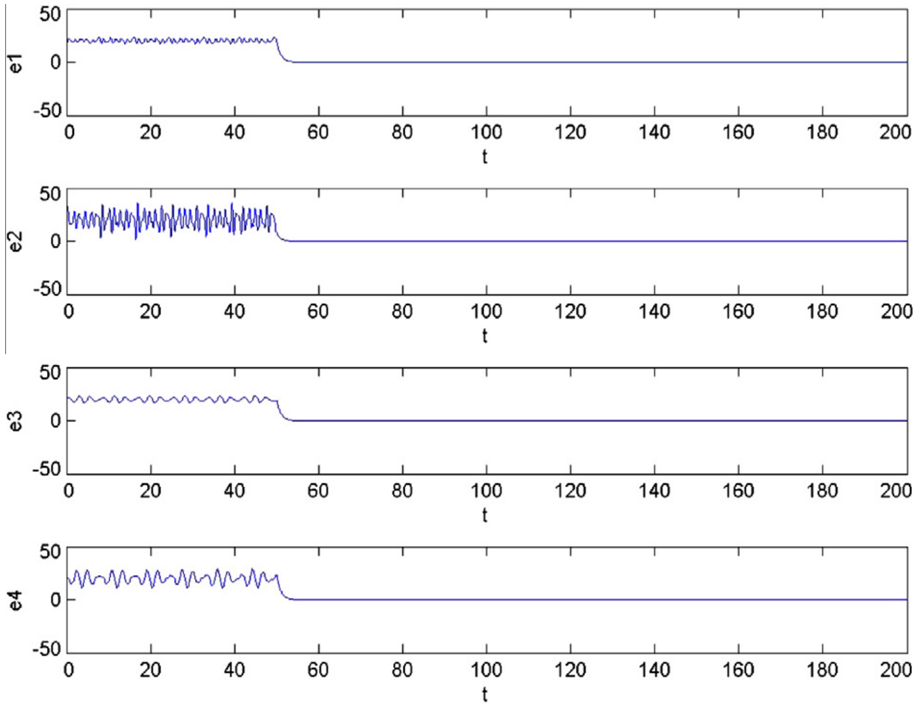

Fig. 8. Time histories of errors for Case I.

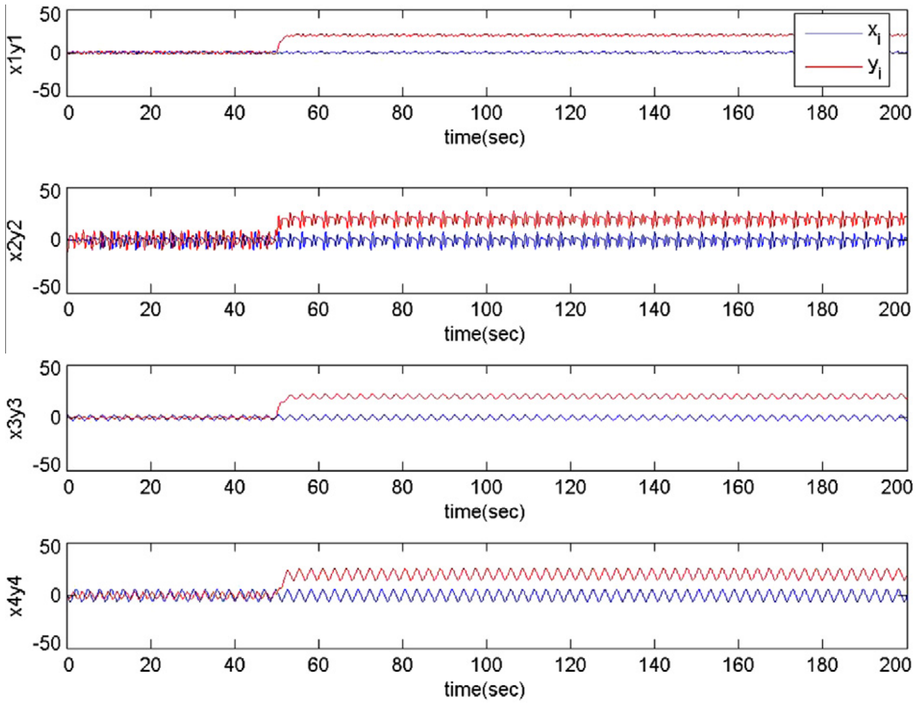

Fig. 9. Time histories of x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 for Case I.

