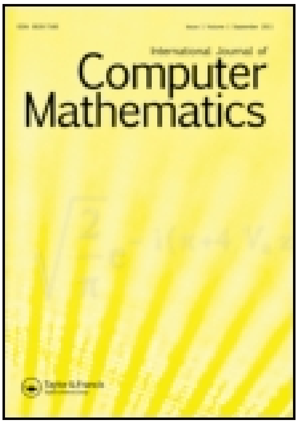

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The r-component connectivity $\kappa_r(G)$ of the non-complete graph G is the minimum number of vertices whose deletion results in a graph with at least r components. So, κ_2 is the usual connectivity. In this paper, we determine the r-component connectivity of the hypercube Q_n for $r = 2, 3, \ldots, n+1$, and we classify all the corresponding optimal solutions.

Keywords: hypercubes; component connectivity

2010 AMS Subject Classifications: 05C75; 05C40

1. Introduction

Let G be a non-complete graph. An *r-component cut* of G is a set of vertices whose deletion results in a graph with at least r components. The *r-component connectivity* or simply r-connectivity $\kappa_r(G)$ of G is the size of the smallest r-component cut of G (if there is no r-component cut of G, then we define $\kappa_r(G)$ to be ∞). So, $\kappa_2(G)$ is the usual connectivity of G. It is clear that $\kappa_m(G) \leq \kappa_{m+1}(G)$ for every positive integer m. In this paper, we determine the r-component connectivity of the hypercube Q_n for $r = 2, 3, \ldots, n+1$. This measure was introduced independently in a number of papers [2,6], and it is a good measure of robustness of interconnection networks.

The hypercube is one of the fundamental interconnection networks. The hypercube Q_n (with $n \geq 2$) is defined as having the vertex set of binary strings of length n. Two vertices are adjacent if their strings differ in exactly 1 bit, that is, their Hamming distance[1] is 1. So, Q_n is an n-regular graph with 2^n vertices.

Component connectivity is an extension of standard connectivity. It can also be viewed as an understanding of the fault resiliency of networks. There are a number of other related concepts in studying how intact the graph is when faults are present. Some of these concepts are related; in

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this paper, we relate some of these results to component connectivity. The most comprehensive results regarding fault resiliency of the hypercube are those presented in a series of papers [7–9], specifically, Theorem 2.2 and Theorem 2.4(1). The results presented in this paper can be viewed as the augmentation and extension of those results. We refer the reader to [7–9] for details and the importance of the hypercubes. (For such fault resiliency treatment for other classes of graphs, see [3,4].) In this paper, we determine $\kappa_r(Q_n)$ for $r = 2, 3, \dots, n+1$, and we classify all the optimal solutions.

2. Determining $\kappa_r(Q_n)$

The first goal of this paper is to determine $\kappa_r(Q_n)$ for $r = 2, 3, \dots, n+1$.

THEOREM 2.1 *Let $n \geq 2$ and $1 \leq k \leq n$. Then, $\kappa_{k+1}(Q_n) = kn - k(k+1)/2 + 1$.*

We prove here that $\kappa_{k+1}(Q_n) \leq kn - k(k+1)/2 + 1$; the other direction will follow from the results that come later in this section. Let u be an arbitrary vertex in Q_n. Then, u has n neighbours, say, $u_1, u_2, \dots, u_n$. (Note that they are mutually non-adjacent as Q_n is bipartite.) Given a set of vertices T, we use $N(T)$ to denote the set of vertices that are not in T but incident to at least one vertex in T. (If $T = \{t\}$, we write $N(t)$ instead of $N(\{t\})$.) Let $1 \leq k \leq n$ and $S = N(\{u_1, u_2, \dots, u_k\})$. Clearly, $u \in S$. Now, each u_i has $n-1$ additional neighbours, but every pair of u_i and u_j shares exactly one neighbour other than u in Q_n. In addition, u is the only common neighbour of any three u_i's. Hence, $|S| = k(n-1) - \binom{k}{2} + 1 = kn - k(k+1)/2 + 1$. It is clear that $Q_n - S$ has at least $k+1$ components where at least k of them are singletons. This finishes the proof of $\kappa_{k+1}(Q_n) \leq kn - k(k+1)/2 + 1$. The difficulty is in proving that $\kappa_{k+1}(Q_n) \geq kn - k(k+1)/2 + 1$.

There are many different results on faulty hypercubes and some of them are related to Theorem 2.1. One such result is the following.

