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Snapback repellers and homoclinic orbits for multi-dimensional maps

Kang-Ling Liao, Chih-Wen Shih *

Department of Applied Mathematics, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, 30050, Taiwan, ROC

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ABSTRACT

Marotto extended Li–Yorke's theorem on chaos from one-dimension to multi-dimension through introducing the notion of *snapback repeller* in 1978. Due to a technical flaw, he redefined *snapback repeller* in 2005 to validate this theorem. This presentation provides two methodologies to facilitate the application of Marotto's theorem. The first one is to estimate the radius of repelling neighborhood for a repelling fixed point. This estimation is of essential and practical significance as combined with numerical computations of snapback points. Secondly, we propose a sequential graphic-iteration scheme to construct homoclinic orbit for a repeller. This construction allows us to track the homoclinic orbit. Applications of the present methodologies with numerical computation to a chaotic neural network and a predator–prey model are demonstrated.

1. Introduction

Exploring and detecting chaotic behaviors have been major tasks in dynamical system theory. These investigations often involve numerical computations. However, due to the instability of orbits in chaotic dynamical systems, effective finding of chaos requires skillful computation techniques. In addition, in solving the application problems, it is often demanded to allocate the chaotic regime (parameter ranges). Therefore, combining analytic theory with valid numerical computation illuminates the development of these investigations. Among the limited analytic theories, Marotto extended Li–Yorke's theorem on chaos from one-dimension to multi-dimension through introducing the notion of snapback repeller.

Let us describe the notion of snapback repellers and Marotto's theorem. We consider a C^1 map $F:\mathbb{R}^n \to \mathbb{R}^n$. Denote by $B_r(\mathbf{x})$ the closed ball in $\mathbb{R}^n$ with center at $\mathbf{x}$ and radius $r > 0$ under certain norm on $\mathbb{R}^n$. A fixed point $\mathbf{z}$ of F is *repelling* if all eigenvalues of $DF(\mathbf{z})$ exceed one in magnitude. If there exist a norm $\|\cdot\|$ on $\mathbb{R}^n$ and $s > 1$ such that $\|F(\mathbf{x}) - F(\mathbf{y})\| > s \cdot \|\mathbf{x} - \mathbf{y}\|$, for all $\mathbf{x}, \mathbf{y} \in B_r(\mathbf{z})$, where $B_r(\mathbf{z})$ is defined under this norm, then $B_r(\mathbf{z})$ is called a *repelling neighborhood* of $\mathbf{z}$. It is known that if $\mathbf{z}$ is a repelling fixed point of F, then there exist a norm and $r > 0$ so that $B_r(\mathbf{z})$ is a repelling neighborhood of $\mathbf{z}$ [16]. However, this property does not necessarily hold for the Euclidean norm in general. In addition, if $\mathbf{z}$ is a fixed point and $B_r(\mathbf{z})$ is a closed ball centered at $\mathbf{z}$, under some norm, such that

$$\left|\lambda(\mathbf{x})\right| > 1, \quad \text{for all eigenvalues } \lambda(\mathbf{x}) \text{ of } DF(\mathbf{x}),\ \text{for all } \mathbf{x} \in B_r(\mathbf{z}), \tag{1.1}$$

then $B_r(\mathbf{z})$ need not be a repelling neighborhood of $\mathbf{z}$. This is due to that the norm constructed for such a property depends on the matrix $DF(\mathbf{x})$ which varies at different points $\mathbf{x}$, as the Mean Value Theorem in multi-dimension is applied.

Recently, Marotto modified the definition of snapback repeller to validate the theorem [14], due to a technical flaw in the original derivation. The revised definition of snapback repeller is stated as follows. Let us denote $\mathbf{x}_k = F^k(\mathbf{x}_0)$ for $k \in \mathbb{N}$ and point $\mathbf{x}_0 \in \mathbb{R}^n$.

* Corresponding author. Fax: +866 3 5724679.
E-mail address: cwshih@math.nctu.edu.tw (C.-W. Shih).

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Definition 1.1. Let $\bar{\mathbf{x}}$ be a repelling fixed point of F. Suppose that there exist a point $\mathbf{x}_0 \neq \bar{\mathbf{x}}$ in a repelling neighborhood of $\bar{\mathbf{x}}$ and an integer $\ell > 1$, such that $\mathbf{x}_\ell = \bar{\mathbf{x}}$ and $\det(DF(\mathbf{x}_k)) \neq 0$ for $1 \leqslant k \leqslant \ell$. Then $\bar{\mathbf{x}}$ is called a *snapback repeller* of F.

The point $\mathbf{x}_0$ in the definition is called a *snapback point* of F. Under this definition, the following theorem by Marotto holds [13,14].

Marotto's theorem. *If F possesses a snapback repeller, then F is chaotic in the following sense: There exist* (i) *a positive integer N, such that F has a point of period p, for each integer $p \geqslant N$,* (ii) *a scrambled set of F, i.e., an uncountable set S containing no periodic points of F, such that*

(a) $F(S) \subset S$,
(b) $\limsup_{k\to\infty} \|F^k(\mathbf{x}) - F^k(\mathbf{y})\| > 0$, *for all* $\mathbf{x}, \mathbf{y} \in S$, *with* $\mathbf{x} \neq \mathbf{y}$,
(c) $\limsup_{k\to\infty} \|F^k(\mathbf{x}) - F^k(\mathbf{y})\| > 0$, *for all* $\mathbf{x} \in S$ *and periodic point* $\mathbf{y}$ *of* F,

(iii) *an uncountable subset S_0 of S, such that* $\liminf_{k\to\infty} \|F^k(\mathbf{x}) - F^k(\mathbf{y})\| = 0$, *for every* $\mathbf{x}, \mathbf{y} \in S_0$.

Marotto's theorem is significant in extending the analytic theory of chaos from one-dimension to multi-dimension. It is also effective in applications, for example, in finding the chaotic regimes (parameter ranges) for dynamical systems. The theorem is valid under the new definition, as that the convergence of preimages of a repeller back to the repeller is guaranteed. However, methodologies for examining the condition of the theorem are demanded for application. Indeed, confirming that some preimage of a repelling fixed point lies in the repelling neighborhood of this fixed point is a nontrivial task; in addition, the existence of snapback repellers and homoclinic orbits are difficult to observe numerically for multi-dimensional maps, due to the unstable structure of these orbits. Focusing on practical applications, we thus propose two directions to confirm that a repelling fixed point is a snapback repeller for multi-dimensional maps. The first one is to find the repelling neighborhood U of the repeller $\bar{\mathbf{x}}$ and a preimage point $\mathbf{x}_0$ of $\bar{\mathbf{x}}$ lying in U, i.e., with $F^\ell(\mathbf{x}_0) = \bar{\mathbf{x}}$, $\mathbf{x}_0 \in U$ and $\mathbf{x}_0 \neq \bar{\mathbf{x}}$, for some $\ell > 1$. Therefore, deriving an estimation of the repelling neighborhood for a repeller becomes the key part in utilizing this theorem. Moreover, a computable norm is needed for practical application. The second direction is to construct the preimages $\{\bar{\mathbf{x}}^{-k}\}_{k=1}^\infty$ of $\bar{\mathbf{x}}$, such that $F(\bar{\mathbf{x}}^{-k}) = \bar{\mathbf{x}}^{-k+1}$, $k \geqslant 2$, $F(\bar{\mathbf{x}}^{-1}) = \bar{\mathbf{x}}$, $\lim_{k\to\infty} F(\bar{\mathbf{x}}^{-k}) = \bar{\mathbf{x}}$. We call such an orbit $\{\bar{\mathbf{x}}^{-k}\}_{k=1}^\infty$ a (degenerate) homoclinic orbit for the repeller $\bar{\mathbf{x}}$. The existence of such a homoclinic orbit guarantees the existence of a snapback point in the repelling neighborhood of repeller $\bar{\mathbf{x}}$. Marotto's theorem thus holds without knowing the repelling region of the fixed point.

This presentation aims at deriving two methodologies to establish the existence of snapback repellers:

(i) estimations for the radius of repelling neighborhood of a repelling fixed point, under the Euclidean norm,
(ii) a sequential graphic-iteration scheme to construct the homoclinic orbit for a repelling fixed point.

For (i), we shall formulate a first-order estimate as well as a second-order estimate. The latter one is especially useful for quadratic maps. The estimation is of essential and practical significance as combined with numerical computations of snapback points. For (ii), the approach more or less bears a sense of employing (1.1), with certain manipulation, to conclude the existence of snapback repeller. This construction allows us to track homoclinic orbit. In some applications, a combination of (i) with (ii) accomplishes the use of Marotto's theorem. These two methodologies can then be combined with numerical computations and the technique of interval computing which controls rigorous computation precision, to conclude chaotic dynamics for the systems.

There have been a number of efforts in modifying the original definition of snapback repeller [5,10]. A different definition based on the existence for a sequence of preimages of the repeller was adopted in [11]. Our second methodology provides a way to construct such a backward orbit. Persistence of snapback repeller and positive entropy for perturbations of maps with snapback repeller have been discussed in [11,12]. Previous works based on the application of Marotto's theorem in the literature may require reconsideration under the valid definition; for example, [1–4,6,7,15,17]. The goal of this study is to provide some methodologies for valid applications of Marotto's theorem. In particular, we shall apply our approaches to confirm the existence of snapback repellers, under the new definition, for a chaotic neural network [2–4,6] and a predator–prey model [15].

The rest of this presentation is organized as follows. In Section 2, we derive two estimations for the radius of repelling neighborhood of a repelling fixed point, under the Euclidean norm. The sequential graphic-iteration scheme is formulated in Section 3. In Section 4, we apply the sequential graphic-iteration scheme to construct the homoclinic orbit in a chaotic neural network. In Section 5, we provide two numerical examples to illustrate the uses of the present methodologies.

2. Repelling neighborhood

In this section, we develop two approaches to estimate the radius of repelling neighborhood for a repelling fixed point. The first one is a first-order estimate and the second one is a second-order estimate. The Euclidean norm is adopted throughout this section.

Let us quote the following lemma from [10]. Note that if an $n \times n$ matrix A is real, then $A^T A$ is symmetric and positive semi-definite.

Lemma 2.1. *(See [10].) Let* $\mathbf{z}$ *be a fixed point of* F *which is continuously differentiable in closed ball* $B_r(\mathbf{z})$*. If*

$$\lambda > 1, \quad \textit{for all eigenvalues } \lambda \textit{ of } \big(DF(\mathbf{z})\big)^T DF(\mathbf{z}), \tag{2.1}$$

then there exist $s > 1$ *and* $r' \in (0, r]$ *such that* $\|F(\mathbf{x}) - F(\mathbf{y})\|_2 > s \cdot \|\mathbf{x} - \mathbf{y}\|_2$*, for all* $\mathbf{x}, \mathbf{y} \in B_{r'}(\mathbf{z})$*, and all eigenvalues of* $(DF(\mathbf{x}))^T DF(\mathbf{x})$ *exceed one for all* $\mathbf{x} \in B_{r'}(\mathbf{z})$*.*

Condition (2.1) was employed in [10] in an attempt to revise the original definition of snapback repeller to validate Marotto's theorem. On the one hand, using (2.1) as a definition is more restrictive, as commented by Marotto [14], due to that a repelling fixed point has the potential to be a snapback repeller, without satisfying the eigenvalue condition (2.1). On the other hand, the formulation in [10] does provide an estimate for the radius of repelling neighborhood of a repelling fixed point, although such an estimate was not elaborated in [10].

Let us present this derivation and estimate. It follows from the fundamental theorem of calculus that $F(\mathbf{y}) - F(\mathbf{x}) = \int_0^1 DF(\mathbf{x} + s(\mathbf{y} - \mathbf{x})) \cdot (\mathbf{y} - \mathbf{x})\, ds$, and

$$\big\|F(\mathbf{y}) - F(\mathbf{x})\big\|_2 \geqslant \big\|DF(\mathbf{z})(\mathbf{y} - \mathbf{x})\big\|_2 - \int_0^1 \big\|DF\big(\mathbf{x} + s(\mathbf{y} - \mathbf{x})\big) - DF(\mathbf{z})\big\|_2\, ds \cdot \|\mathbf{y} - \mathbf{x}\|_2.$$

Notably, $\|DF(\mathbf{z})(\mathbf{y} - \mathbf{x})\|_2 = \sqrt{(\mathbf{y} - \mathbf{x})^T (DF(\mathbf{z}))^T DF(\mathbf{z})(\mathbf{y} - \mathbf{x})} \geqslant s_1 \cdot \|\mathbf{y} - \mathbf{x}\|_2$, where

$$s_1 := \sqrt{\text{minimal eigenvalue of } \big(DF(\mathbf{z})\big)^T DF(\mathbf{z})}. \tag{2.2}$$

Consider the $n \times n$ matrix $B(\mathbf{w}, \mathbf{z}) := DF(\mathbf{w}) - DF(\mathbf{z})$, and set

$$\begin{aligned} \eta_r &:= \max_{\mathbf{w} \in B_r(\mathbf{z})} \big\|B(\mathbf{w}, \mathbf{z})\big\|_2 \\ &= \max_{\mathbf{w} \in B_r(\mathbf{z})} \sqrt{\text{maximal eigenvalue of } \big(B(\mathbf{w}, \mathbf{z})\big)^T B(\mathbf{w}, \mathbf{z})}. \end{aligned} \tag{2.3}$$

Hence, we can estimate $\|F(\mathbf{y}) - F(\mathbf{x})\|_2$ through s_1 and η_r.