CASE II. The generalized synchronization error function is,

$$e_i = x_i - y_i + F\sin wt + 20, \quad (i = 1, 2, 3, 4)$$

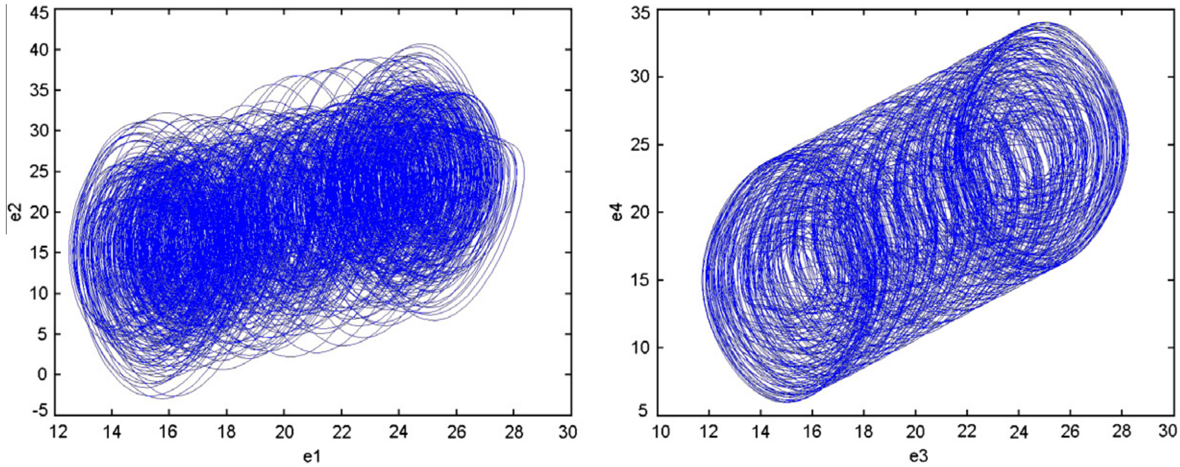

Fig. 10. Phase portraits of error dynamics for Case II.

Our goal is $y_i = x_i + F \sin wt + 20$, i.e. $\lim_{t\to\infty} e_i = \lim_{t\to\infty}(x_i - y_i + F \sin wt + 20) = 0$, $(i = 1,2,3,4)$
The error dynamics becomes

$$\begin{aligned}
\dot{e}_1 &= \dot{x}_1 - \dot{y}_1 = x_2 - y_2 - u_1 + Fw\cos wt \\
\dot{e}_2 &= \dot{x}_2 - \dot{y}_2 = \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos x_1 \sin x_1 - \frac{1}{l}(g + A\eta^2 \sin x_3)(m_1 \sin x_1 - m_2 \cos x_1) - \frac{k}{l^2}x_2\right] - u_2 \\
&\quad - \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos y_1 \sin y_1 - \frac{1}{l}(g + A\eta^2 \sin y_3)(m_1 \sin y_1 - m_2 \cos y_1) - \frac{k}{l^2}y_2\right] + Fw\cos wt \\
\dot{e}_3 &= \dot{x}_3 - \dot{y}_3 = x_4 - y_4 - u_3 + Fw\cos wt \\
\dot{e}_4 &= \dot{x}_4 - \dot{y}_4 = -A\sin x_3 + A\sin y_3 - u_4 + Fw\cos wt
\end{aligned} \tag{45}$$

Let initial states be $(x_1,x_2,x_3,x_4) = (0,0,2,2)$, $(y_1,y_2,y_3,y_4) = (2,2,0,0)$, and $F = 5$, $w = 0.1$, we find the error dynamic always exists in first quadrant as shown in Fig. 10. By GYC partial region asymptotical stability theorem, one can choose a Lyapunov function in the form of a positive definite function in first quadrant:

$$V = e_1 + e_2 + e_3 + e_4 \tag{46}$$

Its time derivative is

$$\begin{aligned}
\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 = (x_2 - y_2 - u_1 + Fw\cos wt) \\
&\quad + \left\{\frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos x_1\sin x_1 - \frac{k}{l^2}x_2 - \frac{1}{l}(g + A\eta^2\sin x_3)(m_1\sin x_1 - m_2\cos x_1)\right]\right. \\
&\quad \left. - \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos y_1\sin y_1 - \frac{1}{l}(g + A\eta^2\sin y_3)(m_1\sin y_1 - m_2\cos y_1) - \frac{k}{l^2}y_2\right] - u_2 + Fw\cos wt\right\} \\
&\quad + (x_4 - y_4 - u_3 + Fw\cos wt) + (-A\sin x_3 + A\sin y_3 - u_4 + Fw\cos wt)
\end{aligned} \tag{47}$$

