Anisotropic power-law inflation for the Dirac-Born-Infeld theory

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We find a new set of the Bianchi type I power-law expanding solutions in a string-motivated Dirac-Born-Infeld theory. Stability analysis shows that these power-law inflationary solutions remain stable with or without the contribution of the Dirac-Born-Infeld effect. We also find a new set of Bianchi type I expanding power-law solutions in a two scalar Dirac-Born-Infeld model with an additional phantom field. It is shown that the inclusion of the phantom field turns the Bianchi type I power-law solutions unstable during the inflationary phase.

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I. INTRODUCTION

An inflationary universe is a nice resolution to many important phenomenona coded with cosmic microwave background radiation that is consistent with the observations by the Wilkinson Microwave Anisotropy Probe [1,2]. Research interests have been pretty active in trying to understand the physical origin of the highly isotropic universe. One of the most important predictions associated with the inflation is the cosmic no-hair conjecture. This conjecture claims that all classical hair should disappear once the vacuum energy dominates. Note that the field equations of the gravitational system with a cosmological constant Λ can always be cast as

$$G_{\mu\nu} - T_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{1.1}$$

with the Einstein tensor $G_{\mu\nu}$ representing the geometric impact of the gravitational effect driven by the energy momentum tensor $T_{\mu\nu}$ and the cosmological constant Λ. Gibbons and Hawking [3] and Hawking and Moss [4] claimed that all models with a positive cosmological constant will approach a late time de Sitter space, which was later named as the cosmic no-hair theorem for Einstein gravity. Robert Wald [5] provided a partial proof of this conjecture. It was shown in Ref. [5] that the universe will eventually evolve towards the late time de Sitter spacetime, at least locally, for all non-type-IX Bianchi spaces under certain physical conditions:

(a) there is a positive cosmological constant coupled to the system,

(b) the matter sources obey both the dominant energy condition and the strong-energy condition (SEC).

Note that the dominant energy condition (strong-energy condition) is defined by the inequality $T_{\mu\nu}t^\mu t^\nu \geq 0$ $[(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)t^\mu t^\nu \geq 0]$ for all timelike vectors t^μ [5] with $T_{\mu\nu}$ and T denoting the energy momentum tensor and its trace for all the fields coupled to the gravitational system. It was also shown in Ref. [5] that the type IX Bianchi space behaves similarly provided that Λ is sufficiently large.

Many known examples have been shown to support the cosmic no-hair theorems under a number of different constraints on the field parameters [5–14]. Counterexamples are, however, known to exist. These are examples that the energy conditions do not hold exactly [15–17]. Many of these solutions can be shown to be unstable [10,18–20]. Therefore, these results all appear to support the Hawking's no-hair conjecture. Consequently, it is very important to examine all existing counterexamples to test the validity of the no-hair conjecture.

Some of the studies focus on the effect of the higher derivative corrections [21–52]. Recently, a new set of anisotropic inflationary solutions seems to act as one more counterexample to the no-hair conjecture [53–58]. It shows that, with a vector field coupled with the inflaton, there could be a small anisotropic expansion in the Bianchi type I (BI) space. This set of newly found anisotropic inflation is also shown to be an attractor solution [59].

Analytic power-law solutions can also be found in a model with an exponential scalar potential motivated by supergravity theory [59]. In this approach, the anisotropic hair seems to persist without the presence of a cosmological constant. The one-scalar-field model studied in Ref. [59] will be referred to as the Kanno-Soda-Watanabe (KSW) model in this paper. In the hope that the no-hair conjecture will prevail one way or the other, a phantom field is introduced. The two-scalar-fields model also admits a new set of power-law solutions. As a result, we can show that the phantom field contribution does lead the new set of solutions to collapse.

Note that the Dirac-Born-Infeld (DBI) model motivated by string theory has attracted much attention lately [60–69]. It is known that DBI inflation is driven by the motion of a D3-brane in a warped throat region of a closed and bounded internal space. In addition to a noncanonical kinetic term, the effective action incorporates a potential arising from the quantum interaction between D-branes. In particular, one of the main reasons for the popularity of this model is due to its large non-Gaussianity. Indeed, it was shown that the DBI model has a strict lower bound on the non-Gaussianity of the cosmic microwave background radiation power spectrum [60,61]. More discussions on this subject can be found in Refs. [62–64,70–87].