Proposition 2.2. *Consider a continuously differentiable map* F *with fixed point* $\mathbf{z}$*. Let* s_1 *and* η_r *be as defined in* (2.2) *and* (2.3)*. If there exists an* $r > 0$ *such that* $s_1 - \eta_r > 1$*, then* $B_r(\mathbf{z})$ *is a repelling neighborhood for* $\mathbf{z}$*, under the Euclidean norm.*

Next, let us present the second estimate which is based on the first and second derivatives of F. This formulation is especially advantageous for quadratic maps as their second derivatives are constants.

Since the eigenvalues of $(DF(\mathbf{x}))^T DF(\mathbf{x})$ are all non-negative. Let $\sigma_i(\mathbf{x})$ and $\beta_{ij}(\mathbf{x})$ be defined as

$$\begin{aligned} \sigma_i(\mathbf{x}) &:= \sqrt{\text{eigenvalues of } \big(DF(\mathbf{x})\big)^T DF(\mathbf{x})}, \\ \beta_{ij}(\mathbf{x}) &:= \text{eigenvalues of Hessian matrix } H_{F_i}(\mathbf{x}) = \big[\partial_k \partial_l F_i(\mathbf{x})\big]_{k \times l}, \end{aligned}$$

where $i, j = 1, 2, \ldots, n$. Let α_r and β_r be defined as

$$\alpha_r := \min_{\mathbf{x} \in B_r(\mathbf{z})} \min_{1 \leqslant i \leqslant n} \big\{\sigma_i(\mathbf{x})\big\}, \tag{2.4}$$

$$\beta_r := \max_{1 \leqslant i \leqslant n} \max_{\mathbf{x} \in B_r(\mathbf{z})} \max_{1 \leqslant j \leqslant n} \big|\beta_{ij}(\mathbf{x})\big|. \tag{2.5}$$

Proposition 2.3. *Consider a* C^2 *map* $F = (F_1, \ldots, F_n)$ *with fixed point* $\mathbf{z}$*. Let* α_r *and* β_r *be as defined in* (2.4) *and* (2.5)*. If there exists an* $r > 0$*, such that*

$$\alpha_r - r\sqrt{n}\beta_r > 1, \tag{2.6}$$

then $B_r(\mathbf{z})$ *is a repelling neighborhood of* $\mathbf{z}$*, under the Euclidean norm.*

Proof. As F is C^2, we have $F(\mathbf{y})-F(\mathbf{x})=DF(\mathbf{x})(\mathbf{y}-\mathbf{x})+\int_0^1(1-\tau)D^2F(\mathbf{x}+\tau(\mathbf{y}-\mathbf{x}))(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})\,d\tau$, for any $\mathbf{x},\mathbf{y}\in B_r(\mathbf{z})$; hence,

$$\big\|F(\mathbf{y})-F(\mathbf{x})\big\|_2 \geqslant \big\|DF(\mathbf{x})(\mathbf{y}-\mathbf{x})\big\|_2-\left\|\int_0^1(1-\tau)D^2F\big(\mathbf{x}+\tau(\mathbf{y}-\mathbf{x})\big)(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})\,d\tau\right\|_2.$$

Step I. Estimate the first-order term $\|DF(\mathbf{x})(\mathbf{y}-\mathbf{x})\|_2$.

There are two ways to derive this estimate; namely, through "Singular Value Decomposition" and "Polar Decomposition" of matrices [8,9]. We present the first one. Indeed, $DF(\mathbf{x})=U(\mathbf{x})\Upsilon(\mathbf{x})(V(\mathbf{x}))^T$, where $U(\mathbf{x})$ and $V(\mathbf{x})$ are unitary matrices, and $\Upsilon(\mathbf{x})$ is the diagonal matrix with diagonal entries $\sigma_i(\mathbf{x})=\sqrt{\text{eigenvalues of }(DF(\mathbf{x}))^TDF(\mathbf{x})}$, $i=1,\dots,n$. For any $\mathbf{w}=(w_1,\dots,w_n)\in\mathbb{R}^n$, since $\sigma_i(\mathbf{x})\geqslant 0$, for all $i=1,\dots,n$, we derive

$$\min_{1\leqslant i\leqslant n}\big\{\sigma_i(\mathbf{x})\big\}\cdot\|\mathbf{w}\|_2\leqslant\big\|\Upsilon(\mathbf{x})\mathbf{w}\big\|_2\leqslant\max_{1\leqslant i\leqslant n}\big\{\sigma_i(\mathbf{x})\big\}\cdot\|\mathbf{w}\|_2. \tag{2.7}$$

From $DF(\mathbf{x})=U(\mathbf{x})\Upsilon(\mathbf{x})(V(\mathbf{x}))^T$, $\|U(\mathbf{x})\Upsilon(\mathbf{x})(V(\mathbf{x}))^T(\mathbf{y}-\mathbf{x})\|_2=\|\Upsilon(\mathbf{x})(V(\mathbf{x}))^T(\mathbf{y}-\mathbf{x})\|_2$ and (2.7), we obtain

$$\begin{aligned}\big\|DF(\mathbf{x})(\mathbf{y}-\mathbf{x})\big\|_2&\geqslant\min_{1\leqslant i\leqslant n}\big\{\sigma_i(\mathbf{x})\big\}\cdot\big\|\big(V(\mathbf{x})\big)^T(\mathbf{y}-\mathbf{x})\big\|_2\\&\geqslant\min_{\mathbf{x}\in B_r(\mathbf{z})}\min_{1\leqslant i\leqslant n}\big\{\sigma_i(\mathbf{x})\big\}\cdot\big\|\big(V(\mathbf{x})\big)^T(\mathbf{y}-\mathbf{x})\big\|_2\\&=\alpha_r\cdot\|\mathbf{y}-\mathbf{x}\|_2,\end{aligned}$$

for any $\mathbf{x},\mathbf{y}\in B_r(\mathbf{z})$.

Step II. Estimate the second-order term.

Notably, $[D^2F(\mathbf{w})\mathbf{hk}]^T=[\mathbf{k}^TH_{F_1}(\mathbf{w})\mathbf{h},\dots,\mathbf{k}^TH_{F_n}(\mathbf{w})\mathbf{h}]$, for $\mathbf{w},\mathbf{h},\mathbf{k}\in\mathbb{R}^n$, where H_{F_i} is the Hessian matrix for F_i. It follows that

$$\big\|D^2F(\mathbf{w})(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})\big\|_2=\left(\sum_{i=1}^n\big|(\mathbf{y}-\mathbf{x})^TH_{F_i}(\mathbf{w})(\mathbf{y}-\mathbf{x})\big|^2\right)^{\frac{1}{2}}. \tag{2.8}$$

On the other hand,

$$\begin{aligned}\big|(\mathbf{y}-\mathbf{x})^TH_{F_i}(\mathbf{w})(\mathbf{y}-\mathbf{x})\big|&\leqslant\Big(\max_{1\leqslant i\leqslant n}\max_{\mathbf{w}\in B_r(\mathbf{z})}\max_{1\leqslant j\leqslant n}\big|\beta_{ij}(\mathbf{w})\big|\Big)\cdot\|\mathbf{y}-\mathbf{x}\|_2^2\\&\leqslant 2r\beta_r\cdot\|\mathbf{y}-\mathbf{x}\|_2,\end{aligned}$$

for any $\mathbf{w}\in B_r(\mathbf{z})$, $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$, due to that Hessian matrix $H_{F_i}(\mathbf{w})$ is symmetric, and $(-\max_{1\leqslant j\leqslant n}|\beta_{ij}(\mathbf{w})|)\cdot\|\mathbf{y}-\mathbf{x}\|_2\leqslant(\mathbf{y}-\mathbf{x})^TH_{F_i}(\mathbf{w})(\mathbf{y}-\mathbf{x})\leqslant(\max_{1\leqslant j\leqslant n}|\beta_{ij}(\mathbf{w})|)\cdot\|\mathbf{y}-\mathbf{x}\|_2$, for each $\mathbf{w}\in B_r(\mathbf{z})$, $i=1,\dots,n$, and

$$\min_{1\leqslant i\leqslant n}\min_{\mathbf{w}\in B_r(\mathbf{z})}\Big\{-\max_{1\leqslant j\leqslant n}\big|\beta_{ij}(\mathbf{w})\big|\Big\}=-\max_{1\leqslant i\leqslant n}\max_{\mathbf{w}\in B_r(\mathbf{z})}\max_{1\leqslant j\leqslant n}\big|\beta_{ij}(\mathbf{w})\big|.$$

Hence, from (2.8), we drive $\|D^2F(\mathbf{w})(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})\|_2\leqslant 2r\beta_r\sqrt{n}\cdot\|\mathbf{y}-\mathbf{x}\|_2$, for any $\mathbf{w}=\mathbf{x}+\tau(\mathbf{y}-\mathbf{x}),\mathbf{x},\mathbf{y}\in B_r(\mathbf{z})$. Thereafter, $\|\int_0^1(1-\tau)D^2F(\mathbf{x}+\tau(\mathbf{y}-\mathbf{x}))(\mathbf{y}-\mathbf{x})(\mathbf{y}-\mathbf{x})\,d\tau\|_2\leqslant r\beta_r\sqrt{n}\cdot\|\mathbf{y}-\mathbf{x}\|_2$.

Finally, combining Steps I and II, we obtain $\|F(\mathbf{y})-F(\mathbf{x})\|_2\geqslant(\alpha_r-r\beta_r\sqrt{n})\cdot\|\mathbf{y}-\mathbf{x}\|_2$, for any $\mathbf{x},\mathbf{y}\in B_r(\mathbf{z})$. This completes the proof. □

3. Sequential graphic-iteration scheme

In this section, we present an approach to exploit the existence of snapback repeller, without estimating the repelling neighborhood. In particular, we develop a scheme to construct homoclinic orbits for repelling fixed point $\bar{\mathbf{x}}$ of F, i.e., we show that there exists $\{\bar{\mathbf{x}}^{-j}\colon j\in\mathbb{N}\}$ such that $F(\bar{\mathbf{x}}^{-1})=\bar{\mathbf{x}}$, $F(\bar{\mathbf{x}}^{-j})=\bar{\mathbf{x}}^{-j+1}$, for $j\geqslant 2$, $\lim_{j\to\infty}F(\bar{\mathbf{x}}^{-j})=\bar{\mathbf{x}}$. Notably, the existence of such an orbit guarantees the existence of snapback point in the repelling neighborhood of $\bar{\mathbf{x}}$, and thus leads to Marotto's theorem. The present scheme utilizes the local structure of F and employs lower and upper bounds of F on restricted regions sequentially. We shall illustrate the use of this scheme to a chaotic neural network in Section 4.

Consider a C^1 map $F:\mathbb{R}^n\to\mathbb{R}^n$ with $F=(F_1,\dots,F_n)$. We assume that there exists a compact, connected and convex region $\Omega\subset\mathbb{R}^n$ on which F is one-to-one and has a fixed point $\bar{\mathbf{x}}$. For simplicity, we consider $\Omega=\prod_{i=1}^n\Omega_i:=\prod_{i=1}^n[a_i,b_i]$, with $a_i < b_i$. Notably, a sufficient condition for F to be one-to-one on Ω is

$$\left|\frac{\partial F_i}{\partial x_i}(\mathbf{x})\right| > \sum_{j=1,\, j\neq i}^{n}\left|\frac{\partial F_i}{\partial x_j}(\mathbf{x})\right|, \quad \text{for all } i=1,\dots,n,\ \mathbf{x}\in\Omega. \tag{3.1}$$

Herein, we shall actually employ the condition

$$\left|\frac{\partial F_i}{\partial x_i}(\mathbf{x})\right| > 1+\sum_{j=1,\, j\neq i}^{n}\left|\frac{\partial F_i}{\partial x_j}(\mathbf{x})\right|, \quad \text{for all } i=1,\dots,n,\ \mathbf{x}\in\Omega. \tag{3.2}$$

Then, by Gerschgorin's theorem, all eigenvalues $\lambda(\mathbf{x})$ of $DF(\mathbf{x})$ satisfy $|\lambda(\mathbf{x})| > 1$, for all $\mathbf{x}\in\Omega$.

Hence, under (3.2), $\bar{\mathbf{x}}$ is a repelling fixed point of F. In addition, (3.2) implies (3.1). Under condition (3.2), we denote

$$\mathcal{A} := \left\{ i\in\{1,\dots,n\} \;\middle|\; \frac{\partial F_i}{\partial x_i}(\mathbf{x}) > 1+\sum_{j=1,\, j\neq i}^{n}\left|\frac{\partial F_i}{\partial x_j}(\mathbf{x})\right|,\ \text{for all } \mathbf{x}\in\Omega \right\},$$

$$\mathcal{B} := \left\{ i\in\{1,\dots,n\} \;\middle|\; \frac{\partial F_i}{\partial x_i}(\mathbf{x}) < -1-\sum_{j=1,\, j\neq i}^{n}\left|\frac{\partial F_i}{\partial x_j}(\mathbf{x})\right|,\ \text{for all } \mathbf{x}\in\Omega \right\}.$$

In the following derivation, we will apply Brouwer's fixed point theorem to construct successive preimages of $\bar{\mathbf{x}}$ lying in the designated regions. Let us sketch the scheme in the following steps (I)–(VII). Let $\ell \geqslant 2$.