Choose

$$\begin{aligned}
u_1 &= x_2 - y_2 + Fw\cos wt + e_1 \\
u_2 &= \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos x_1\sin x_1 - \frac{1}{l}(g + A\eta^2\sin x_3)(m_1\sin x_1 - m_2\cos x_1) - \frac{k}{l^2}x_2\right] - \frac{1}{m_1+m_2} \\
&\quad \left[(m_1-m_2)\eta^2\cos y_1\sin y_1 - \frac{1}{l}(g + A\eta^2\sin y_3)(m_1\sin y_1 - m_2\cos y_1) - \frac{k}{l^2}y_2\right] + Fw\cos wt + e_2 \\
u_3 &= x_4 - y_4 + Fw\cos wt + e_3 \\
u_4 &= -A\sin x_3 + A\sin y_3 + Fw\cos wt + e_4
\end{aligned} \tag{48}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0 \tag{49}$$

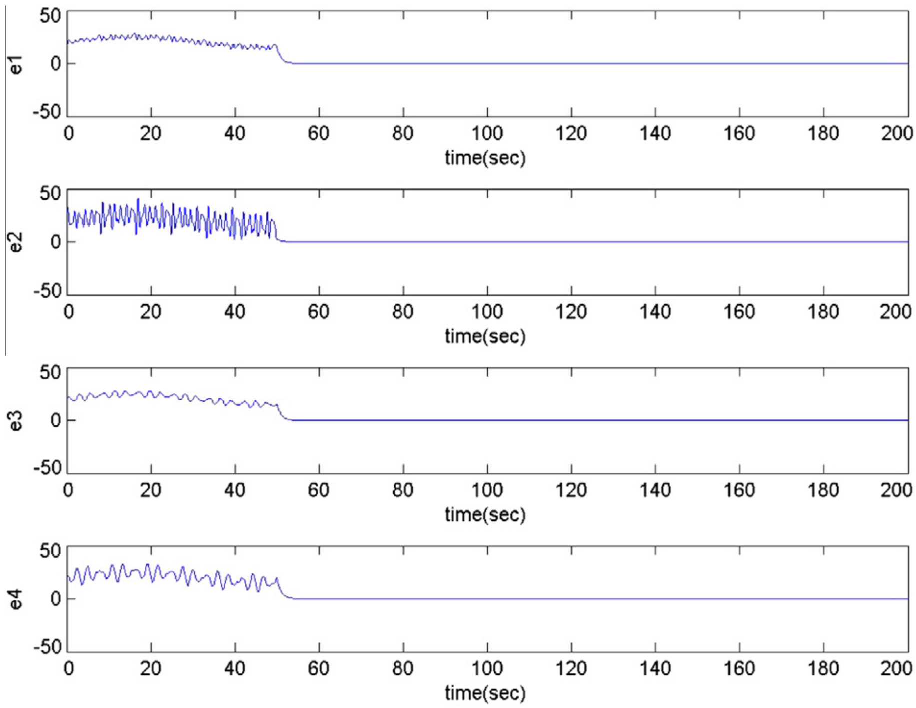

Fig. 11. Time histories of errors for Case II.

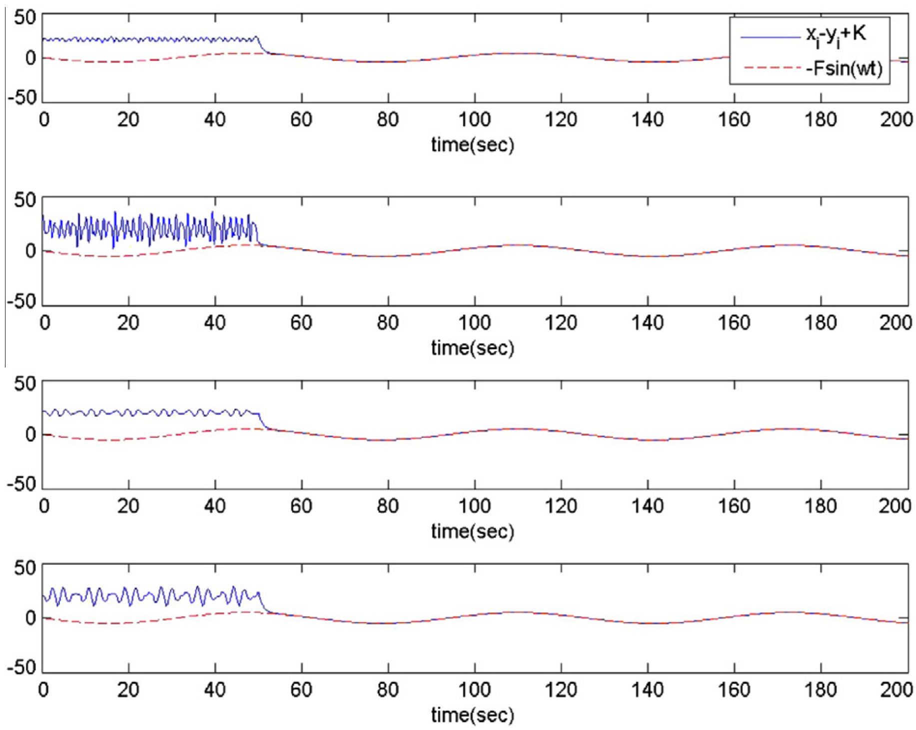

Fig. 12. Time histories of $x_i - y_i + K$ and $-F\sin\Omega t$ for Case II.