Therefore, we would like to study the effect of the DBI scalar field on the KSW model. A new set of power-law solutions will be shown to exist shortly in this paper. To investigate the stability of the obtained anisotropic power-law inflation, we will extend the method proposed by Ref. [65] for the stability analysis of the DBI field in the isotropic universe. Stability analysis shows, however, that this new set of solutions is still stable under the power-law perturbations. In fact, we can extract the large-f effect of the perturbation equations. The result shows that this set of inflationary solutions remains stable even if the f term is pretty large.

Therefore, we will turn our attention to the two-scalar-fields DBI model with an additional phantom field coupled to the system [88–90]. We will study the effect of this new model on the stability of the BI space. Indeed, we will show that a new set of power-law anisotropic expanding solutions does exist in the BI space. A detailed stability analysis will also be performed. The result shows that the phantom field does lead to the collapse of this new set of solutions as expected.

This paper will be organized as follows: (i) A brief review of the motivation of this research has been given in Sec. I, (ii) in Sec. II, a one-scalar-field DBI model will be introduced and analyzed, (iii) anisotropic Bianchi type I solutions will be solved in Sec. III, (iv) in Sec. IV, we will show that this new set of inflationary solutions is a set of attractor solutions and remains stable in the presence of the DBI scalar field, (v) the two-scalar-fields DBI model will be discussed in Sec. V. In particular, we will show that the system is unstable in the inflationary phase when the phantom field is introduced. (vi) Finally, concluding remarks will be given in Sec. VI. In addition, some detailed calculations related to the two-scalar-fields model will be presented in the Appendix.

II. DIRAC-BORN-INFELD MODEL

The action of the Dirac-Born-Infeld model ϕ [60–67] we will be interested in is given by

$$S = \int d^4x\sqrt{g}\left[\frac{1}{2}R - \frac{1}{f(\phi)}\left(\sqrt{1+f(\phi)\partial_\mu\phi\partial^\mu\phi} - 1\right) - V(\phi) - \frac{1}{4}h^2(\phi)F_{\mu\nu}F^{\mu\nu}\right], \tag{2.1}$$

with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ the $U(1)$ field tensor, and $V(\phi)$ a non-negative potential arising from quantum interactions between a D3-brane associated with ϕ and other D-branes [65]. $h(\phi)$ is a scalar field potential coupled to the $U(1)$ field [59]. We have also set the Planck scale $M_p = 1$ for convenience. Note that the above DBI f-dependent model will be equivalent to the one-scalar-field model

$$S \to S_{\rm KSW} = \int d^4x\sqrt{g}\left[\frac{1}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - \frac{1}{4}h^2(\phi)F_{\mu\nu}F^{\mu\nu}\right] \tag{2.2}$$

when we take the limit $f \to 0$. This one-scalar-field model will be referred to as the KSW model [59] in this paper.

Similar to Ref. [59], we will also focus on the BI metric given by

$$ds^2 = -dt^2 + \exp[2\alpha(t) - 4\sigma(t)]dx^2 + \exp[2\alpha(t) + 2\sigma(t)](dy^2 + dz^2). \tag{2.3}$$

In addition, the scalar field ϕ and the vector field A_μ will be chosen as $\phi = \phi(t)$ and $A_\mu = (0, A_x(t), 0, 0)$, respectively, consistent with the Bianchi type I homogeneous space.