(I) Locating the $(\ell-1)$-th preimage point $\bar{\mathbf{x}}^{-\ell+1}$ of $\bar{\mathbf{x}}$, which lies outside of Ω: First, for $i=1,\dots,n$, we set

$$\hat{f}_{i,(1)}(\xi) := \sup\{F_i(x'_1,\dots,x'_{i-1},\xi,x'_{i+1},\dots,x'_n):\ x'_j\in\Omega_j,\ j\in\{1,\dots,n\}/\{i\}\},$$

$$\check{f}_{i,(1)}(\xi) := \inf\{F_i(x'_1,\dots,x'_{i-1},\xi,x'_{i+1},\dots,x'_n):\ x'_j\in\Omega_j,\ j\in\{1,\dots,n\}/\{i\}\},$$

for $\xi\in\mathbb{R}^1$. For each $\mathbf{x}'=(x'_1,\dots,x'_n)\in\Omega$ and $i=1,\dots,n$, we define

$$\tilde{f}_{\mathbf{x}',i,(1)}(\xi) = F_i(x'_1,\dots,x'_{i-1},\xi,x'_{i+1},\dots,x'_n), \quad \text{for } \xi\in\mathbb{R}^1.$$

Then, for each $\mathbf{x}'\in\Omega$,

$$\check{f}_{i,(1)}(\xi) \leqslant \tilde{f}_{\mathbf{x}',i,(1)}(\xi) \leqslant \hat{f}_{i,(1)}(\xi), \quad \text{for all } \xi\in\mathbb{R}^1,\ i=1,\dots,n. \tag{3.3}$$

Under (3.2), for each $\mathbf{x}'\in\Omega$, $\check{f}'_{i,(1)}(\xi)=\tilde{f}'_{\mathbf{x}',i,(1)}(\xi)=\hat{f}'_{i,(1)}(\xi)>1$, if $i\in\mathcal{A}$, and $\check{f}'_{i,(1)}(\xi)=\tilde{f}'_{\mathbf{x}',i,(1)}(\xi)=\hat{f}'_{i,(1)}(\xi)<-1$, if $i\in\mathcal{B}$, for all $\xi\in[a_i,b_i]$. We denote by $\hat{f}^{-1}_{i,(1)}(y)$, $\check{f}^{-1}_{i,(1)}(y)$, and $\tilde{f}^{-1}_{\mathbf{x}',i,(1)}(y)$ the preimages of y under $\hat{f}_{i,(1)}$, $\check{f}_{i,(1)}$, and $\tilde{f}_{\mathbf{x}',i,(1)}$ lying in Ω_i, respectively. Hence, for each $\mathbf{x}'\in\Omega$

$$\hat{f}^{-1}_{i,(1)}(y) \leqslant \tilde{f}^{-1}_{\mathbf{x}',i,(1)}(y) \leqslant \check{f}^{-1}_{i,(1)}(y), \quad \text{for } y\in[\hat{f}_{i,(1)}(a_i),\check{f}_{i,(1)}(b_i)],\ \text{if } i\in\mathcal{A}, \tag{3.4}$$

$$\check{f}^{-1}_{i,(1)}(y) \leqslant \tilde{f}^{-1}_{\mathbf{x}',i,(1)}(y) \leqslant \hat{f}^{-1}_{i,(1)}(y), \quad \text{for } y\in[\hat{f}_{i,(1)}(b_i),\check{f}_{i,(1)}(a_i)],\ \text{if } i\in\mathcal{B}, \tag{3.5}$$

and $\tilde{f}^{-1}_{\mathbf{x}',i,(1)}$, $\hat{f}^{-1}_{i,(1)}$ and $\check{f}^{-1}_{i,(1)}$ are increasing on $[\hat{f}_{i,(1)}(a_i),\check{f}_{i,(1)}(b_i)]$, if $i\in\mathcal{A}$, and decreasing on $[\hat{f}_{i,(1)}(b_i),\check{f}_{i,(1)}(a_i)]$, if $i\in\mathcal{B}$.

As we plan to construct an orbit, $\{\bar{\mathbf{x}}^{-k}\}_{k=1}^{\infty}$, such that $\bar{\mathbf{x}}^{-\ell+1}\notin\Omega$, and $\bar{\mathbf{x}}^{-k}\in\operatorname{int}(\Omega)$, for all $k\geqslant\ell$, we further assume that

$$\bar{x}_i^{-\ell+1} \in (\hat{f}_{i,(1)}(a_i),\check{f}_{i,(1)}(b_i)) \setminus [a_i,b_i], \quad \text{if } i\in\mathcal{A}, \tag{3.6}$$

$$\bar{x}_i^{-\ell+1} \in (\hat{f}_{i,(1)}(b_i),\check{f}_{i,(1)}(a_i)) \setminus [a_i,b_i], \quad \text{if } i\in\mathcal{B}. \tag{3.7}$$

Under conditions (3.6)–(3.7), $\bar{\mathbf{x}}^{-\ell+1}\notin\Omega$, and a further preimage $\bar{\mathbf{x}}^{-\ell}$ lying in the interior of Ω can be found. As there may be other possibility for this step, we could also assume, instead of (3.6)–(3.7),

$$\bar{\mathbf{x}}^{-\ell+1}\in\mathbb{R}^n\setminus\Omega, \qquad \bar{\mathbf{x}}^{-\ell}\in\operatorname{int}(\Omega). \tag{3.8}$$

In some applications, this condition can be verified. Notably, when $\ell\geqslant 3$, if $\bar{\mathbf{x}}^{-\ell+2}\in\mathbb{R}^n\setminus\Omega$, then we allow $\bar{\mathbf{x}}^{-\ell+1}\in\partial\Omega$ in conditions (3.6)–(3.7), or (3.8). With (3.6)–(3.7), we can find intervals $[a_{i,(0)},b_{i,(0)}]\supseteq\Omega_i$, such that

$$\hat{f}_{i,(1)}(a_i) < a_{i,(0)} \leqslant \bar{x}_i^{-\ell+1} \leqslant b_{i,(0)} < \check{f}_{i,(1)}(b_i), \quad \text{for } i\in\mathcal{A}, \tag{3.9}$$

$$\hat{f}_{i,(1)}(b_i) < a_{i,(0)} \leqslant \bar{x}_i^{-\ell+1} \leqslant b_{i,(0)} < \check{f}_{i,(1)}(a_i), \quad \text{for } i\in\mathcal{B}, \tag{3.10}$$

and take $a_{i,(1)} := \hat{f}^{-1}_{i,(1)}(a_{i,(0)})$, $b_{i,(1)} := \check{f}^{-1}_{i,(1)}(b_{i,(0)})$, for $i\in\mathcal{A}$, and $b_{i,(1)} := \check{f}^{-1}_{i,(1)}(b_{i,(0)})$, $a_{i,(1)} := \hat{f}^{-1}_{i,(1)}(a_{i,(0)})$, if $i\in\mathcal{B}$. Thus, by (3.9), we have $a_i < \hat{f}^{-1}_{i,(1)}(a_{i,(0)}) = a_{i,(1)}$ and $b_{i,(1)} = \check{f}^{-1}_{i,(1)}(b_{i,(0)}) < b_i$, if $i\in\mathcal{A}$. Hence, $\Omega_{i,(1)} := [a_{i,(1)},b_{i,(1)}]\subset\Omega_i$, for $i\in\mathcal{A}$.

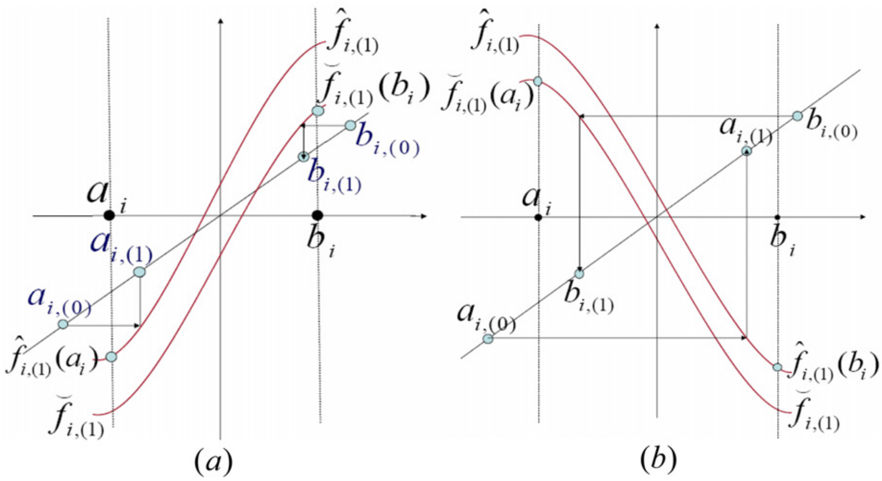

Fig. 1. (a) Configuration for case $\mathcal{A}$ in step (I). (b) Configuration for case $\mathcal{B}$ in step (I).

Similarly, we have $\Omega_{i,(1)} := [b_{i,(1)}, a_{i,(1)}] \subset \Omega_i$, for $i \in \mathcal{B}$, as depicted in Fig. 1. If condition (3.8), instead of (3.6)–(3.7), is assumed, then suitable $a_{i,(1)}$, $b_{i,(1)}$ can also be chosen. For convenience of expression and without loss of generality, we assume $\ell = 2$ in the sequel.

(II) Finding the preimage $\bar{\mathbf{x}}^{-2}$ of $\bar{\mathbf{x}}^{-1}$ under F, which lies in $\prod_{i=1}^n \Omega_{i,(1)}$: By (3.4)–(3.7), for each $\mathbf{x}' \in \prod_{i=1}^n \Omega_{i,(1)}$, there exists ξ_i with

$$\xi_i \in \left[\hat{f}^{-1}_{i,(1)}\left(\bar{x}_i^{-1}\right), \breve{f}^{-1}_{i,(1)}\left(\bar{x}_i^{-1}\right)\right] \subseteq \left[\hat{f}^{-1}_{i,(1)}(a_{i,(0)}), \breve{f}^{-1}_{i,(1)}(b_{i,(0)})\right] = \Omega_{i,(1)}, \quad \text{if } i \in \mathcal{A},$$
$$\xi_i \in \left[\breve{f}^{-1}_{i,(1)}\left(\bar{x}_i^{-1}\right), \hat{f}^{-1}_{i,(1)}\left(\bar{x}_i^{-1}\right)\right] \subseteq \left[\breve{f}^{-1}_{i,(1)}(b_{i,(0)}), \hat{f}^{-1}_{i,(1)}(a_{i,(0)})\right] = \Omega_{i,(1)}, \quad \text{if } i \in \mathcal{B},$$

such that $\bar{x}_i^{-1} = \tilde{f}_{\mathbf{x}',i,(1)}(\xi_i) = F_i(x_1', \dots, x_{i-1}', \xi_i, x_{i+1}', \dots, x_n')$, $i = 1, \dots, n$.

Next, we define a function $H_{(1)} = (H_{1,(1)}, \dots, H_{n,(1)}) : \prod_{i=1}^n \Omega_{i,(1)} \to \prod_{i=1}^n \Omega_{i,(1)}$, by

$$H_{i,(1)}\left(x_1', \dots, x_n'\right) = \xi_{i,(1)}, \tag{3.11}$$

where $\xi_{i,(1)}$ satisfies

$$\bar{x}_i^{-1} = \tilde{f}_{\mathbf{x}',i,(1)}(\xi_{i,(1)}) = F_i\left(x_1', \dots, x_{i-1}', \xi_{i,(1)}, x_{i+1}', \dots, x_n'\right), \quad i = 1, \dots, n. \tag{3.12}$$

To show that mapping $H_{(1)}$ is C^1, we consider the following map: $G_{(1)} = (G_{1,(1)}, \dots, G_{n,(1)}) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$G_{i,(1)}\left(\mathbf{x}', \mathbf{x}\right) = \bar{x}_i^{-1} - F_i\left(x_1', \dots, x_{i-1}', x_i, x_{i+1}', \dots, x_n'\right), \quad i = 1, \dots, n. \tag{3.13}$$

Then, $G_{(1)}(\mathbf{x}', H_{(1)}(\mathbf{x}')) = 0$, and $\det \frac{\partial G_{(1)}}{\partial \mathbf{x}}(\mathbf{x}', \mathbf{x}) \neq 0$, for any $\mathbf{x}, \mathbf{x}' \in \Omega$, thanks to (3.2). Hence $H_{(1)}$ is C^1 on Ω. Therefore, by Brouwer's fixed point theorem, $H_{(1)}$ has a fixed point $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n) \in \prod_{i=1}^n \Omega_{i,(1)}$, i.e., $\bar{x}_i^{-1} = F_i(\tilde{x}_1, \dots, \tilde{x}_n)$, with $\tilde{x}_i \in \Omega_{i,(1)}$, $i = 1, \dots, n$. Restated, this $\tilde{\mathbf{x}}$ is a preimage of $\bar{\mathbf{x}}^{-1}$ under F, which shall be denoted by $\bar{\mathbf{x}}^{-2}$. Again, $\bar{\mathbf{x}}^{-2} \in \prod_{i=1}^n \Omega_{i,(1)} \subset \prod_{i=1}^n \Omega_i = \Omega$.