which is negative definite function in first quadrant. Error states versus time and time histories of $x_i - y_i + 20$ and $-F\ \sin wt$ are shown in Figs. 11 and 12.

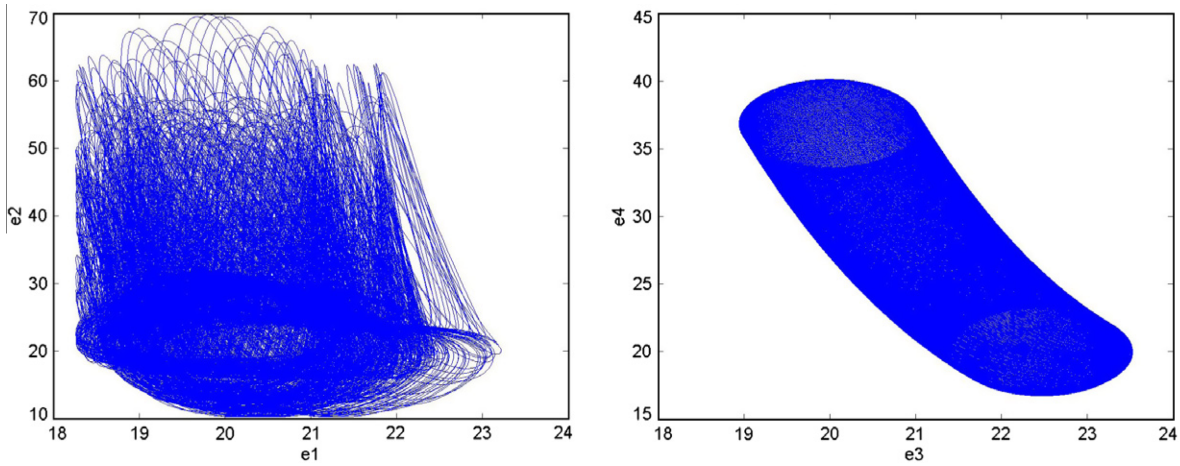

Fig. 13. Phase portraits of error dynamics for Case III.

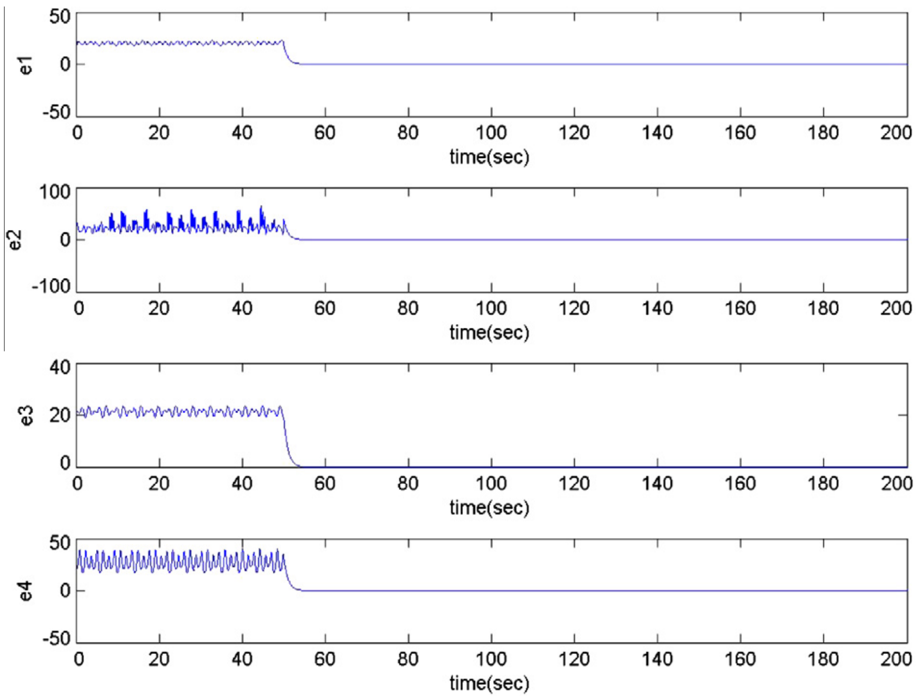

Fig. 14. Time histories of error for Case III.