THEOREM 2.2 ([8]) *Let $n \geq 4$. Let $1 \leq k \leq n-2$ and S be a set of vertices in Q_n such that $|S| \leq kn - k(k+1)/2$. Then, $Q_n - S$ is either connected or has one large component plus a number of small components with at most $k-1$ vertices in total.*

We can observe that the special case $k = 1$ implies that Q_n has connectivity n. We note that Theorem 2.1 follows directly from Theorem 2.2 for $n \geq 4$ and $1 \leq k \leq n-2$. If $Q_n - S$ has one large component plus a number of small components with $k-1$ vertices in total, then the number of components is maximized when the small components are $k-1$ singletons. Hence, $\kappa_{k+1}(Q_n) \geq kn - k(k+1)/2 + 1$ for $k \leq n-2$. Obviously, one can easily check the validity of Theorem 2.1 for $n = 2$ and $n = 3$. Thus, the missing cases are $k = n-1$ and $k = n$ for $n \geq 4$. One may wonder why only extend the range by two more cases for each n. The reason is that the formula $kn - k(k+1)/2 + 1$ does not hold for $k = n+2$ as proved in the next result.

PROPOSITION 2.3 *Let $n \geq 4$. Then, $\kappa_{n+2}(Q_n) > (n+1)n - (n+1)(n+2)/2 + 1$.*

Proof Suppose that $\kappa_{n+2}(Q_n) \leq (n+1)n - (n+1)(n+2)/2 + 1$. Since $(n+1)n - (n+1)(n+2)/2 + 1 < (n-1)n - (n-1)n/2 + 1 \leq \kappa_n(Q_n)$, this implies $\kappa_{n+2}(Q_n) < \kappa_n(Q_n)$, which is a contradiction. ■

We now point out that the formula $kn - k(k+1)/2 + 1$ gives the same value for $k = n-1$ and $k = n$. Hence, if the formula holds for $k = n-1$, then this implies that the formula holds for

$k = n$ as well, since $n^2 - n(n+1)/2 + 1 \geq \kappa_{n+1}(Q_n) \geq \kappa_n(Q_n) = (n-1)n - (n-1)(n)/2 + 1$. So, the only missing case is when $n \geq 4$ and $k = n - 1$. We note that Theorem 2.2 does not hold for $k = n - 1$ as shown by the following example: Let u be an arbitrary vertex with n neighbours $u_1, u_2, \ldots, u_n$. Let S be the set of vertices that are adjacent to at least one of $u_1, u_2, \ldots, u_n$ excluding u. Then, $|S| = n(n-1) - n(n-1)/2 = n(n-1)/2$, and we have a component having $n+1$ vertices in $Q_n - S$, thus violating the conclusion of Theorem 2.2. We call a set S given as above an *exceptional set*. Fortunately, this is the only exceptional case.

THEOREM 2.4 *Let $n \geq 4$. Let S be a set of vertices of size at most $n(n-1)/2$.*

(1) *If $|S| \leq n(n-1)/2 - 1$, then $Q_n - S$ is either connected or has one large component plus a number of small components with at most $n - 2$ vertices in total.*
(2) *If $|S| = n(n-1)/2$ and S is not exceptional, then $Q_n - S$ is either connected or has one large component plus a number of small components with at most $n - 2$ vertices in total. If $|S| = n(n-1)/2$ and S is exceptional, then $Q_n - S$ has exactly two components, one of which is $K_{1,n}$.*

Theorem 2.4(1) was proved in [8], so we only need to prove Theorem 2.4(2).

Proof of Theorem 2.4(2) We apply induction on n. We first note that the statement is true for $n = 4$, since Q_4 has 16 vertices and the graph is symmetrical, so the claim can be checked by brute force.

Let H_0 (H_1, respectively) be the subgraph of Q_n induced by vertices with 0 (1, respectively) in the last position. Then, H_0 and H_1 are isomorphic to Q_{n-1}. Let S_0 and S_1 be the set of elements of S that are vertices in H_0 and H_1, respectively. We have two cases.

Case 1. Either $|S_0|$ or $|S_1|$ is at least $(n-1)(n-2)/2$. Without loss of generality, assume that $|S_1| \geq (n-1)(n-2)/2$.

Case 1a. $|S_1| \geq (n-1)(n-2)/2 + 1$. Then, $|S_0| \leq n(n-1)/2 - (n-1)(n-2)/2 - 1 = n - 2$, thus $H_0 - S_0$ is connected. Let Y be the component of $Q_n - S$ containing $H_0 - S_0$. Suppose v is a vertex of $H_1 - S_1$. Then, v is part of Y if v's unique neighbour in H_0 is not in S_0. Since $|S_0| \leq n - 2$, there are at most $n - 2$ vertices of $H_1 - S_1$ that are not part of Y and we are done.

Case 1b. $|S_1| = (n-1)(n-2)/2$. Hence, $|S_0| = n - 1$.

Subcase 1b(i). $H_0 - S_0$ is connected. We define Y to be the same as before. We apply the induction hypothesis. The first case is that $H_1 - S_1$ has one large component and a number of small components with at most $n - 3$ vertices in total. In other words, S_1 is not exceptional. Now, the large component in $H_1 - S_1$ has at least $2^{n-1} - (n-1)(n-2)/2 - (n-3)$ vertices and $|S_0| = n - 1$. So, if $2^{n-1} - (n-1)(n-2)/2 - (n-3) > n - 1$, then the large component is part of Y. This inequality holds for $n \geq 5$, and hence $Q_n - S$ has one large component and a number of small components with at most $n - 3$ vertices in total.