(III) Constructing a sequence of nested regions $\{\prod_{i=1}^n \Omega_{i,(k)}\}_{k=1}^\infty$, $\Omega_{i,(k)} \subseteq \Omega_{i,(k-1)}$: For $k \geqslant 2$, $i = 1, \dots, n$, we set

$$\hat{f}_{i,(k)}(\xi) := \sup\left\{F_i\left(x_1', \dots, x_{i-1}', \xi, x_{i+1}', \dots, x_n'\right) : x_j' \in \Omega_{j,(k-1)}, \ j \in \{1, \dots, n\}/\{i\}\right\},$$
$$\breve{f}_{i,(k)}(\xi) := \inf\left\{F_i\left(x_1', \dots, x_{i-1}', \xi, x_{i+1}', \dots, x_n'\right) : x_j' \in \Omega_{j,(k-1)}, \ j \in \{1, \dots, n\}/\{i\}\right\},$$

for $\xi \in \mathbb{R}^1$. In addition, for each $\mathbf{x}' = (x_1', \dots, x_n') \in \prod_{i=1}^n \Omega_{i,(k-1)}$, we define

$$\tilde{f}_{\mathbf{x}',i,(k)}(\xi) = F_i\left(x_1', \dots, x_{i-1}', \xi, x_{i+1}', \dots, x_n'\right), \quad \xi \in \mathbb{R}^1, \ i = 1, \dots, n.$$

Then, for any $\mathbf{x}' \in \prod_{i=1}^n \Omega_{i,(k-1)}$, and $\xi \in \mathbb{R}^1$

$$\breve{f}_{i,(1)}(\xi) \leqslant \breve{f}_{i,(k)}(\xi) \leqslant \tilde{f}_{\mathbf{x}',i,(k)}(\xi) \leqslant \hat{f}_{i,(k)}(\xi) \leqslant \hat{f}_{i,(1)}(\xi), \quad i = 1, \dots, n.$$

Similar to step (I), for each $\mathbf{x}' \in \prod_{i=1}^n \Omega_{i,(k-1)}$, $\breve{f}'_{i,(k)} = \tilde{f}'_{\mathbf{x}',i,(k)} = \hat{f}'_{i,(k)} > 1$, if $i \in \mathcal{A}$, and $\breve{f}'_{i,(k)} = \tilde{f}'_{\mathbf{x}',i,(k)} = \hat{f}'_{i,(k)} < -1$, if $i \in \mathcal{B}$, on $[a_i, b_i]$.

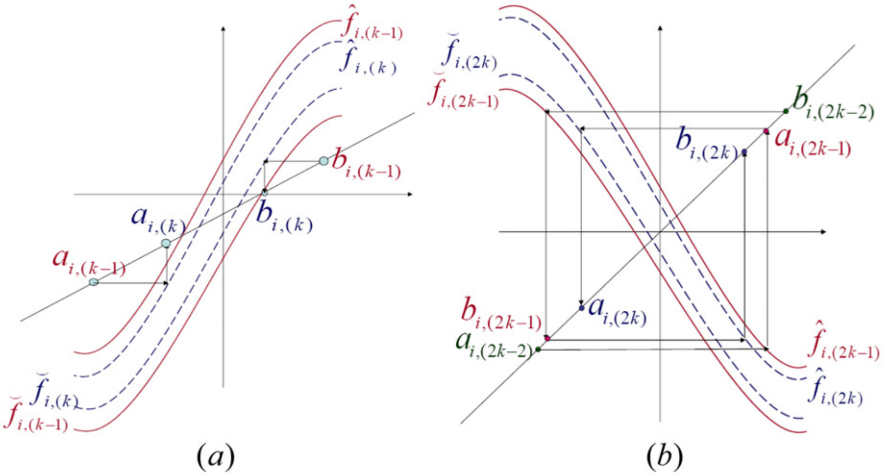

Fig. 2. (a) Configuration for case $\mathcal{A}$ in step (III). (b) Configuration for case $\mathcal{B}$ in step (III).

We further assume that $\hat{f}_{i,(1)}$ (resp. $\check{f}_{i,(1)}$) has fixed point $\hat{x}_{i,(1)}$ (resp. $\check{x}_{i,(1)}$) lying in Ω_i. Accordingly, $\hat{f}_{i,(k)}$ (resp. $\check{f}_{i,(k)}$) also has fixed point $\hat{x}_{i,(k)}$ (resp. $\check{x}_{i,(k)}$) lying in Ω_i, and

$$a_{i,(0)} \leqslant a_i \leqslant \hat{x}_{i,(1)} \leqslant \hat{x}_{i,(k)} \leqslant \check{x}_{i,(k)} \leqslant \check{x}_{i,(1)} \leqslant b_i \leqslant b_{i,(0)}, \quad \text{if } i \in \mathcal{A},$$
$$a_{i,(0)} \leqslant a_i \leqslant \check{x}_{i,(1)} \leqslant \check{x}_{i,(k)} \leqslant \hat{x}_{i,(k)} \leqslant \hat{x}_{i,(1)} \leqslant b_i \leqslant b_{i,(0)}, \quad \text{if } i \in \mathcal{B},$$

for all $k \geqslant 2$. We denote by $\hat{f}^{-1}_{i,(k)}(y)$, $\check{f}^{-1}_{i,(k)}(y)$, and $\tilde{f}^{-1}_{\mathbf{x}',i,(k)}(y)$ the preimages of y under $\hat{f}_{i,(k)}$, $\check{f}_{i,(k)}$, and $\tilde{f}_{\mathbf{x}',i,(k)}$ lying in Ω_i, for all $k \geqslant 2$, respectively. Hence, for any $\mathbf{x}' \in \prod_{i=1}^n \Omega_{i,(k-1)}$,

$$\begin{aligned} &\hat{f}^{-1}_{i,(k)}(y) \leqslant \tilde{f}^{-1}_{\mathbf{x}',i,(k)}(y) \leqslant \check{f}^{-1}_{i,(k)}(y), \quad \text{for } y \in \big[\hat{f}_{i,(1)}(a_i), \check{f}_{i,(1)}(b_i)\big], \text{ if } i \in \mathcal{A}, \\ &\check{f}^{-1}_{i,(k)}(y) \leqslant \tilde{f}^{-1}_{\mathbf{x}',i,(k)}(y) \leqslant \hat{f}^{-1}_{i,(k)}(y), \quad \text{for } y \in \big[\hat{f}_{i,(1)}(b_i), \check{f}_{i,(1)}(a_i)\big], \text{ if } i \in \mathcal{B}, \end{aligned} \tag{3.14}$$

and $\tilde{f}^{-1}_{\mathbf{x}',i,(k)}$, $\hat{f}^{-1}_{i,(k)}$ and $\check{f}^{-1}_{i,(k)}$ are all increasing on $[\hat{f}_{i,(1)}(a_i), \check{f}_{i,(1)}(b_i)]$, for $i \in \mathcal{A}$, and all decreasing on $[\hat{f}_{i,(1)}(b_i), \check{f}_{i,(1)}(a_i)]$, for $i \in \mathcal{B}$. Thus, we set

$$\Omega_{(k)} := \prod_{i=1}^{n} \Omega_{i,(k)}, \quad \text{for } k \geqslant 1,$$

where $\Omega_{i,(k)} := [a_{i,(k)}, b_{i,(k)}]$, for $i \in \mathcal{A}$, $\Omega_{i,(k)} := [b_{i,(k)}, a_{i,(k)}]$, for $k =$ odd, $i \in \mathcal{B}$, $\Omega_{i,(k)} := [a_{i,(k)}, b_{i,(k)}]$, for $k =$ even, $i \in \mathcal{B}$, where

$$\begin{aligned} &a_{i,(k)} := \hat{f}^{-1}_{i,(k)}(a_{i,(k-1)}), \qquad b_{i,(k)} := \check{f}^{-1}_{i,(k)}(b_{i,(k-1)}), \quad \text{for } i \in \mathcal{A}, \\ &a_{i,(k)} := \hat{f}^{-1}_{i,(k)}(a_{i,(k-1)}), \qquad b_{i,(k)} := \check{f}^{-1}_{i,(k)}(b_{i,(k-1)}), \quad \text{for } k = \text{odd}, \ i \in \mathcal{B}, \\ &a_{i,(k)} := \check{f}^{-1}_{i,(k)}(a_{i,(k-1)}), \qquad b_{i,(k)} := \hat{f}^{-1}_{i,(k)}(b_{i,(k-1)}), \quad \text{for } k = \text{even}, \ i \in \mathcal{B}, \end{aligned}$$

as depicted in Fig. 2. Obviously, by induction, we have $a_{i,(k)} \leqslant b_{i,(k)}$, for all $k \geqslant 0$ if $i \in \mathcal{A}$, and for even k if $i \in \mathcal{B}$; in addition, $b_{i,(k)} \leqslant a_{i,(k)}$, for odd k if $i \in \mathcal{B}$. Moreover, with these settings, it can be shown that

$$\Omega_{i,(k)} \subseteq \Omega_{i,(k-1)} \subset \Omega_i, \quad \text{for } k \geqslant 2, \ i = 1, \ldots, n. \tag{3.15}$$

Subsequently, for all $\mathbf{x}' \in \Omega_{(k-1)}$, $\xi \in \mathbb{R}^1$, $i = 1, \ldots, n$, $k \geqslant 3$,

$$\check{f}_{i,(1)}(\xi) \leqslant \check{f}_{i,(k-1)}(\xi) \leqslant \check{f}_{i,(k)}(\xi) \leqslant \tilde{f}_{\mathbf{x}',i,(k)}(\xi) \leqslant \hat{f}_{i,(k)}(\xi) \leqslant \hat{f}_{i,(k-1)}(\xi) \leqslant \hat{f}_{i,(1)}(\xi).$$

Now we prove (3.15), for $i \in \mathcal{A}$. First, we have shown that the $\Omega_{i,(1)} \subset \Omega_i$ in step (I). Next, for $k = 2$, observe that $a_{i,(0)} \leqslant \hat{x}_{i,(1)}$, $b_{i,(0)} \geqslant \check{x}_{i,(1)}$, $a_{i,(1)} \leqslant \hat{x}_{i,(1)} \leqslant \hat{x}_{i,(2)}$, and $b_{i,(1)} \geqslant \check{x}_{i,(1)} \geqslant \check{x}_{i,(2)}$. Accordingly, $a_{i,(2)} = \hat{f}^{-1}_{i,(2)}(a_{i,(1)}) \geqslant a_{i,(1)}$, $b_{i,(2)} = \check{f}^{-1}_{i,(2)}(b_{i,(1)}) \leqslant b_{i,(1)}$. Therefore, $\Omega_{i,(2)} \subseteq \Omega_{i,(1)}$, for all $i \in \mathcal{A}$. Next, assume that the assertion holds for j, then (3.15) with $k = j+1$ follows from $a_{i,(j)} \leqslant \hat{x}_{i,(j)} \leqslant \hat{x}_{i,(j+1)}$, and $b_{i,(j)} \geqslant \check{x}_{i,(j)} \geqslant \check{x}_{i,(j+1)}$. Therefore $a_{i,(j+1)} = \hat{f}^{-1}_{i,(j+1)}(a_{i,(j)}) \geqslant a_{i,(j)}$, $b_{i,(j+1)} = \check{f}^{-1}_{i,(j+1)}(b_{i,(j)}) \leqslant b_{i,(j)}$. Hence, $\Omega_{i,(j+1)} \subseteq \Omega_{i,(j)}$ for all $i \in \mathcal{A}$.

(IV) Constructing successive preimages $\bar{\mathbf{x}}^{-k}$ of $\bar{\mathbf{x}}$ under F, with $\bar{\mathbf{x}}^{-k} \in \prod_{i=1}^{n} \Omega_{i,(k-1)}$, for $k \geqslant 3$: Herein, we only illustrate the case of $i \in \mathcal{A}$. Since $\bar{x}_i^{-k} \in \Omega_{i,(k-1)} \subseteq \Omega_i$, by (3.14), there exists ξ_i with

$$\xi_i \in \left[\hat{f}_{i,(k)}^{-1}\left(\bar{x}_i^{-k}\right), \check{f}_{i,(k)}^{-1}\left(\bar{x}_i^{-k}\right)\right] \subseteq \left[\hat{f}_{i,(k)}^{-1}(a_{i,(k-1)}), \check{f}_{i,(k)}^{-1}(b_{i,(k-1)})\right] = \Omega_{i,(k)}, \quad \text{for } i \in \mathcal{A},$$

such that $\bar{x}_i^{-k} = \tilde{f}_{\mathbf{x}',i,(k)}(\xi_i) = F_i(x_1', \ldots, x_{i-1}', \xi_i, x_{i+1}', \ldots, x_n')$, $i = 1, \ldots, n$. Suppose $\bar{\mathbf{x}}^{-k}$, $k \geqslant 2$ have been defined, we formulate functions $H_{(k)}$ and $G_{(k)}$ through replacing $\Omega_{i,(1)}$, $\bar{x}_i^{-1}$, and $\xi_{i,(1)}$ by $\Omega_{i,(k)}$, $\bar{x}_i^{-k}$, and $\xi_{i,(k)}$, respectively, in (3.11), (3.12) and (3.13). Then, similar to step (II), by Brouwer's fixed point theorem, there exists a preimage $\bar{\mathbf{x}}^{-k-1} \in \prod_{i=1}^{n} \Omega_{i,(k)} \subseteq \prod_{i=1}^{n} \Omega_{i,(k-1)}$ of $\bar{\mathbf{x}}^{-k}$.