CASE III. The generalized synchronization error function is $e_i = \frac{1}{2}x_i^2 - y_i + 20,\ (i = 1,2,3,4)$

Our goal is $y_i = \frac{1}{2}x_i^2 + 20$, i.e. $\lim_{t\to\infty} e_i = \lim_{t\to\infty}\left(\frac{1}{2}x_i^2 - y_i + 20\right) = 0,\ (i = 1,2,3,4)$.
The error dynamics becomes

$$
\begin{aligned}
&\dot{e}_1 = x_1\dot{x}_1 - \dot{y}_1 = x_1x_2 - y_2 - u_1\\
&\dot{e}_2 = x_2\dot{x}_2 - \dot{y}_2 = x_2\frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos x_1\sin x_1 - \frac{1}{l}(g + A\eta^2\sin x_3)(m_1\sin x_1 - m_2\cos x_1)\right.\\
&\left.-\frac{k}{l^2}x_2\right] - \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos y_1\sin y_1 - \frac{1}{l}(g + A\eta^2\sin y_3)(m_1\sin y_1 - m_2\cos y_1) - \frac{k}{l^2}y_2\right] - u_2\\
&\dot{e}_3 = x_3\dot{x}_3 - \dot{y}_3 = x_3x_4 - y_4 - u_3\\
&\dot{e}_4 = x_4\dot{x}_4 - \dot{y}_4 = x_4(-A\sin x_3) + A\sin y_3 - u_4
\end{aligned}
\tag{50}
$$

Let initial states be $(x_1,x_2,x_3,x_4) = (0,0,2,2)$, $(y_1,y_2,y_3,y_4) = (2,2,0,0)$ and, we find the error dynamics always exists in first quadrant as shown in Fig. 13. By GYC partial region asymptotical stability theorem, one can choose a Lyapunov function in the form of a positive definite function in first quadrant:

$$V = e_1 + e_2 + e_3 + e_4 \tag{51}$$

Its time derivative is

$$\begin{aligned}\dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 = (x_1x_2 - y_2 - u_1) \\ &+ \left\{x_2 \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2 \cos x_1 \sin x_1 - \frac{1}{l}(g + A\eta^2 \sin x_3)(m_1 \sin x_1 - m_2 \cos x_1) - \frac{k}{l^2}x_2\right]\right. \\ &\left. - \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2 \cos y_1 \sin y_1 - \frac{1}{l}(g + A\eta^2 \sin y_3)(m_1 \sin y_1 - m_2 \cos y_1) - \frac{k}{l^2}y_2\right] - u_2\right\} \\ &+ (x_3x_4 - y_4 - u_3) + [x_4(-A\sin x_3) + A\sin y_3 - u_4]\end{aligned} \tag{52}$$

Choose

$$\begin{aligned}u_1 &= x_1x_2 - y_2 + e_1 \\ u_2 &= x_2 \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2 \cos x_1 \sin x_1 - \frac{1}{l}(g + A\eta^2 \sin x_3)(m_1 \sin x_1 - m_2 \cos x_1) - \frac{k}{l^2}x_2\right] \\ &\quad - \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2 \cos y_1 \sin y_1 - \frac{1}{l}(g + A\eta^2 \sin y_3)(m_1 \sin y_1 - m_2 \cos y_1) - \frac{k}{l^2}y_2\right] + e_2 \\ u_3 &= x_3x_4 - y_4 + e_3 \\ u_4 &= x_4(-A\sin x_3) + A\sin y_3 + e_4\end{aligned} \tag{53}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0 \tag{54}$$

which is negative definite function in first quadrant. Error states versus time and time histories of $\frac{1}{2}x_i^2 + 20$ and y_i are shown in Fig. 14 and in Fig. 15.