The second case from the induction hypothesis is that $H_1 - S_1$ has two components, one of which is $K_{1,n-1}$. So, S_1 is exceptional. So, each component has at least n vertices as $n \geq 5$. (One component has n vertices and the other has $2^{n-1} - n - (n-1)(n-2)/2$ vertices.) But $|S_0| = n - 1$; therefore, each of these components in $H_1 - S_1$ is part of Y, and hence $Q_n - S$ is connected.

Subcase 1b(ii). $H_0 - S_0$ is disconnected. Then, $H_0 - S_0$ consists of two components, one of which is a singleton, say, w. (Apply Theorem 2.2 with $k = 2$.) Let Y be the component in $Q_n - S$ containing the larger component, X, of $H_0 - S_0$. Now, we apply the induction hypothesis on $H_1 - S_1$. The first case is that $H_1 - S_1$ has one large component and a number of small components with at most $n - 3$ vertices in total. So, S_1 is not exceptional. Now, the large component in $H_1 - S_1$ has at least $2^{n-1} - (n-1)(n-2)/2 - (n-3)$ vertices, $|S_0| = n - 1$, and there is exactly one vertex in $H_0 - S_0$ that is not in X. So, if $2^{n-1} - (n-1)(n-2)/2 - (n-3) > n - 1 + 1$, then this large component is part of Y. The inequality holds for $n \geq 5$; hence, $Q_n - S$ has one large component and a number of small components with at most $n-2$ vertices in total. (Up to $n-3$ vertices from the small components in $H_1 - S_1$ and possibly w.)

The second case from the induction hypothesis is that $H_1 - S_1$ has two components, one of which is $K_{1,n-1}$ with, say, y as the centre. This corresponds to the case when S_1 is exceptional. Recall that $H_0 - S_0$ has exactly two components, one of which is a singleton w. The large component Z in $H_1 - S_1$ has $2^{n-1} - (n-1)(n-2)/2 - n$ vertices, $|S_0| = n-1$, and there is exactly one vertex in $H_0 - S_0$ that is not in X. So, if $2^{n-1} - (n-1)(n-2)/2 - n > n - 1 + 1$, then this large component is part of Y. The inequality holds for $n \geq 6$, so both X and Z are part of Y if $n \geq 6$. Suppose $n = 5$. We know that S_0 is exactly the set of neighbours of w and $S_1 = N(N(y)) - \{y\}$. Since the hypercube is vertex transitive, we may fix the choice for y and there are 16 choices for w. Hence, there are 16 cases, and in each case, it can be easily verified that $Q_5 - S$ satisfies Theorem 2.4(2). Henceforth, we may assume that $n \geq 6$. If w is adjacent to y, then we are done as $Q_n - S$ will either be connected or has two components, one of which is $K_{1,n}$. The latter implies that S is exceptional. Let $z_1, z_2, \ldots, z_{n-1}$ be the neighbours of y in H_1. Suppose that w is adjacent to z_1. Then, it is clear that z_2 is adjacent to a vertex in the large component of $H_0 - S_0$, and hence $Q_n - S$ is connected. (To see this, suppose that z_2 is not adjacent to a vertex in the large component of $H_0 - S_0$. Then z_2 is adjacent to a vertex in S_0. But S_0 are precisely the vertices in H_0 that are adjacent to w. Let this vertex be w'. Now, we have found a 5-cycle, namely, $y - z_1 - w - w' - z_2 - y$, which is a contradiction as Q_n is bipartite.) So, we may assume that w is not adjacent to any vertex of $K_{1,n-1}$ in $H_1 - S_1$. We now claim that at least one of $z_1, z_2, \ldots, z_{n-1}$ is adjacent to a vertex in the large component of $H_0 - S_0$. Otherwise, they are adjacent to the neighbours of w in H_0, which implies that w and y are adjacent, which is a contradiction as w is not adjacent to any vertex of this $K_{1,n-1}$ whose centre is y. Hence, $Q_n - S$ satisfies Theorem 2.4(2).

Case 2. $|S_0|, |S_1| \leq (n-1)(n-2)/2 - 1$. So, $|S_0|, |S_1| \geq n$. Now, $(n-1)(n-2)/2 - 1 = (n-3)(n-1) - (n-3)(n-2)/2$, so we may apply Theorem 2.2 on S_0 and S_1 since $n \geq 5$. We choose a to be the smallest integer such that $|S_0| \leq a(n-1) - a(a+1)/2$ and b to be the smallest integer such that $|S_1| \leq b(n-1) - b(b+1)/2$. Since $|S_0|, |S_1| \geq n$, we get $a, b \geq 2$. By the choice of a, we get $|S_0| \geq (a-1)(n-1) - (a-1)a/2 + 1$, and hence $|S_1| \leq n(n-1)/2 - (a-1)(n-1) + (a-1)a/2 - 1$. Now, by Theorem 2.2, $H_0 - S_0$ is either connected or has a large component and a number of small components with at most $a-1$ vertices in total. It suffices to show that $H_1 - S_1$ is either connected or has a large component and a number of small components with at most $n-a-1$ vertices in total. This can be accomplished by using Theorem 2.2 and showing that $|S_1| \leq (n-a)(n-1) - (n-a)(n-a+1)/2$. Observe that $(n-a)(n-1) - (n-a)(n-a+1)/2 - (n(n-1)/2 - (a-1)(n-1) + (a-1)a/2 - 1) = an + a + 2 - 2n - a^2 = (a-2)(n-a-1) \geq 0$ since $2 \leq a \leq n-3$. This completes the proof. ■