We have thus constructed in (II)–(IV) a sequence of regions $\{\prod_{i=1}^{n} \Omega_{i,(k)}\}_{k=1}^{\infty}$, which satisfy $\Omega_{i,(k)} \subseteq \Omega_{i,(k-1)} \subset \Omega_i$, for all $k \geqslant 2$, $i = 1, \ldots, n$, and an orbit $\{\bar{\mathbf{x}}^{-k}\}_{k=1}^{\infty}$, such that $\bar{\mathbf{x}}^{-k} \in \prod_{i=1}^{n} \Omega_{i,(k-1)}$, for all $k \geqslant 2$.

(V) Confirming that $\bar{\mathbf{x}}$ lies in $\Omega_{(k)}$, for every k: For any $\mathbf{x}' = (x_1', \ldots, x_n') \in \Omega$, since $\bar{x}_i \in \Omega_i \subseteq [a_{i,(0)}, b_{i,(0)}]$, by (3.3)–(3.5), (3.9), (3.10), there exists some $\xi_i \in \Omega_{i,(1)}$, such that $\bar{x}_i = \tilde{f}_{\mathbf{x}',i,(1)}(\xi_i)$, for all $i = 1, \ldots, n$. Following the arguments as in (II), we can find a first preimage $\bar{\mathbf{x}}^{-1}$ of $\bar{\mathbf{x}}$, which lies in $\prod_{i=1}^{n} \Omega_{i,(1)} \subseteq \Omega$. Let us denote it by $\bar{\mathbf{x}}^{-1,*}$. We repeat this process as in steps (II), (IV), and construct successive preimages of $\bar{\mathbf{x}}$: $\{\bar{\mathbf{x}}^{-k,*}\}_{k=1}^{\infty}$, with $\bar{\mathbf{x}}^{-k,*} \in \prod_{i=1}^{n} \Omega_{i,(k)} \subseteq \prod_{i=1}^{n} \Omega_i = \Omega$, for all $k \geqslant 1$. Then each $\bar{\mathbf{x}}^{-k,*}$ is equal to $\bar{\mathbf{x}}$, since F is one-to-one on Ω. Hence, we conclude

$$\bar{\mathbf{x}} = \bar{\mathbf{x}}^{-k,*} \in \prod_{i=1}^{n} \Omega_{i,(k)}, \quad \text{for all } k \geqslant 1. \tag{3.16}$$

(VI) Convergence of $\bar{\mathbf{x}}^{-k}$ to $\bar{\mathbf{x}}$, as $k \to \infty$: According to the construction in steps (I)–(V), this convergence follows from the condition

$$\|\Omega_{i,(k)}\| \to 0, \quad \text{as } k \to \infty, \text{ for all } i = 1, \ldots, n. \tag{3.17}$$

Therefore, as $\bar{\mathbf{x}}^{-k} \in \prod_{i=1}^{n} \Omega_{i,(k-1)}$, for all $k \geqslant 2$, with (3.16) and (3.17), the orbit $\{\bar{\mathbf{x}}^{-k}\}_{k=1}^{\infty}$ is a homoclinic orbit for the repelling fixed point $\bar{\mathbf{x}}$.

(VII) $\bar{\mathbf{x}}$ is a snapback repeller: As $\bar{\mathbf{x}}$ is a repelling fixed point, there exist a norm $\|\cdot\|_*$ and $r > 0$, such that $B_r^*(\bar{\mathbf{x}}) := \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \bar{\mathbf{x}}\|_* \leqslant r\} \subseteq \Omega$ is a repelling neighborhood of $\bar{\mathbf{x}}$. Since $\bar{\mathbf{x}}^{-k} \to \bar{\mathbf{x}}$, as $k \to \infty$, there must exist $\bar{\mathbf{x}}^{-k_0} \in \{\bar{\mathbf{x}}^{-k}\}_{k=1}^{\infty}$, such that $\bar{\mathbf{x}}^{-k_0} \in B_r^*(\bar{\mathbf{x}})$, i.e., this $\bar{\mathbf{x}}^{-k_0}$ is a snapback point of $\bar{\mathbf{x}}$. Moreover, $\det(DF(\bar{\mathbf{x}})) \neq 0$, and $\det(DF(\bar{\mathbf{x}}^{-k})) \neq 0$, for $k \geqslant \ell$, according to (3.2). If, furthermore,

$$\det\left(DF\left(\bar{\mathbf{x}}^{-k}\right)\right) \neq 0, \quad \text{for } 1 \leqslant k \leqslant \ell - 1, \tag{3.18}$$

then $\bar{\mathbf{x}}$ is a snapback repeller and Marotto's theorem holds. We summarize the above derivation.

Theorem 3.1. *Assume that C^1 map $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies* (3.2) *and has a repelling fixed point $\bar{\mathbf{x}}$ in a compact, connected, convex region $\Omega \subset \mathbb{R}^n$, $\hat{f}_{i,(1)}$ and $\check{f}_{i,(1)}$ both have fixed points in Ω_i, for all $i = 1, \ldots, n$, and* (3.6), (3.7), *or* (3.8) *hold. Then there exist a sequence of nested regions $\{\Omega_{(k)}\}_{k=1}^{\infty}$ with $\Omega_{(k+1)} \subseteq \Omega_{(k)} \subset \Omega$, and preimages $\bar{\mathbf{x}}^{-k-1} \in \Omega_{(k)}$ of $\bar{\mathbf{x}}$, $k \in \mathbb{N}$. If, furthermore, $\|\Omega_{i,(k)}\| \to 0$, as $k \to \infty$, for all $i = 1, \ldots, n$, then $\{\bar{\mathbf{x}}^{-k}\}_{k=1}^{\infty}$ is a homoclinic orbit for $\bar{\mathbf{x}}$. Moreover, if* (3.18) *holds, then $\bar{\mathbf{x}}$ is a snapback repeller and F is chaotic in the sense of Marotto's theorem.*

Remark. The conditions in Theorem 3.1 are formulated for DF and the associated one-dimensional maps (the upper and lower maps), hence are easy to examine in applications.

4. Application to TCNN

In studying the chaotic behaviors of a transiently chaotic neural network (TCNN), the associated n-dimensional map $F = (F_1, \ldots, F_n)$, with

$$F_i(\mathbf{x}) = \alpha x_i + w_{ii} g_{c_i}(x_i) + \sum_{j=1,\, j \neq i}^{n} w_{ij} g_0(x_j) + d_i, \quad i = 1, \ldots, n, \tag{4.1}$$

has been investigated. Herein, for $c \in \mathbb{R}$, $\epsilon > 0$, $g_c(\xi) := (1 + e^{-\frac{\xi}{\epsilon}})^{-1} - c$, for $\xi \in \mathbb{R}$. In particular, the existence of snapback repellers for (4.1) has been analyzed in [2–4,6], under Marotto's original definition. These arguments are therefore insufficient as discussed in Section 1. In this section, we shall apply the sequential graphic-iteration scheme developed in Section 3 to complete the justification for the existence of snapback repellers for (4.1).

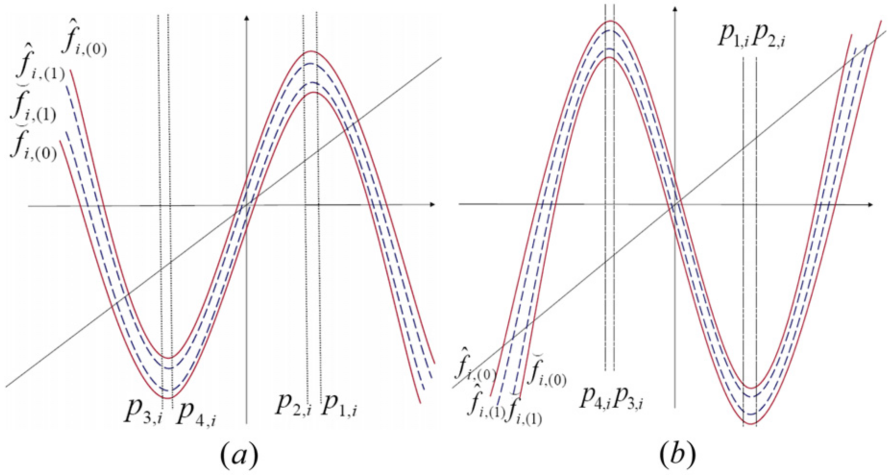

Fig. 3. (a) Configuration satisfying (PC-1-a). (b) Configuration satisfying (PC-2-a).

Let us recall the formulation in [6]. Consider first the single-neuron map: $f(\xi) = \alpha\xi + wg_c(\xi) + \gamma$, for $\xi \in \mathbb{R}$, where $\alpha, w, c, \gamma \in \mathbb{R}$. The following two sets of conditions for parameters $(\epsilon, \alpha, w, c, \gamma)$, labelled by (PC-1-) and (PC-2-), were adopted for the designated configurations of f; firstly,

$$w > 0, \qquad \left(1 + \alpha + \frac{\gamma}{4\epsilon}\right) < 0, \qquad 4\epsilon\left(-1 + \alpha - \frac{\gamma}{4\epsilon}\right) + w > 0, \tag{PC-1-a}$$

$$w < 0, \qquad \left(-1 + \alpha - \frac{\gamma}{4\epsilon}\right) > 0, \qquad 4\epsilon\left(1 + \alpha + \frac{\gamma}{4\epsilon}\right) + w < 0. \tag{PC-2-a}$$

It was shown in Lemma 3.1 of [6] that under (PC-1-a), there exist p_1, p_2, p_3, p_4 with $p_1 > p_2 > p_4 > p_3$ such that $f'(\xi) > 1 + (\frac{\gamma}{4\epsilon})$ for $p_4 < \xi < p_2$, $f'(\xi) < -1 - (\frac{\gamma}{4\epsilon})$ for $\xi > p_1$ and $\xi < p_3$, and under (PC-2-a), there exist p_1, p_2, p_3, p_4 with $p_2 > p_1 > p_3 > p_4$ such that $f'(\xi) < -1 - (\frac{\gamma}{4\epsilon})$ for $p_3 < \xi < p_1$, $f'(\xi) > 1 + (\frac{\gamma}{4\epsilon})$ for $\xi > p_2$ and $\xi < p_4$, cf. Fig. 3. Accordingly, $\mathbb{R}$ can be partitioned by these points, namely,

$$\tilde{\Omega}^{\mathrm{l}} := \{\xi \in \mathbb{R} \mid \xi \leqslant p_3\}, \qquad \tilde{\Omega}^{\mathrm{m}} := \{\xi \in \mathbb{R} \mid p_4 \leqslant \xi \leqslant p_2\}, \qquad \tilde{\Omega}^{\mathrm{r}} := \{\xi \in \mathbb{R} \mid \xi \geqslant p_1\}, \tag{4.2}$$

$$\tilde{\Omega}^{\mathrm{l}} := \{\xi \in \mathbb{R} \mid \xi \leqslant p_4\}, \qquad \tilde{\Omega}^{\mathrm{m}} := \{\xi \in \mathbb{R} \mid p_3 \leqslant \xi \leqslant p_1\}, \qquad \tilde{\Omega}^{\mathrm{r}} := \{\xi \in \mathbb{R} \mid \xi \geqslant p_2\}, \tag{4.3}$$

corresponding to conditions (PC-1-a) and (PC-2-a), respectively. Herein, "l", "m", and "r" mean "left", "middle", and "right" respectively. For fixed α, w, $0 < c < 1$, and $\gamma > 0$, we define $\hat{f}(\xi) = \alpha\xi + wg_c(\xi) + \gamma$, $\check{f}(\xi) = \alpha\xi + wg_c(\xi) - \gamma$, $f_h(\xi) = \alpha\xi + wg_c(\xi) + h$, for $-\gamma \leqslant h \leqslant \gamma$. Further parameter conditions for the existence of fixed points for f_h were formulated as follows:

$$g_c(p_1) > \frac{1-\alpha}{w}p_1 + \frac{\gamma}{w}, \qquad g_c(p_3) < \frac{1-\alpha}{w}p_3 - \frac{\gamma}{w}, \tag{PC-1-b}$$

$$g_c(p_2) > \frac{1-\alpha}{w}p_2 - \frac{\gamma}{w}, \qquad g_c(p_4) < \frac{1-\alpha}{w}p_4 + \frac{\gamma}{w}. \tag{PC-2-b}$$

Let $\hat{f}^{-1,\mathrm{l}}(\eta)$, $\hat{f}^{-1,\mathrm{m}}(\eta)$, $\hat{f}^{-1,\mathrm{r}}(\eta)$ represent the preimages of η lying in $\tilde{\Omega}^{\mathrm{l}}$, $\tilde{\Omega}^{\mathrm{m}}$, $\tilde{\Omega}^{\mathrm{r}}$ respectively, under $\hat{f}$; $\hat{x}^{\mathrm{l}}$, $\hat{x}^{\mathrm{m}}$, $\hat{x}^{\mathrm{r}}$ represent the fixed points of $\hat{f}$ lying in $\tilde{\Omega}^{\mathrm{l}}$, $\tilde{\Omega}^{\mathrm{m}}$, $\tilde{\Omega}^{\mathrm{r}}$ respectively. Similar notations are designed for $\check{f}$.