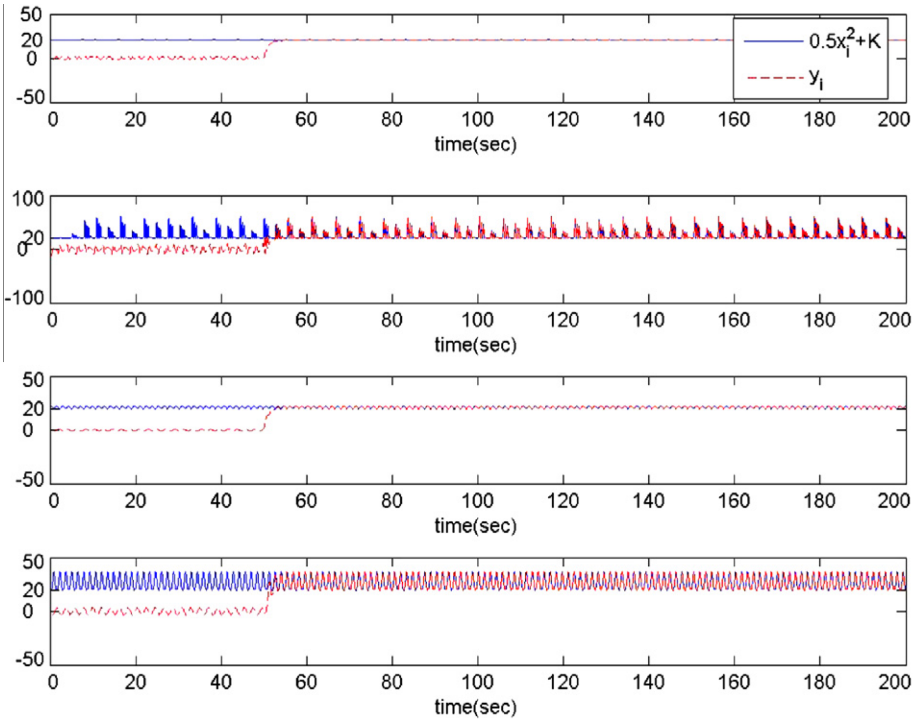

Fig. 15. Time histories of $\frac{1}{2}x_i^2 + 20$ and y_i for Case III.

CASE IV. The generalized synchronization error function is $e_i = x_i - y_i + z_i + 50$, where $z_i (i = 1,2,3,4)$ are chaotic states of new Mathieu-Van der Pol system.

The goal system for synchronization is new Mathieu-Van der Pol system and initial states is $(z_1,z_2,z_3,z_4) = (0.1,-0.5,0.1,-0.5)$, system parameters $a_1 = 10$, $b_1 = 3$, $c_1 = 0.4$, $d_1 = 70$, $e_1 = 1$, $f_1 = 5$, $g_1 = 0.1$.

$$\begin{cases} \dot{z}_1 = z_2; \quad \dot{z}_2 = -(a_1 + b_1 z_3)z_1 - (a_1 + b_1 z_3)z_1^3 - c_1 z_2 + d_1 z_3 \\ \dot{z}_3 = z_4; \quad \dot{z}_4 = -e_1 z_3 + f_1(1 - z_3^2)z_4 + g_1 z_1 \end{cases} \tag{55}$$

We have $\lim_{t\to\infty} e = \lim_{t\to\infty} (x - y + z + 50) = 0$

The error dynamics becomes

$$\begin{aligned} \dot{e}_1 &= \dot{x}_1 - \dot{y}_1 + \dot{z}_1 = x_2 - y_2 + z_2 - u_1 \dot{e}_2 = \dot{x}_2 - \dot{y}_2 + \dot{z}_2 \\ &= \frac{1}{m_1 + m_2}\left[(m_1 - m_2)\eta^2 \cos x_1 \sin x_1 - \frac{1}{l}(g + A\eta^2 \sin x_3)(m_1 \sin x_1 - m_2 \cos x_1) - \frac{k}{l^2} x_2\right] \\ &\quad - \frac{1}{m_1 + m_2}\left[(m_1 - m_2)\eta^2 \cos y_1 \sin y_1 - \frac{1}{l}(g + A\eta^2 \sin y_3)(m_1 \sin y_1 - m_2 \cos y_1) - \frac{k}{l^2} y_2\right] - (a_1 + b_1 z_3)z_1 \\ &\quad - (a + bz_3)z_1^3 - c_1 z_2 + d_1 z_3 - u_2 \dot{e}_3 \\ &= \dot{x}_3 - \dot{y}_3 + \dot{z}_3 = x_4 - y_4 + z_4 - u_3 \dot{e}_4 = \dot{x}_4 - \dot{y}_4 + \dot{z}_4 = -A \sin x_3 + A \sin y_3 - e_1 z_3 + f_1(1 - z_3^2)z_4 + g_1 z_1 - u_4 \end{aligned} \tag{56}$$