Now, the claim $\kappa_{k+1}(Q_n) \geq kn - k(k+1)/2 + 1$ for $k = n$ follows directly from Theorem 2.4. This finishes the proof of Theorem 2.1.

3. Classification of optimal solutions

In the previous section, we have shown that $\kappa_{r+1}(Q_n) = rn - r(r+1)/2 + 1$ for $n \geq 2$ and $1 \leq r \leq n$. It will be interesting to determine all the optimal solutions, that is, find all the optimal $(r+1)$-component cuts of size $rn - r(r+1)/2 + 1$ in Q_n. For $1 \leq r \leq n$, we can pick X to be a set of r neighbours of an arbitrary vertex u. We have already seen that $N(X)$ is an optimal solution for every such set X. These are called the *trivial* solutions. We now show the following result (of which, the case $r = 1$ has been known already).

THEOREM 3.1 *Let $n \geq 2$ and $1 \leq r \leq n-2$. Then, every optimal $(r+1)$-component cut of Q_n is trivial.*

Proof Let X be an optimal $(r+1)$-component cut. Then,$|X| = rn - r(r+1)/2 + 1$. We will establish this result in three steps.

The first step is to show that $Q_n - X$ has a large component and r singletons. Suppose $r \leq n-3$. Let $k = r+1$ in Theorem 2.2. Since $rn - r(r+1)/2 + 1 \leq (r+1)n - (r+1)(r+2)/2$ as $r+1 \leq n-2$, we can conclude that $Q_n - X$ has a large component plus a number of small components with r vertices in total. But $|X|$ is an $(r+1)$-component cut. So, $Q_n - X$ consists of a large component and r singletons. Now, suppose $r = n-2$. We note that $|X| = (n-2)n - (n-2)(n-1)/2 + 1 = n(n-1)/2$. So, we apply Theorem 2.4(2). Since X is not exceptional as X is an $(r+1)$-component cut, $Q_n - X$ has a large component plus a number of small components with $n-2$ vertices in total. But $|X|$ is an $(n-1)$-component cut, so $Q_n - X$ consists of a large component and $n-2$ singletons.

The first step is complete, that is, we have shown that $Q_n - X$ has a large component and r singletons. Let A be the set of these r vertices.

The second step is to prove that every pair of these r vertices has distance 2 in Q_n. Suppose not. Then, there are two vertices x and y whose distance is at least 3. Since x and y differ in at least one position, we may assume, without loss of generality, that x is in H_0 and y is in H_1 (using the usual definition of H_i's). Let $X_0 = X \cap V(H_0)$ and $X_1 = X \cap V(H_1)$. Clearly, if z is in H_0 (H_1, respectively) and z is an isolated vertex in $Q_n - X$, then z is an isolated vertex in $H_0 - X_0$ ($H_1 - X_1$, respectively). Let $a_0 = |A \cap V(H_0)|$ and $a_1 = |A \cap V(H_1)|$, so $a_0 + a_1 = r$. Then, $H_0 - X_0$ has at least $a_0 + 1$ components and $H_1 - X_1$ has at least $a_1 + 1$ components. Thus, by Theorem 2.1, $|X_0| \geq a_0(n-1) - a_0(a_0+1)/2 + 1$ and $|X_1| \geq a_1(n-1) - a_1(a_1+1)/2 + 1$. We have two cases.

Case 1. $a_0, a_1 \geq 2$. We first note that $(a_0(n-1) - a_0(a_0+1)/2 + 1) + (a_1(n-1) - a_1(a_1+1)/2 + 1) > rn - r(r+1)/2 + 1$ is equivalent to $(a_0 - 1)(a_1 - 1) > 0$, which is clearly true for $a_0, a_1 \geq 2$. Using the definition of X, X_0, X_1 and the bounds on $|X_0|$ and $|X_1|$, we get a contradiction as follows:

$$|X| = |X_0| + |X_1| \geq \left(a_0(n-1) - \frac{a_0(a_0+1)}{2} + 1\right) + \left(a_1(n-1) - \frac{a_1(a_1+1)}{2} + 1\right)$$
$$> rn - \frac{r(r+1)}{2} + 1 = |X|.$$

Case 2. $a_0 = 1$ and $a_1 = r-1$. Clearly, $|X_0| \geq n-1$ since x is isolated in $H_0 - X_0$. But y is also isolated in $Q_n - X$ and the distance of x and y is at least 3. Therefore, the unique neighbour of y in H_0 is not a member of $N_{H_0}(x)$, implying that $|X_0| \geq n$. By Theorem 2.1, $|X_1| \geq (r-1)(n-1) - (r-1)r/2 + 1$, hence

$$|X| = |X_0| + |X_1| = \left((r-1)(n-1) - \frac{(r-1)r}{2} + 1\right) + n > rn - \frac{r(r+1)}{2} + 1 = |X|,$$

which is a contradiction.