The following conditions allow us to find the preimages of fixed points of f_h in designated regions:

$$\check{f}^{-1,\mathrm{l}}(\check{x}^{\mathrm{m}}) > \max\{\hat{f}(p_3), \hat{f}(p_4)\}, \tag{PC-1-c(i)}$$

$$\hat{f}^{-1,\mathrm{r}}(\hat{x}^{\mathrm{m}}) < \min\{\check{f}(p_1), \check{f}(p_2)\}, \tag{PC-1-c(ii)}$$

$$\hat{f}^{-1,\mathrm{m}}(\hat{x}^{\mathrm{l}}) < \min\{\check{f}(p_3), \check{f}(p_4)\}, \tag{PC-2-c(i)}$$

$$\check{f}^{-1,\mathrm{m}}(\check{x}^{\mathrm{r}}) > \max\{\hat{f}(p_1), \hat{f}(p_2)\}, \tag{PC-2-c(ii)}$$

$$\hat{f}^{-1,\mathrm{m}}(\hat{f}^{-1,\mathrm{l}}(\check{x}^{\mathrm{m}})) < p_1, \qquad \check{f}^{-1,\mathrm{m}}(\hat{f}^{-1,\mathrm{m}}(\hat{f}^{-1,\mathrm{l}}(\check{x}^{\mathrm{m}}))) > p_3, \tag{PC-2-c(iii)}$$

$$\check{f}^{-1,\mathrm{m}}(\check{f}^{-1,\mathrm{r}}(\hat{x}^{\mathrm{m}})) > p_3, \qquad \hat{f}^{-1,\mathrm{m}}(\check{f}^{-1,\mathrm{m}}(\check{f}^{-1,\mathrm{r}}(\hat{x}^{\mathrm{m}}))) < p_1. \tag{PC-2-c(iv)}$$

Now, we come back to the n-dimensional map F. The following upper and lower maps for each component F_i in (4.1) were employed in [6]: $\hat{f}_i(\xi) = \alpha\xi + w_{ii}g_{c_i}(\xi) + \gamma_i$ and $\check{f}_i(\xi) = \alpha\xi + w_{ii}g_{c_i}(\xi) - \gamma_i$, for $\xi \in \mathbb{R}^1$, where γ_i is a number greater than $\sum_{j=1,\, j\neq i}^{n} |w_{ij}| + |d_i|$, for each i. Indeed, for every $i = 1, \ldots, n$, $\check{f}_i(x'_i) \leqslant F_i(\mathbf{x}') \leqslant \hat{f}_i(x'_i)$, for all $\mathbf{x}' = (x'_1, \ldots, x'_n) \in \mathbb{R}^n$.

In this presentation, we adopt the following refined upper and lower maps for each component F_i:

$$\hat{f}_{i,(0)}(\xi) := \sup\{F_i(x'_1, \ldots, x'_{i-1}, \xi, x'_{i+1}, \ldots, x'_n):\ x'_j \in \mathbb{R}^1,\ j \in \{1, \ldots, n\}/\{i\}\}$$
$$= \alpha\xi + w_{ii}g_{c_i}(\xi) + \hat{B}_{i,(0)} + d_i, \quad \xi \in \mathbb{R}^1, \tag{4.4}$$
$$\check{f}_{i,(0)}(\xi) := \inf\{F_i(x'_1, \ldots, x'_{i-1}, \xi, x'_{i+1}, \ldots, x'_n):\ x'_j \in \mathbb{R}^1,\ j \in \{1, \ldots, n\}/\{i\}\}$$
$$= \alpha\xi + w_{ii}g_{c_i}(\xi) + \check{B}_{i,(0)} + d_i, \quad \xi \in \mathbb{R}^1, \tag{4.5}$$

where $\hat{B}_{i,(0)} := \sum_{j=1,\, j\neq i}^{n} \sup_{\xi\in\mathbb{R}}\{w_{ij}g_0(\xi)\}$, and $\check{B}_{i,(0)} := \sum_{j=1,\, j\neq i}^{n} \inf_{\xi\in\mathbb{R}}\{w_{ij}g_0(\xi)\}$. Notably, conditions (PC-1-c) and (PC-2-c) will be considered for the upper and lower maps (4.4), (4.5) for each i-component. Indeed, for every $i = 1, \ldots, n$, we have $\check{f}_i(x'_i) \leqslant \check{f}_{i,(0)}(x'_i) \leqslant F_i(\mathbf{x}') \leqslant \hat{f}_{i,(0)}(x'_i) \leqslant \hat{f}_i(x'_i)$, for all $\mathbf{x}' = (x'_1, \ldots, x'_n) \in \mathbb{R}^n$.

We assume that for each i, parameters $(\epsilon, \alpha, w_{ii}, c_i, \gamma_i)$ satisfy either condition (PC-1-a) or (PC-2-a). Then $\mathbb{R}^n$ can be partitioned by 4^n points $p_{1,i},\ p_{2,i},\ p_{3,i},\ p_{4,i}$, according to the above setting. For $* \in \{\mathrm{l}, \mathrm{m}, \mathrm{r}\}$, let $\hat{f}_{i,(0)}^{-1,*}(y)$ (resp. $\check{f}_{i,(0)}^{-1,*}(y)$) denote the preimage of y lying in region $\tilde{\Omega}_i^*$ under $\hat{f}_{i,(0)}$ (resp. $\check{f}_{i,(0)}$), and $\hat{x}_{i,(0)}^*$ (resp. $\check{x}_{i,(0)}^*$) is the fixed point of $\hat{f}_{i,(0)}$ (resp. $\check{f}_{i,(0)}$) in region $\tilde{\Omega}_i^*$, where the definition of $\tilde{\Omega}_i^*$ is similar to $\tilde{\Omega}^*$ in (4.2), (4.3), for each component i. We then define the regions:

$$\Omega^{j_1\cdots j_n} := \Omega_1^{j_1} \times \cdots \times \Omega_n^{j_n}, \quad j_i \in \{\mathrm{l}, \mathrm{m}, \mathrm{r}\},\ i = 1, \ldots, n,$$
$$\Omega_i^{\mathrm{l}} := [\check{f}_{i,(0)}^{-1,\mathrm{l}}(\check{x}_{i,(0)}^{\mathrm{m}}), p_{3,i}], \qquad \Omega_i^{\mathrm{m}} := [p_{4,i}, p_{2,i}], \qquad \Omega_i^{\mathrm{r}} := [p_{1,i}, \hat{f}_{i,(0)}^{-1,\mathrm{r}}(\hat{x}_{i,(0)}^{\mathrm{m}})],$$
$$\Omega_i^{\mathrm{l}} := [\hat{x}_{i,(0)}^{\mathrm{l}}, p_{4,i}], \qquad \Omega_i^{\mathrm{m}} := [p_{3,i}, p_{1,i}], \qquad \Omega_i^{\mathrm{r}} := [p_{2,i}, \check{x}_{i,(0)}^{\mathrm{r}}],$$

corresponding to (PC-1-a) and (PC-2-a) respectively.

In particular, if i is the index that parameters $(\epsilon, \alpha, w_{ii}, c_i, \gamma_i)$ satisfy condition (PC-1-a) (resp. (PC-2-a)), then $(\partial F_i/\partial x_i)(\mathbf{x}) > 1 + \sum_{j=1,\, j\neq i}^{n} |\frac{\partial F_i}{\partial x_j}(\mathbf{x})|$ (resp. $< -1 - \sum_{j=1,\, j\neq i}^{n} |\frac{\partial F_i}{\partial x_j}(\mathbf{x})|$), for all $\mathbf{x} \in \Omega^{\mathrm{m}\cdots\mathrm{m}}$, and we shall denote by $i \in \mathcal{I}$ (resp. $\mathcal{J}$) for such indices i. They correspond to the notation of index sets $\mathcal{A}$ and $\mathcal{B}$ in Section 3 respectively. Such a correspondence of notation also holds for the other regions $\Omega^{j_1\cdots j_n}$, depending on the slopes of $\hat{f}_{i,(0)}, \check{f}_{i,(0)}$ in the corresponding ranges.

It has been shown in Theorem 4.1 of [6] that if parameters $(\epsilon, \alpha, w_{ii}, c_i, \gamma_i)$ satisfy either (PC-1-a, b) or (PC-2-a, b), for $i = 1, \ldots, n$, then there exist 3^n fixed points and each of them lies in $\Omega^{j_1\cdots j_n}$ respectively. In addition, all these 3^n fixed points are repelling and F is one-to-one on each $\Omega^{j_1\cdots j_n}$, $j_i \in \{\mathrm{l}, \mathrm{m}, \mathrm{r}\}$, since (3.2) holds for each $\Omega^{j_1\cdots j_n}$. However, it is not sufficient to conclude that these fixed points are snapback repellers by employing merely the setting of upper and lower maps. One needs to elaborate on constructing sequential upper and lower maps to locate the preimages of the fixed point, so that these preimages converge back to the fixed point. In the following, we provide a rigorous justification to show that the fixed point in the middle region $\Omega^{\mathrm{m}\cdots\mathrm{m}}$ is a snapback repeller through constructing a homoclinic orbit with the sequential graphic-iteration scheme. The repellers in other regions can be treated similarly.

We denote this fixed point in region $\Omega^{\mathrm{m}\cdots\mathrm{m}}$ by $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_n)$ in the following discussion. We further introduce the following notations:

$$L_i^{\mathrm{m}} = \max\{g'_0(\xi):\ \xi \in \Omega_i^{\mathrm{m}}\}, \qquad L_{\max}^{\mathrm{m}} := \max\{L_i^{\mathrm{m}}:\ 1 \leqslant i \leqslant n\},$$
$$\tilde{L}_i^{\mathrm{m}} = \max\{\alpha + w_{ii}g'_{c_i}(\xi):\ \xi \in \Omega_i^{\mathrm{m}}\}, \qquad \tilde{L}_{\max(\mathcal{J})}^{\mathrm{m}} = \max\{\tilde{L}_i^{\mathrm{m}} \mid i \in \mathcal{J}\},$$
$$\tilde{l}_i^{\mathrm{m}} = \min\{\alpha + w_{ii}g'_{c_i}(\xi):\ \xi \in \Omega_i^{\mathrm{m}}\}, \qquad \tilde{l}_{\min(\mathcal{I})}^{\mathrm{m}} = \min\{\tilde{l}_i^{\mathrm{m}} \mid i \in \mathcal{I}\}.$$

The following assumptions will be used to show $\Omega_{i,(k)}^{\mathrm{m}} \to \{\bar{x}_i\}$ as $k \to \infty$, for all i:

$$1 + L_{\max}^{\mathrm{m}} \sum_{j=1,\, j\neq i}^{n} |w_{ij}| < \tilde{l}_{\min(\mathcal{I})}^{\mathrm{m}}, \tag{PC-1-d}$$

$$\tilde{L}_{\max(\mathcal{J})}^{\mathrm{m}} < -1 - L_{\max}^{\mathrm{m}} \sum_{j=1,\, j\neq i}^{n} |w_{ij}|. \tag{PC-2-d}$$

Theorem 4.1. *Assume that the parameters $(\epsilon, \alpha, w_{ii}, c_i, \gamma_i)$ satisfy either conditions* (PC-1-a, b, c(i), c(ii), d) *or* (PC-2-a, b, c(iii), c(iv), d)*. Then, there exist homoclinic orbits for the fixed point $\bar{\mathbf{x}}$ lying in $\Omega^{\mathrm{m}\cdots\mathrm{m}}$ and $\bar{\mathbf{x}}$ is a snapback repeller.*

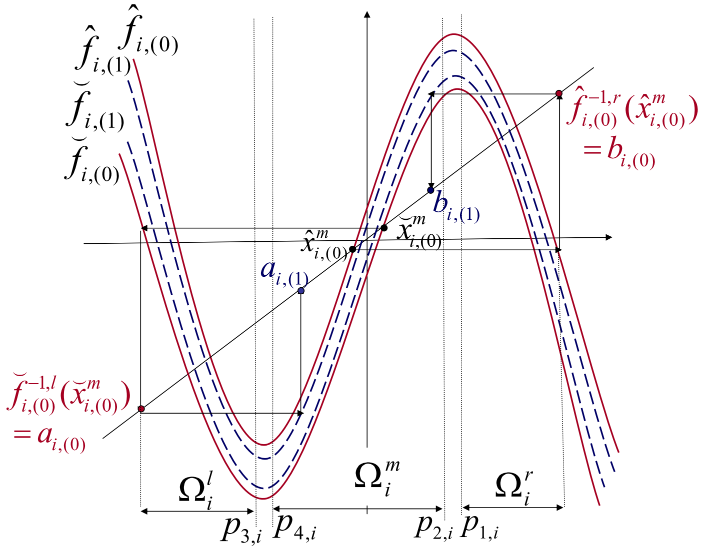

Fig. 4. Configuration for $a_{i,(k)}$ and $b_{i,(k)}$, $k = 0, 1$, for parameters $(\epsilon, \alpha, w_{ii}, c_i, \gamma_i)$ satisfies (PC-1-a).