Let initial states be $(x_1,x_2,x_3,x_4) = (0,0,2,2)$, $(y_1,y_2,y_3,y_4) = (2,2,0,0)$ and, we find the error dynamics always exists in first quadrant as shown in Fig. 16. By GYC partial region asymptotical stability theorem, one can choose a Lyapunov function in the form of a positive definite function in first quadrant:

$$V = e_1 + e_2 + e_3 + e_4 \tag{57}$$

Its time derivative is

$$\begin{aligned} \dot{V} &= \dot{e}_1 + \dot{e}_2 + \dot{e}_3 + \dot{e}_4 = (x_2 - y_2 + z_2 - u_1) \\ &\quad + \left\{\frac{1}{m_1 + m_2}\left[(m_1 - m_2)\eta^2 \cos x_1 \sin x_1 - \frac{k}{l^2} x_2 - \frac{1}{l}(g + A\eta^2 \sin x_3)(m_1 \sin x_1 - m_2 \cos x_1)\right]\right. \\ &\quad - \frac{1}{m_1 + m_2}\left[(m_1 - m_2)\eta^2 \cos y_1 \sin y_1 - \frac{1}{l}(g + A\eta^2 \sin y_3)(m_1 \sin y_1 - m_2 \cos y_1) - \frac{k}{l^2} y_2\right] \\ &\quad \left. - (a_1 + b_1 z_3)z_1 - (a + bz_3)z_1^3 - c_1 z_2 + d_1 z_3 - u_2\right\} + (x_4 - y_4 + z_4 - u_3) + [-A \sin x_3 + A \sin y_3 - e_1 z_3 \\ &\quad + f_1(1 - z_3^2)z_4 + g_1 z_1 - u_4] \end{aligned} \tag{58}$$

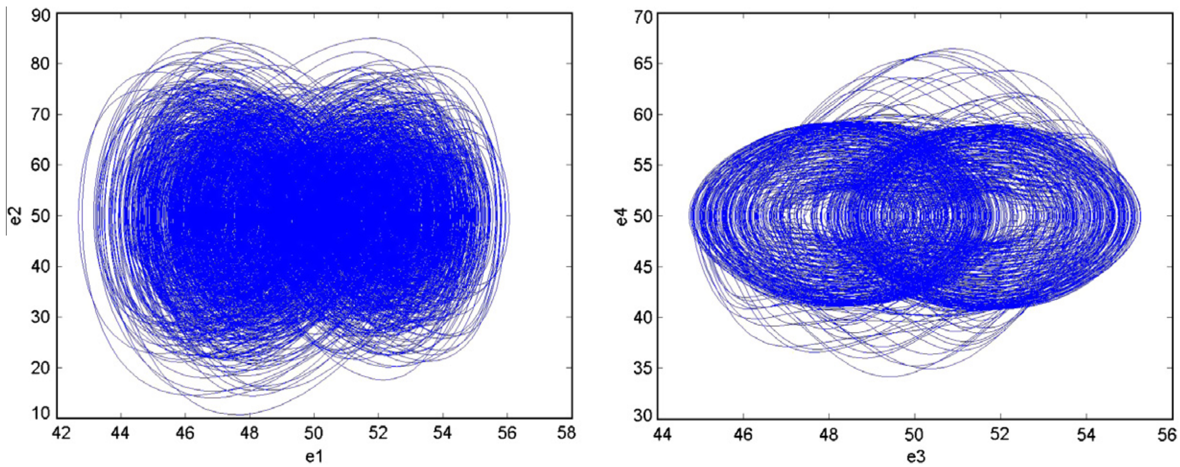

Fig. 16. Phase portraits of error dynamics for Case IV.

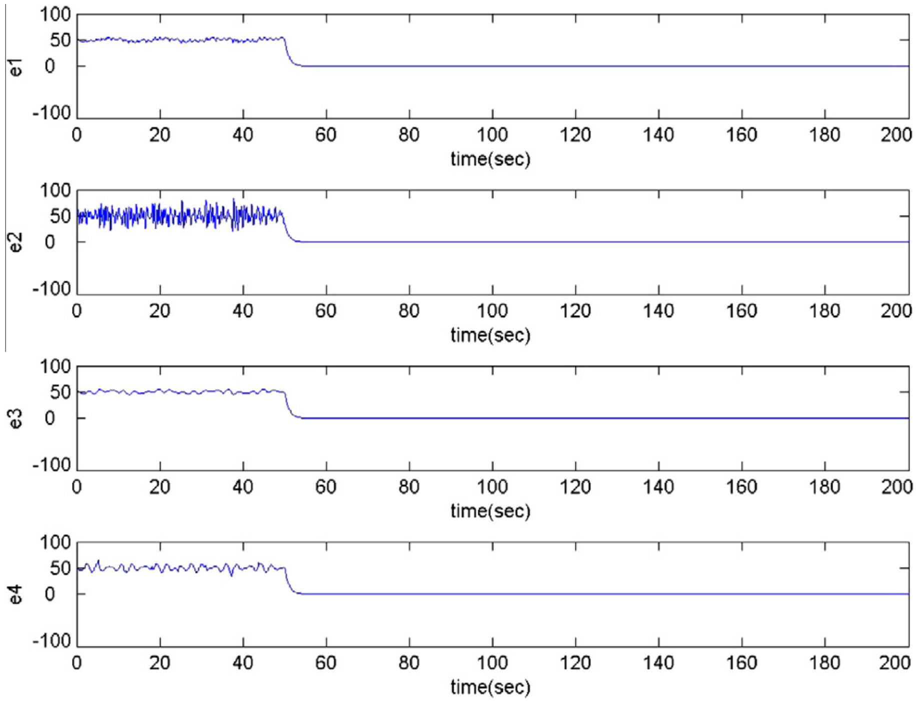

Fig. 17. Time histories of errors for Case IV.