The third step is to show that X is trivial, that is, the vertices in $A = \{v_1, v_2, \ldots, v_r\}$ have a common neighbour. Without loss of generality, we may assume that $v_1 = 00\ldots0$. Now, since v_2 is of distance 2 from v_1, v_2 has exactly two 1's and we may assume that $v_2 = 110\ldots0$. Similarly, v_3 has exactly two 1's. But v_3 is also of distance 2 from v_2. So, we may assume that $v_3 = 1010\ldots0$.

Repeat the argument for v_4, and we get two choices for v_4. It is either $0110\ldots0$ or $10010\ldots0$. We have two cases.

Case 1. $v_4 = 0110\ldots0$. If $r \geq 5$, then it is impossible to find another vertex that is of distance 2 to each of v_1, v_2, v_3 and v_4. So, we only need to consider $r = 4$. But $|N(A)| = 4n - 8 > 4n - 9 = |X|$.

Case 2. $v_4 = 10010\ldots0$. Then, in order for v_5 to be of distance 2 to each of v_1, v_2, v_3 and v_4, we may assume $v_5 = 100010\ldots0$. Indeed, $v_6, \ldots, v_r$ must be of a similar form. Hence, they have a common neighbour $100\ldots0$. The proof is complete. ■

The remaining task is to classify optimal n-component cuts and optimal $(n+1)$-component cuts. Let u be a vertex in Q_n and $u_1, u_2, \ldots, u_n$ be its neighbours. Pick any $n-1$ of them, say, $u_1, u_2, \ldots, u_{n-1}$, then $N(\{u_1, u_2, \ldots, u_{n-1}\})$ is a trivial n-component cut. However, every neighbour of u_n is in $N(\{u_1, u_2, \ldots, u_{n-1}\})$, that is, $N(\{u_1, u_2, \ldots, u_{n-1}\}) = N(\{u_1, u_2, \ldots, u_n\})$. So, if X is a trivial n-component cut, then $Q_n - X$ has n singletons, not $n-1$ singletons. In fact, $Q_n - X$ has a large component and n singletons, giving $n+1$ components in total. This is not surprising as $\kappa_n(Q_n) = \kappa_{n+1}(Q_n)$. Suppose we have proved that all n-component cuts are trivial. Then, since they are also $(n+1)$-component cuts and every optimal $(n+1)$-component cut is also an optimal n-component cut, every optimal $(n+1)$-component cut is trivial. Hence, it remains to be proven that every n-component cut is trivial. The following result establishes this.

THEOREM 3.2 *Let $n \geq 4$ and let S be a set of vertices of size at most $n(n-1)/2 + 1$. Then, the following are the only possibilities.*

(1) *$Q_n - S$ has one large component plus a number of small components with at most $n-2$ vertices in total. (This includes the case that $Q_n - S$ is connected.)*
(2) *$Q_n - S$ has exactly two components, one of which is $K_{1,n}$.*
(3) *$Q_n - S$ has exactly two components, one of which is $K_{1,n-1}$.*
(4) *$Q_n - S$ has exactly two components, one of which is $K_{1,n-2}$.*
(5) *S is a trivial $(n+1)$-component cut, and $Q_n - S$ has one large component plus n singletons.*

Proof We apply induction on n. We first note that the statement is true for $n = 4$, since Q_4 has 16 vertices and it is symmetrical, so the claim can be checked by brute force. (Note also that Theorem 2.2 for $k = 2$ is useful here.) Now, assume that $n \geq 5$ and the claim is true for $n-1$. As before, let H_0 (H_1, respectively) be the subgraph of Q_n induced by vertices with 0 (1, respectively) in the last position. Then, H_0 and H_1 are isomorphic to Q_{n-1}. Let S_0 and S_1 be the set of elements of S that are vertices in H_0 and H_1, respectively. We have two cases.

Case 1. Either $|S_0|$ or $|S_1|$ is at least $|S| = (n-1)(n-2)/2 + 1$. Without loss of generality, we may assume that $|S_1| \geq (n-1)(n-2)/2 + 1$.