Proof. We divide the proof of Theorem 4.1 into steps (I)–(VII) which correspond to the ones in the sequential graphic-iteration scheme in Section 3. To shorten the presentation, we set $w_{ii} = w$, for all $i = 1, \dots, n$ and illustrate the arguments only for the case that parameters $(\epsilon, \alpha, w_{ii}, c_i, \gamma_i)$ satisfy (PC-1-a, b, c, d) for every i. In addition, Ω, $\Omega_{(k)}$, Ω_i, $\Omega_{i,(k)}$, $\hat{f}^{-1}_{i,(k)}$, and $\breve{f}^{-1}_{i,(k)}$ in Section 3 are $\Omega^{\mathrm{m}\cdots\mathrm{m}}$, $\Omega^{\mathrm{m}\cdots\mathrm{m}}_{(k)}$, Ω^{m}_i, $\Omega^{\mathrm{m}}_{i,(k)}$, $\hat{f}^{-1,\mathrm{m}}_{i,(k)}$, and $\breve{f}^{-1,\mathrm{m}}_{i,(k)}$ herein, respectively, for all $k \geqslant 1$; $\hat{f}^{-1,\mathrm{m}}_{i,(k)}(y)$ (resp. $\breve{f}^{-1,\mathrm{m}}_{i,(k)}(y)$) represents the preimage of y lying in region Ω^{m}_i under $\hat{f}_{i,(k)}$ (resp. $\breve{f}_{i,(k)}$), for all $k \geqslant 1$.

For each $\mathbf{x}' = (x'_1, \dots, x'_n) \in \mathbb{R}^n$ and $i = 1, \dots, n$, we define $\tilde{f}_{\mathbf{x}',i,(0)}(\xi) = \alpha\xi + wg_{c_i}(\xi) + \sum_{j=1,\, j\neq i}^n w_{ij} g_0(x'_j) + d_i$, for $\xi \in \mathbb{R}^n$. Subsequently $\breve{f}_{i,(0)}(\xi) \leqslant \tilde{f}_{\mathbf{x}',i,(0)}(\xi) \leqslant \hat{f}_{i,(0)}(\xi)$, for all $\xi \in \mathbb{R}$, $i = 1, \dots, n$, where $\hat{f}_{i,(0)}, \breve{f}_{i,(0)}$ are defined in (4.4), (4.5). Now we set $a_{i,(0)} := \breve{f}^{-1,\mathrm{l}}_{i,(0)}(\breve{x}^{\mathrm{m}}_{i,(0)})$, $b_{i,(0)} := \hat{f}^{-1,\mathrm{r}}_{i,(0)}(\hat{x}^{\mathrm{m}}_{i,(0)})$, for all i. These initial settings will be used to locate the first preimage point $\bar{\mathbf{x}}^{-1}$ of $\bar{\mathbf{x}}$, cf. Fig. 4.

(I) Locating the first preimage point $\bar{\mathbf{x}}^{-1}$ of $\bar{\mathbf{x}}$ with $\bar{\mathbf{x}}^{-1} \notin \prod_{i=1}^n \Omega^{\mathrm{m}}_i$: Consider a label set $\{j_1, \dots, j_n\}$, $j_i \in \{\mathrm{r}, \mathrm{l}\}$. By (PC-1-c(i), c(ii)), as $\bar{x}_i \in \Omega^{\mathrm{m}}_i \cap [p_{4,i}, \breve{x}^{\mathrm{m}}_{i,(0)}] \subset [\breve{f}^{-1,\mathrm{l}}_{i,(0)}(\breve{x}^{\mathrm{m}}_{i,(0)}), \breve{x}^{\mathrm{m}}_{i,(0)}]$ (resp. $\Omega^{\mathrm{m}}_i \cap [\hat{x}^{\mathrm{m}}_{i,(0)}, p_{2,i}] \subset [\hat{x}^{\mathrm{m}}_{i,(0)}, \hat{f}^{-1,\mathrm{r}}_{i,(0)}(\hat{x}^{\mathrm{m}}_{i,(0)})]$), for each $(x'_1, \dots, x'_n) \in \Omega^{j_1\cdots j_n}$, there exists $\xi_i \in \Omega^{\mathrm{l}}_i$ (resp. Ω^{r}_i) if $j_i = \mathrm{l}$ (resp. r) such that $\bar{x}_i = \alpha\xi_i + wg_{c_i}(\xi_i) + \sum_{j=1,\, j\neq i}^n w_{ij} g_0(x'_j) + d_i$, for all i. Next, we define functions $H_{(0)}$ and $G_{(0)}$ as replacing $\Omega_{i,(1)}$, $\bar{x}^{-1}_i$, and $\xi_{i,(1)}$ by $\Omega^{j_i}_i$, $\bar{x}_i$, and ξ_i, respectively, in (3.11), (3.12), and (3.13). Therefore, according to Brouwer's fixed point theorem, there exists a first preimage $\bar{\mathbf{x}}^{-1} = (\bar{x}^{-1}_1, \dots, \bar{x}^{-1}_n)$, where $\bar{x}^{-1}_i$ can be chosen in Ω^{l}_i or Ω^{r}_i, for each $i = 1, \dots, n$. That is, the preimage $\bar{\mathbf{x}}^{-1} \notin \prod_{i=1}^n \Omega^{\mathrm{m}}_i$. Moreover, $\bar{x}^{-1}_i$ satisfies (3.6), for all i, and $a_{i,(0)}, b_{i,(0)}$ satisfy $[a_{i,(0)}, b_{i,(0)}] \supseteq \Omega^{\mathrm{m}}_i$, (3.9), due to (PC-1-c(i), c(ii)).

(II)–(V): By (PC-1-b), $\hat{f}_{i,(1)}$ and $\breve{f}_{i,(1)}$ both have fixed point in Ω^{m}_i, for all $i = 1, \dots, n$, and all conditions in steps (II)–(V) of Section 3 are satisfied, by (PC-1-b, c(i), c(ii)). For this TCNN map, we formulate $\hat{f}_{i,(k)}(\xi) := \alpha\xi + wg_{c_i}(\xi) + \hat{B}_{i,(k)} + d_i$ and $\breve{f}_{i,(k)}(\xi) := \alpha\xi + wg_{c_i}(\xi) + \breve{B}_{i,(k)} + d_i$, where $\hat{B}_{i,(k)} := \sum_{j=1,\, j\neq i}^n \max_{\xi\in\Omega^{\mathrm{m}}_{j,(k-1)}}\{w_{ij} g_0(\xi)\}$, $\breve{B}_{i,(k)} := \sum_{j=1,\, j\neq i}^n \min_{\xi\in\Omega^{\mathrm{m}}_{j,(k-1)}}\{w_{ij} g_0(\xi)\}$, for $k \geqslant 2$, $i = 1, \dots, n$. The scenario is similar to Fig. 4 which is drawn for $k = 0, 1$. Therefore, we can construct a sequence of nested domains $\prod_{i=1}^n \Omega_{i,(k)}$, with $\prod_{i=1}^n \Omega_{i,(k)} \subseteq \prod_{i=1}^n \Omega_{i,(k-1)}$, $k \geqslant 2$. In addition, we can construct an orbit $\{\bar{\mathbf{x}}^{-k}\}_{k=1}^\infty$, such that $\bar{\mathbf{x}}^{-k} \in \prod_{i=1}^n \Omega^{\mathrm{m}}_{i,(k-1)}$, for all $k \geqslant 2$, and show that $\bar{\mathbf{x}} \in \Omega^{\mathrm{m}\cdots\mathrm{m}}_{(k)}$, for all $k \geqslant 1$.

(VI) Convergence of $\bar{\mathbf{x}}^{-k}$ to $\bar{\mathbf{x}}$, as $k \to \infty$: We shall show that $\|\Omega^{\mathrm{m}}_{i,(k)}\| \to 0$, as $k \to \infty$, for all i. This is where condition (PC-1-d) is needed. Notably, for all $i = 1, \dots, n$, $k \in \mathbb{N}$,

$$\hat{f}_{i,(k)}(a_{i,(k)}) = \alpha a_{i,(k)} + wg_{c_i}(a_{i,(k)}) + \left[\sum_{j=1,\, j\neq i}^n \max_{x_j\in\Omega^{\mathrm{m}}_{j,(k-1)}} \{w_{ij} g_0(x_j)\}\right] + d_i = a_{i,(k-1)}, \tag{4.6}$$

$$\breve{f}_{i,(k)}(b_{i,(k)}) = \alpha b_{i,(k)} + wg_{c_i}(b_{i,(k)}) + \left[\sum_{j=1,\, j\neq i}^n \min_{x_j\in\Omega^{\mathrm{m}}_{j,(k-1)}} \{w_{ij} g_0(x_j)\}\right] + d_i = b_{i,(k-1)}. \tag{4.7}$$

There exist $\sigma_{i,(k)} \in [a_{i,(k)}, b_{i,(k)}]$, and $\bar{\sigma}_{j,(k-1)} \in [a_{j,(k-1)}, b_{j,(k-1)}]$ so that the difference of (4.6) and (4.7) yields

$$\mathcal{X}_{i,(k)} = \sum_{j=1}^{n} \frac{\beta_{i,j,(k-1)}}{\alpha + wg'_{c_i}(\sigma_{i,(k)})} \mathcal{X}_{j,(k-1)}, \quad \text{for } i = 1, \dots, n, \ k \in \mathbb{N},$$

where $\mathcal{X}_{i,(k)} := b_{i,(k)} - a_{i,(k)} \geqslant 0$, $\beta_{i,j,(k-1)} = |w_{ij}| g'_0(\bar{\sigma}_{j,(k-1)})$ if $j \neq i$, $= 1$ if $j = i$, by the Mean Value Theorem. Now, let $M_{(k)} := \max\{\mathcal{X}_{i,(k)} \mid i = 1, \dots, n\} = \mathcal{X}_{i'_k,(k)}$, for some $i'_k \in \{1, \dots, n\}$. Since $\beta_{i,j,(k-1)} \geqslant 0$ and $\alpha + wg'_{c_i}(\sigma_{i,(k)}) > 1$, for all $i, j \in \{1, \dots, n\}$, we thus have

$$M_{(k)} \leqslant \sum_{j=1}^{n} \frac{\beta_{i'_k,j,(k-1)}}{\alpha + wg'_{c_{i'_k}}(\sigma_{i'_k,(k)})} M_{(k-1)},$$

for all $k \geqslant 1$. Therefore, $M_{(k)} \leqslant \mathcal{R} M_{(k-1)}$, if we set

$$\mathcal{R} := \sup\left\{ \sum_{j=1}^{n} \frac{\beta_{i,j,(k-1)}}{\alpha + wg'_{c_i}(\sigma_{i,(k)})} : \ i = 1, \dots, n, \ k \in \mathbb{N} \right\}.$$

The assertion will hold if $0 \leqslant \mathcal{R} < 1$. Let us elaborate. It is obvious that $\mathcal{R} \geqslant 0$. Next, for every $i = 1, \dots, n$, $k \in \mathbb{N}$, we compute

$$\sum_{j=1}^{n} \frac{\beta_{i,j,(k-1)}}{\alpha + wg'_{c_i}(\sigma_{i,(k)})} \leqslant \frac{1}{\tilde{l}^{\mathrm{m}}_{\min(\mathcal{I})}} \left(1 + L^{\mathrm{m}}_{\max} \sum_{j=1,\, j\neq i}^{n} |w_{ij}| \right) < 1,$$

by (PC-1-d) and $\tilde{l}^{\mathrm{m}}_{\min(\mathcal{I})} \geqslant 1$. Therefore, we conclude that $b_{i,(k)} - a_{i,(k)} \to 0$ as $k \to \infty$, for all $i = 1, \dots, n$.