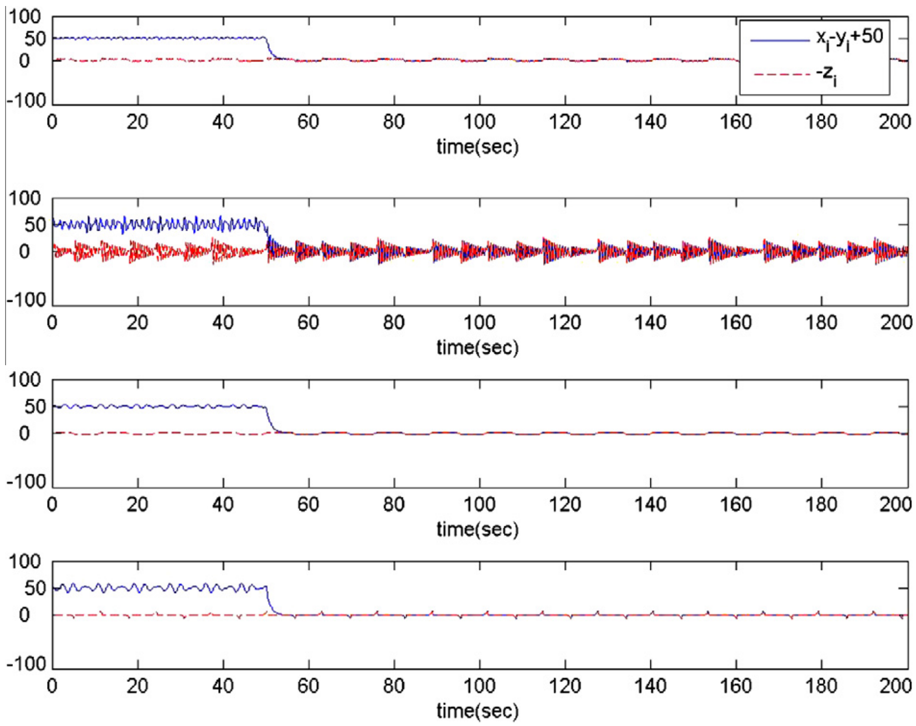

Fig. 18. Time histories of $x_i - y_i + 50$ and $-z_i$ for Case IV.

Choose

$$\begin{aligned}
u_1 &= x_2 - y_2 + z_2 + e_1 \\
u_2 &= \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos x_1 \sin x_1 - \frac{1}{l}(g + A\eta^2 \sin x_3)(m_1 \sin x_1 - m_2\cos x_1) - \frac{k}{l^2}x_2\right] \\
&\quad - \frac{1}{m_1+m_2}\left[(m_1-m_2)\eta^2\cos y_1 \sin y_1 - \frac{1}{l}(g + A\eta^2 \sin y_3)(m_1 \sin y_1 - m_2\cos y_1) - \frac{k}{l^2}y_2\right] \\
&\quad - (a_1 + b_1 z_3)z_1 - (a + bz_3)z_1^3 - c_1 z_2 + d_1 z_3 + e_2 \\
u_3 &= x_4 - y_4 + z_4 + e_3 \\
u_4 &= -A\sin x_3 + A\sin y_3 - e_1 z_3 + f_1(1 - z_3^2)z_4 + g_1 z_1 + e_4
\end{aligned} \tag{59}$$

We obtain

$$\dot{V} = -e_1 - e_2 - e_3 - e_4 < 0 \tag{60}$$

which is negative definite function in first quadrant. Error states versus time and time histories of $x_i - y_i + 50$ are shown in Figs. 17 and 18.

5. Conclusions

In this paper, a new strategy to achieve chaos generalized synchronization by GYC partial region stability theory is proposed. By using the GYC partial region stability theory, the Lyapunov function is a simple linear homogeneous function of error states and the controllers are more simple and have less simulation error because they are in lower order than that of traditional controllers. A inertial tachometer system and a new Mathieu-Van der Pol system are used as simulation examples which effectively confirm the scheme.

Acknowledgment

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