Case 1a. $|S_1| \geq (n-1)(n-2)/2 + 2$. Then, $|S_0| \leq n(n-1)/2 + 1 - (n-1)(n-2)/2 - 2 = n-2$, thus $H_0 - S_0$ is connected. Let Y be the component of $Q_n - S$ containing $H_0 - S_0$. Suppose v is a vertex of $H_1 - S_1$. Then, v is part of Y if v's unique neighbour in H_0 is not in S_0. Since $|S_0| \leq n-2$, there are at most $n-2$ vertices in $H_1 - S_1$ that are not part of Y and we are done.

Case 1b. $|S_1| = (n-1)(n-2)/2 + 1$. Hence, $|S_0| = n-1$.

Subcase 1b(i). $H_0 - S_0$ is connected. We define Y as before. We apply the induction hypothesis. The first case is that $H_1 - S_1$ has one large component and a number of small components with at most $n-3$ vertices in total. Now, the large component in $H_1 - S_1$ has at least $2^{n-1} - (n-1)(n-2)/2 - 1 - (n-3)$ vertices, and $|S_0| = n-1$. So, if $2^{n-1} - (n-1)(n-2)/2 - 1 - (n-3) > n-1$, then the large component is part of Y. This inequality holds for $n \geq 5$, and hence $Q_n - S$ has one large component and a number of small components with at most $n-3$ vertices in total.

The second case from the induction hypothesis is that $H_1 - S_1$ has two components, one of which is $K_{1,n-1}$. So, each component has at least n vertices if $n \geq 6$. So, we may assume that $n \geq 6$. But $|S_0| = n-1$. Therefore, each of these components in $H_1 - S_1$ is part of Y and hence $Q_n - S$ is connected. If $n = 5$, then H_1 has 16 vertices, $|S_1| = 7$, and the two components of $H_1 - S_1$ are $K_{1,4}$ and $K_{1,3}$. The $K_{1,4}$ is part of Y. Regardless of whether $K_{1,3}$ is part of Y, $Q_5 - S$ satisfies the claim.

The third case from the induction hypothesis is that $H_1 - S_1$ has two components, one of which is $K_{1,n-2}$. So, it has $n-1$ vertices. However, the other component has at least n vertices as $n \geq 5$. But $|S_0| = n-1$. Therefore, $Q_n - S$ is connected or has two components, one of which is $K_{1,n-2}$.

The fourth case from the induction hypothesis is that $H_1 - S_1$ has two components, one of which is $K_{1,n-3}$. So, it has $n-2$ vertices. However, the other component has at least n vertices as $n \geq 5$. But $|S_0| = n-1$. Therefore, $Q_n - S$ is connected or has two components, one of which is $K_{1,n-3}$. So, $Q_n - S$ has a large component and small components with at most $n-2$ vertices.

The fifth case from the induction hypothesis is that S_1 is a trivial n-component cut of H_1, so $H_1 - S_1$ has a large component and $n-1$ singletons. It is easy to see that the large component has at least n vertices, so it is part of Y. If at least one of these singletons is part of Y, then $Q_n - S$ has small components with at most $n-2$ vertices, and we are done. So, assume none of these singletons are part of Y. Since S_1 is a trivial n-component cut in H_1, there exists a vertex u in H_1 with neighbours $u_1, u_2, \ldots, u_{n-1}$ such that $S_1 = N(\{u_1, u_2, \ldots, u_{n-1}\})$. We will use v' to denote the vertex in H_0 adjacent to v in H_1. If $u_1, u_2, \ldots, u_{n-1}$ remain singletons in $Q_n - S$, then $u'_1, u'_2, \ldots, u'_{n-1} \in S_0$. But $|S_0| = n-1$, so we have identified S_0. But this means that $H_0 - S_0$ is disconnected, which is a contradiction as $H_0 - S_0$ is connected.

Subcase 1b(ii). $H_0 - S_0$ is disconnected. Then, $H_0 - S_0$ consists of two components, one of which is a singleton, say, w. (Apply Theorem 2.2 with $k = 2$.) Let Y be the component in $Q_n - S$ containing the larger component, X, of $H_0 - S_0$. Now, we apply the induction hypothesis on $H_1 - S_1$.

The first case is that $H_1 - S_1$ has one large component and a number of small components with at most $n-3$ vertices in total. Now, the large component in $H_1 - S_1$ has at least $2^{n-1} - (n-1)(n-2)/2 - 1 - (n-3)$ vertices, $|S_0| = n-1$, and there is exactly one vertex in $H_0 - S_0$ that is not in X. So, if $2^{n-1} - (n-1)(n-2)/2 - 1 - (n-3) > n - 1 + 1$, then this large component is part of Y. The inequality holds for $n \geq 5$, and hence $Q_n - S$ has one large component and a number of small components with at most $n-2$ vertices in total. (There are up to $n-3$ vertices from the small components in $H_1 - S_1$ and possibly w.)