(VII) $\bar{\mathbf{x}}$ is a snapback repeller: As (PC-1-a) implies that (3.2) holds in $\prod_{i=1}^{n} \Omega_i^{j_i}$, $j_i \in \{\mathrm{l}, \mathrm{m}, \mathrm{r}\}$, we conclude $\det(DF(\mathbf{x})) \neq 0$, for any $\mathbf{x} \in \prod_{i=1}^{n} \Omega_i^{j_i}$, $j_i \in \{\mathrm{l}, \mathrm{m}, \mathrm{r}\}$. Moreover, we have shown that there exist a homoclinic orbit $\{\bar{\mathbf{x}}^{-k}\}_{k=1}^{\infty}$ of $\bar{\mathbf{x}}$, where $\bar{\mathbf{x}}^{-1} \in \Omega^{j_1 \cdots j_n}$, for some $\{j_1, \dots, j_n\}$, $j_i \in \{\mathrm{l}, \mathrm{r}\}$, and $\bar{\mathbf{x}}^{-k} \in \Omega^{\mathrm{m} \cdots \mathrm{m}}$ for all $k \geqslant 2$. Thus $\det(DF(\bar{\mathbf{x}}^{-k})) \neq 0$, for all $k \geqslant 1$. Hence, (3.18) holds. Therefore $\bar{\mathbf{x}}$ is a snapback repeller. □

Remark. (i) As conditions (PC-1-c) and (PC-2-c) involve preimage relations, they can be replaced by the following direct relations, as remarked in Lemma 3.2 of [6]:

$$\check{f}(\hat{f}(p_3)) > p_2, \qquad \check{f}(\hat{f}(p_4)) > p_2, \tag{PC-1-e(i)}$$

$$\hat{f}(\check{f}(p_1)) < p_4, \qquad \hat{f}(\check{f}(p_2)) < p_4, \tag{PC-1-e(ii)}$$

$$g_c(\hat{f}(p_1)) > \frac{1-\alpha}{w} \hat{f}(p_1) - \frac{\gamma}{w}, \qquad \hat{f}(p_1) < p_3, \qquad \min\{\check{f}(p_3), \check{f}(p_4)\} > p_1, \tag{PC-2-e(i)}$$

$$g_c(\check{f}(p_3)) < \frac{1-\alpha}{w} \check{f}(p_3) + \frac{\gamma}{w}, \qquad p_1 < \check{f}(p_3), \qquad \max\{\hat{f}(p_1), \hat{f}(p_2)\} < p_3, \tag{PC-2-e(ii)}$$

$$p_3 > \hat{f}(\hat{f}(p_1)), \qquad \check{f}(p_3) > p_1, \qquad \hat{f}(p_1) < p_4, \tag{PC-2-e(iii)}$$

$$p_1 < \check{f}(\check{f}(p_3)), \qquad \hat{f}(p_1) < p_3, \qquad \check{f}(p_3) > p_2. \tag{PC-2-e(iv)}$$

Indeed, (PC-1-e(i), e(ii)) can replace (PC-1-c(i), c(ii)), respectively, when (PC-1-a, b) hold, and (PC-2-e(i))–(PC-2-e(iv)) can replace (PC-2-c(i))–(PC-2-c(iv)), respectively, when (PC-2-a, b) hold.

(ii) If parameters $(\epsilon, \alpha, w_{ii}, c_i, \gamma_i)$ satisfy condition (PC-2-a), then we set $a_{i,(0)} := \hat{f}^{-1,\mathrm{l}}_{i,(0)}(\check{x}^{\mathrm{m}}_{i,(0)})$, $b_{i,(0)} := \check{f}^{-1,\mathrm{r}}_{i,(0)}(\hat{x}^{\mathrm{m}}_{i,(0)})$. Hence, by (PC-2-c(iii)) and (PC-2-c(iv)), $a_{i,(0)}$ and $b_{i,(0)}$ satisfy $[a_{i,(0)}, b_{i,(0)}] \supseteq \Omega_i^{\mathrm{m}}$, and (3.10).

(iii) More generally, if we can construct preimages $\{\bar{\mathbf{x}}^{-k}\}_{k=1}^{-\ell+1}$, such that $\bar{x}_i^{-\ell+1} \in \Omega_i^{\mathrm{l}}$ (resp. Ω_i^{r}), then when parameters $(\epsilon, \alpha, w_{ii}, c_i, \gamma_i)$ satisfy (PC-1-a, b, c(i)) (resp. (PC-1-a, b, c(ii))) or (PC-2-a, b, c(iii)) (resp. (PC-2-a, b, c(iv))), we can construct $\bar{x}_i^{-\ell} \in \Omega_i^{\mathrm{m}}$.

(iv) We can consider the other region $\prod_{i=1}^{n} \Omega_i^{j_i}$, $j_i \in \{\mathrm{r}, \mathrm{m}, \mathrm{l}\}$ and modify the conditions (PC-1-c, d), (PC-2-c, d), so that Theorem 4.1 is valid for repeller in these regions.

5. Numerical examples

In this section, two numerical examples are presented. We employ the estimation of repelling neighborhood for the repeller introduced in Section 2 and the sequential graphic-iteration scheme in Section 3 to investigate the chaotic behaviors in a chaotic neural network and a discrete-time predator–prey model. In the examples, we use the software Mathematica with 16-digit working-precision to examine the conditions of the theory in Sections 2–4 for the considered maps. In addition, the "Interval Newton Computing" is adopted when rigorous computation precision is required.

Example 5.1. Consider the two-dimensional map (4.1), $F=(F_1,F_2)$ where

$$F_1(x_1,x_2)=\alpha x_1+w_{11}g_{c_1}(x_1)+w_{12}g_0(x_2)+d_1,$$
$$F_2(x_1,x_2)=\alpha x_2+w_{22}g_{c_2}(x_2)+w_{21}g_0(x_1)+d_2.$$

Case I. F with parameters $\epsilon=1$, $c_1=c_2=0.5$, $\alpha=-2.5$, $w_{11}=w_{22}=50$, $w_{12}=0.1$, $w_{21}=0.2$, $d_1=-0.05$, $d_2=-0.1$.

We take $\gamma_1=1>|w_{12}|+|d_1|=0.15$, $\gamma_2=1>|w_{21}|+|d_2|=0.3$, and $p_{3,i}=-3.637$, $p_{4,i}=-2.419$, $p_{2,i}=2.419$, $p_{1,i}=3.637$, $i=1,2$. Set $\Omega^{\mathrm{mm}}=\prod_{i=1}^{2}\Omega_i^{\mathrm{m}}$, with $\Omega_i^{\mathrm{m}}=[p_{4,i},p_{2,i}]$, $i=1,2$. Computations show that F_i satisfies (PC-1-a, b, d, e(i), e(ii)), for $i=1,2$, and $\bar{\mathbf{x}}=(0,0)$ is a repelling fixed point of F lying in Ω^{mm}. Moreover, we compute $\bar{\mathbf{x}}^{-1}\approx(10.019,10.039)\notin\Omega^{\mathrm{mm}}$, $\bar{\mathbf{x}}^{-2}\approx(1.134,1.133)\in\Omega^{\mathrm{mm}}_{(1)}\approx[-1.144,1.144]\times[-1.154,1.154]\subset\Omega^{\mathrm{mm}}$, $\bar{\mathbf{x}}^{-3}\approx(0.113,0.113)\in\Omega^{\mathrm{mm}}_{(2)}\approx[-0.117,0.117]\times[-0.121,0.121]\subset\Omega^{\mathrm{mm}}_{(1)}$, $\ldots$, $\bar{\mathbf{x}}^{-11}\approx(1.111\times10^{-9},1.084\times10^{-9})\in\Omega^{\mathrm{mm}}_{(10)}\approx[-1.457\times10^{-9},1.457\times10^{-9}]\times[-1.723\times10^{-9},1.723\times10^{-9}]\subset\Omega^{\mathrm{mm}}_{(9)}$. Hence, by Theorem 4.1, $\bar{\mathbf{x}}$ is a snapback repeller. Thus, map F is chaotic.

Case II. F with parameters $\epsilon=1$, $c_1=c_2=0.5$, $\alpha=-4$, $w_{11}=w_{22}=150$, $w_{12}=6$, $w_{21}=2$, $d_1=-3$, and $d_2=-1$.

We take $\gamma_1=9.1$, $\gamma_2=3.1$. Similar to Case I, we take $p_{3,1}=-5.323$, $p_{4,1}=-2.921$, $p_{2,1}=2.921$, $p_{1,1}=5.323$, $p_{3,2}=-4.181$, $p_{4,2}=-3.175$, $p_{2,2}=3.175$, $p_{1,2}=4.181$. Then, as in Case I, F_i also satisfies (PC-1-a, b, e(i), e(ii)), and $\bar{\mathbf{x}}=(0,0)$ is a repelling fixed point of F lying in Ω^{mm}. However, the parameters do not satisfy (PC-1-d). Hence, we shall use the expanding condition (2.6) to find a repelling neighborhood $B_{1.07}(\bar{\mathbf{x}})$, due to $\alpha_r-r\sqrt{2}\beta_r>1$, if $0\leqslant r\leqslant1.07$.

We replace Ω^{mm} by $\tilde{\Omega}^{\mathrm{mm}}:=\prod_{i=1}^{2}\tilde{\Omega}_i^{\mathrm{m}}\subseteq B_{1.07}(\bar{\mathbf{x}})$, with $\tilde{\Omega}_i^{\mathrm{m}}=[\tilde{p}_{4,i},\tilde{p}_{2,i}]=[-0.75,0.75]$, for $i=1,2$. Moreover, in the new region $\tilde{\Omega}^{\mathrm{mm}}$, F_i still satisfies (PC-1-a, b, e(i), e(ii)), for $i=1,2$. Hence, we can construct preimages $\{\bar{\mathbf{x}}^{-k}\}$, with $\bar{\mathbf{x}}^{-1}\notin\tilde{\Omega}^{\mathrm{mm}}$, $\bar{\mathbf{x}}^{-2}\in\tilde{\Omega}^{\mathrm{mm}}$, by steps (I)–(II) of Section 4. Furthermore, since $\tilde{\Omega}^{\mathrm{mm}}$ is a repelling neighborhood of $\bar{\mathbf{x}}$, we automatically have $\bar{\mathbf{x}}^{-k}\to\bar{\mathbf{x}}$, as $k\to\infty$. In fact, the computation shows $\bar{\mathbf{x}}^{-1}\approx(19.500,19.000)\notin\tilde{\Omega}^{\mathrm{mm}}$, $\bar{\mathbf{x}}^{-2}\approx(0.574,0.576)\in\tilde{\Omega}^{\mathrm{mm}}$, $\ldots$, $\bar{\mathbf{x}}^{-11}\approx(6.711\times10^{-15},9.703\times10^{-15})$. Hence, the fixed point $\bar{\mathbf{x}}=(0,0)$ is a snapback repeller and the map F is chaotic.

Example 5.2. We consider the discrete-time predator–prey system $F=(F_1,F_2)$ with

$$F(x_1,x_2)=\big(F_1(x_1,x_2),F_2(x_1,x_2)\big)=\big(x_1e^{b(1-x_1)-ax_2},x_1\big(1-e^{-ax_2}\big)\big),\tag{5.1}$$

where $a=5$ and $b=3$. It can be computed that map (5.1) has three fixed points $(0,0)$, $(0.4214296448,0.3471422131)$, and $(1,0)$. Herein, we shall estimate the repelling neighborhood for the repelling fixed point $\bar{\mathbf{x}}=(1,0)$, as (2.1) only holds for this fixed point. Our methodology can be further extended to treat the other fixed points of the map. In respecting (2.2), (2.3) and taking $s_1=\sqrt{27-\sqrt{629}}\cong1.38569$, and $r=0.009$, we compute $s_1-\eta_r>1$. Hence, $B_{0.009}(\bar{\mathbf{x}})$ is a repelling neighborhood of the fixed point $\bar{\mathbf{x}}$.

Next, we find numerical preimages of $\bar{\mathbf{x}}$: $\bar{\mathbf{x}}_q^{-1}=(\bar{x}_q^{-1},0)$, $\ldots$, $\bar{\mathbf{x}}_q^{-10}=(\bar{x}_q^{-10},0)$ with $\bar{x}_q^{-1}=0.0595202092926404$, $\bar{x}_q^{-10}=1.0045817501837184$, and $\bar{\mathbf{x}}_q^{-10}\in B_{0.009}(\bar{\mathbf{x}})$. We can actually use the Multi-Shooting Method and Interval Newton Computing Method to show that there exists a true orbit $\{\bar{\mathbf{x}}^{-1},\ldots,\bar{\mathbf{x}}^{-10}\}$ near the numerical orbit $\{\bar{\mathbf{x}}_q^{-1},\ldots,\bar{\mathbf{x}}_q^{-10}\}$, with $\bar{\mathbf{x}}^{-10}\in B_{0.009}(\bar{\mathbf{x}})$, $F(\bar{\mathbf{x}}^{-k-1})=\bar{\mathbf{x}}^{-k}$, $k=1,2,\ldots,9$, $F(\bar{\mathbf{x}}^{-1})=\bar{\mathbf{x}}$, and $\det(DF(\bar{\mathbf{x}}^{-k}))\neq0$, for $1\leqslant k\leqslant10$. Thus, if we take $\bar{\mathbf{x}}^{-10}$ as the snapback point, then it lies in the repelling neighborhood of $\bar{\mathbf{x}}$, according to Proposition 2.2. Thus, $\bar{\mathbf{x}}$ is a snapback repeller, and F is chaotic.

6. Conclusions

We have derived two methodologies to establish the existence of snapback repeller, hence chaos, for multi-dimensional maps, under Marotto's theorem. The first one is to estimate the radius of repelling neighborhood for a repelling fixed point, under the Euclidean norm. The first-order as well as the second-order formulations have been derived for the estimation. These estimates are essential and useful in various computations involving the local property of the repeller. Secondly, we have proposed a sequential graphic-iteration scheme to construct the homoclinic orbit for a repeller. The scheme employs local structure and iterated upper and lower bounds for each component of the map to track the preimages of a repeller. The application of the scheme to a chaotic neural network has been illustrated. We also demonstrated the use of the estimate for the repelling neighborhood in this chaotic neural network and a predator–prey model. The conditions for the present theories are all computable numerically and can be combined with other computation techniques such as Interval Computing, as demonstrated in our examples. The present investigation has reconfirmed the existence of snapback repellers in several works in the literature, under valid conditions. The formulations along with their extensions provide effective approaches to confirm the existence of snapback repellers and facilitate the use of Marotto's theorem in other systems in the literature.

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