The second case from the induction hypothesis is that $H_1 - S_1$ has two components, one of which is $K_{1,n-1}$ with y as the centre. Recall that $H_0 - S_0$ has exactly two components, one of which is a singleton w. The large component Z in $H_1 - S_1$ has $2^{n-1} - (n-1)(n-2)/2 - 1 - n$ vertices, $|S_0| = n-1$, and there is exactly one vertex in $H_0 - S_0$ that is not in X. So, if $2^{n-1} - (n-1)(n-2)/2 - 1 - n > n - 1 + 1$, then this large component is part of Y. The inequality holds for $n \geq 6$, so both X and Z are part of Y if $n \geq 6$. The case $n = 5$ can be handled separately either by brute force or by an *ad hoc* argument. (A brute force approach includes observing that S_0 is the set of neighbours of w and $S_1 = (N(N(y)) - \{y\}) \cup \{t\}$ for some vertex t. Since the hypercube is vertex transitive, we may fix the choice for y, and then there are 16 choices for w and five choices for t.) Henceforth, we may assume that $n \geq 6$. Since $|S_0| = n-1$, the $K_{1,n-1}$ in $H_1 - S_1$ will be part of Y unless for every v of this $K_{1,n-1}$, its unique neighbour in H_0 is either w or the deleted neighbours of w in H_0. But this can only happen if y and w are adjacent, so $Q_n - S$ will either be connected, or have two components, one of which is $K_{1,n}$.

The third case from the induction hypothesis is that $H_1 - S_1$ has two components, one of which is $K_{1,n-2}$, which has $n-1$ vertices. The other component has at least $n+1$ vertices if $n \geq 6$, so it is part of Y. If w is also part of Y, then we are done. As in the second case, one can check that if neither $K_{1,n-2}$ nor w is part of Y, then they form $K_{1,n-1}$ in $Q_n - S$, so the statement is verified. For $n = 5$, the other component in $H_1 - S_1$ is $K_{1,n-1}$, so this reduces to the second case.

The fourth case from the induction hypothesis is that $H_1 - S_1$ has two components, one of which is $K_{1,n-3}$, which has $n-2$ vertices. The other component has at least $n+1$ vertices as $n \geq 5$, so it must be part of Y, and then the claim follows in the usual way.

The fifth case from the induction hypothesis is that S_1 is a trivial n-component cut of H_1, so $H_1 - S_1$ has a large component and $n-1$ singletons. It is easy to see that the large component is part of Y for $n \geq 6$, and for $n = 5$, the large component is $K_{1,4}$, so the case can be easily checked just as the second case. Since S is a trivial n-component cut, there exists a vertex u in H_1 with neighbours $u_1, u_2, \ldots, u_{n-1}$ such that $S_1 = N(\{u_1, u_2, \ldots, u_{n-1}\})$. We will use v' to denote the vertex in H_0 adjacent to v in H_1. We consider three possibilities.

The first possibility is when none of $u_1, u_2, \ldots, u_{n-1}$ are part of Y in $Q_n - S$. Then, there are two additional scenarios. The first scenario is when $u'_1, u'_2, \ldots, u'_{n-1} \in S_0$. But $|S_0| = n-1$, so we have identified S_0. Indeed, $w = u'$, so S is a trivial $(n+1)$-component cut and $Q_n - S$ has a large component and n singletons. The second scenario is when $u'_1, u'_2, \ldots, u'_{n-2} \in S_0$ and u_{n-1} and w are adjacent. But $S_0 = N(w)$, so this is not a possible configuration in Q_n when $n \geq 5$.

The second possibility is when at most $n-3$ of $u_1, u_2, \ldots, u_{n-1}$ are not part of Y in $Q_n - S$. Now, even if w does not belong to Y, we get that $Q_n - S$ has one large component and a number of small components with at most $n-2$ vertices in total.

The third possibility is when $u_1, u_2, \ldots, u_{n-2}$ are not part of Y in $Q_n - S$, but u_{n-1} is part of Y in Q_n. If w is also part of Y, then $Q_n - S$ has one large component and a number of small components with at most $n-2$ vertices in total. If w is not part of Y, then $w' \in \{u_1, u_2, \ldots, u_{n-2}\} \cup S_1$, so we may assume that $u'_1, u'_2, \ldots, u'_{n-3} \in S_0$. But $S_0 = N(w)$, so this is not a possible configuration in Q_n when $n \geq 5$.

Case 2. $|S_1| = (n-1)(n-2)/2$. Hence, $|S_0| = n$. We apply Theorem 2.4(2) to S_1. So, we have to consider two possibilities.

The first possibility is when $H_1 - S_1$ has a large component together with small components of at most $n-3$ vertices in total. Now, $H_0 - S_0$ is either connected or has two components, one of which is a singleton. It is easy to see that the largest component in $H_1 - S_1$ and the larger component in $H_0 - S_0$ (or $H_0 - S_0$ itself if it is connected) are part of the same component in $Q_n - S$. Hence, $Q_n - S$ has a large component and small components with at most $n-2$ vertices in total.

The second possibility is when $H_1 - S_1$ has two components, one of which is $K_{1,n-1}$. Call the other component X. Suppose $H_0 - S_0$ is connected, and let Y be the component in $Q_n - S$ containing it. It is easy to see that X is part of Y. If the $K_{1,n-1}$ component of $H_1 - S_1$ is also part of Y, then we are done as $Q_n - S$ will be connected. Otherwise, the vertices in S_0 must correspond to the neighbours of the $K_{1,n-1}$ component of $H_1 - S_1$, and hence $Q_n - S$ has two components, one of which is $K_{1,n-1}$.

Case 3. $|S_0|, |S_1| \leq (n-1)(n-2)/2 - 1$. So, $|S_0|, |S_1| \geq n+1$. Now, $(n-1)(n-2)/2 - 1 = (n-3)(n-1) - (n-3)(n-2)/2$, so we may apply Theorem 2.2 on S_0 and S_1. We choose a to be the smallest integer such that $|S_0| \leq a(n-1) - a(a+1)/2$ and b to be the smallest integer such that $|S_1| \leq b(n-1) - b(b+1)/2$. Since $|S_0|, |S_1| \geq n$, $a, b \geq 2$. By the choice of a, $|S_0| \geq (a-1)(n-1) - (a-1)a/2 + 1$. Hence, $|S_1| \leq n(n-1)/2 + 1 - (a-1)(n-1) + (a-1)a/2 - 1$. Now, by Theorem 2.2, $H_0 - S_0$ is either connected or has a large component and a number of small components with at most $a-1$ vertices in total. It suffices to show that $H_1 - S_1$ is either connected or has a large component and a number of small components with at most $n-a-1$ vertices in total. This can be accomplished by using Theorem 2.2 and showing that $|S_1| \leq (n-a)(n-1) - (n-a)(n-a+1)/2$. Note that $(n-a)(n-1) - (n-a)(n-a+1)/2 - (n(n-1)/2 - (a-1)(n-1) + (a-1)a/2) = an + a + 2 - 2n - a^2 + 1 = n(a-2) - (a-2)(a+1) - 1 = (a-2)(n-a-1) - 1 \geq 0$ in the given range $2 \leq a \leq n-3$ unless $a = 2$. We repeat the argument from b. Hence, the only exceptional case is $a = b = 2$. But then $|S_0|, |S_1| \leq 2(n-1) - 3 = 2n-5$. Hence, $|S| \leq 4n - 10$. Now, by Theorem 2.4 (with $k = 4$), $Q_n - S$ is either connected or has a large components and a number of small components with at most three vertices in total. Since $n \geq 5$, $3 \leq n - 2$, it satisfies the statement. This completes the proof. ■

To summarize the results of this section, we proved the following.

THEOREM 3.3 *Let $n \geq 2$ and $1 \leq r \leq n$. Then, every optimal $(r + 1)$-component cut of Q_n is trivial.*

4. Conclusion

In this paper, we studied the component connectivity of the hypercube which can be viewed as results that complement the results given in [7–9]. Open problems in this area include finding component connectivity of other interconnection networks such as the star graph [1] and the alternating group graph [5]. However, unlike the hypercube, no corresponding results to those given in [7–9] are known. The closest results for the star graph and the alternating group graph were given in [3,4], but these are asymptotic results.

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Note

1. The *Hamming distance* of two binary strings of the same length is the number of bits that they differ.

References

[1] S.B. Akers, D. Harel, and B. Krishnamurthy, *The star graph: An attractive alternative to the n-cube*, Proceedings of International Conference on Parallel Processing, 1987, pp. 393–400.
[2] G. Chartrand, S.F. Kapoor, L. Lesniak, and D.R. Lick, *Generalized connectivity in graphs*, Bull. Bombay Math. Colloq. 2 (1984), pp. 1–6.
[3] E. Cheng and L. Lipták, *Linearly many faults in Cayley graphs generated by transposition trees*, Inform. Sci. 177 (2007), pp. 4877–4882.
[4] E. Cheng, L. Lipták, and F. Sala, *Linearly many faults in 2-tree-generated networks*, Networks 55 (2010), pp. 90–98.
[5] J.S. Jwo, S. Lakshmivarahan, and S.K. Dhall, *A new class of interconnection networks based on the alternating group*, Networks 23 (1993), pp. 315–326.
[6] E. Sampathkumar, *Connectivity of a graph – A generalization*, J. Combin. Inform. System Sci. 9 (1984), pp. 71–78.
[7] X. Yang, D.J. Evans, and G.M. Megson, *On the maximal connected component of hypercube with faulty vertices (II)*, Int. J. Comput. Math. 81 (2004), pp. 1175–1185.
[8] X. Yang, D.J. Evans, and G.M. Megson, *On the maximal connected component of a hypercube with faulty vertices III*, Int. J. Comput. Math. 83 (2006), pp. 27–37.
[9] X. Yang, D.J. Evans, B. Chen, G.M. Megson, and H. Lai, *On the maximal connected component of hypercube with faulty vertices*, Int. J. Comput. Math. 81 (2004), pp. 515–525.