The field equations of this model can be shown to be

$$\partial_\mu[\sqrt{-g}(h^2(\phi)F^{\mu\nu})] = 0, \tag{2.4}$$

$$\ddot{\phi} = -\frac{3\dot{\alpha}}{\gamma^2}\dot{\phi} - \frac{\partial_\phi V}{\gamma^3} - \frac{\partial_\phi f}{2f}\frac{(\gamma+2)(\gamma-1)}{(\gamma+1)\gamma}\dot{\phi}^2 - \frac{h\partial_\phi h}{2\gamma^3}F_{\mu\nu}F^{\mu\nu}, \tag{2.5}$$

$$\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) - \gamma\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left[\frac{1}{f(\phi)}\left(\frac{1}{\gamma} - 1\right) + V(\phi) + \frac{1}{4}h^2(\phi)F^{\rho\sigma}F_{\rho\sigma}\right] - h^2(\phi)F_{\mu\gamma}F_\nu^\gamma = 0. \tag{2.6}$$

Note that a new variable γ is defined as $\gamma \equiv 1/\sqrt{1 - f(\phi)\dot{\phi}^2}$ characterizing the motion of the brane [65]. It is clear that $\gamma \geq 1$ for non-negative $f(\phi)$ [65].

Equation (2.4) can be solved directly to give the solution of the gauge field as

$$\dot{A}_x(t) = h^{-2}(\phi)\exp[-\alpha - 4\sigma]p_A, \tag{2.7}$$

with p_A a constant of integration. As a result, the scalar field equation (2.5) can be shown to be

$$\ddot{\phi} = -\frac{3\dot{\alpha}}{\gamma^2}\dot{\phi} - \frac{\partial_\phi V}{\gamma^3} - \frac{\partial_\phi f}{2f}\frac{(\gamma+2)(\gamma-1)}{(\gamma+1)\gamma}\dot{\phi}^2 + \frac{h^{-3}\partial_\phi h}{\gamma^3}\exp[-4\alpha - 4\sigma]p_A^2. \tag{2.8}$$

In addition, the metric equation (2.6) reduces to the following component field equations:

$$\dot{\alpha}^2 = \dot{\sigma}^2 + \frac{1}{3}\left[\frac{\gamma^2}{\gamma+1}\dot{\phi}^2 + V(\phi) + \frac{1}{2}h^{-2}\exp[-4\alpha - 4\sigma]p_A^2\right], \tag{2.9}$$

$$\ddot{\alpha} = -3\dot{\alpha}^2 + \left[\frac{\gamma(\gamma-1)}{2(\gamma+1)}\dot{\phi}^2 + V(\phi)\right] + \frac{1}{6}h^{-2}\exp[-4\alpha - 4\sigma]p_A^2, \tag{2.10}$$

$$\ddot{\sigma} = -3\dot{\alpha}\,\dot{\sigma} + \tfrac{1}{3}h^{-2}\exp[-4\alpha - 4\sigma]p_A^2. \tag{2.11}$$

Note that we have used the identity

$$f = \frac{\gamma^2 - 1}{\gamma^2\dot{\phi}^2} \tag{2.12}$$

in deriving the above equations. Note again that these equations will agree with the field equations of the conventional scalar field KSW model discussed in [59] once we take the limit $\gamma \to 1$.

III. ANISOTROPIC POWER-LAW SOLUTIONS

We will focus on the model with the exponential scalar potential of the following form:

$$V(\phi) = V_0\exp[\lambda\phi]. \tag{3.1}$$

The gauge kinetic potentials will also be chosen as

$$h(\phi) = h_0\exp[\rho\phi], \tag{3.2}$$

$$f(\phi) = f_0\exp[\tau\phi]. \tag{3.3}$$

Here we have chosen constants V_0, h_0, f_0, ρ, λ, and τ carrying the boundary information of these potentials.

Note that the DBI theory motivated by string theory has been studied in Refs. [60–64]. In such a theory, inflation can be shown to be driven by the motion of a D3-brane in a warped throat domain of closed and bounded internal space. In particular, $f(p)$ is the inverse of the D3-brane tension with geometrical information about the throat in the internal space [60–65]. The coupled potentials can be of many different forms. In particular, exponential potentials $f(\phi)$ and $V(\phi)$ given in Eqs. (3.1) and (3.3) have been studied in detail in Ref. [65] for their effect on the evolution of the flat Friedmann-Robertson-Walker space. The other forms of potential such as $f(\phi) = \nu|\phi|^{-p}$ and $V(\phi) = \sigma|\phi|^p$ have also been discussed in [65] too. The case where $f(\phi) =$ constant has also been analyzed in Ref. [69].

We will try to find a set of solutions following the ansatz given by [59]:

$$\alpha = \zeta\log(t); \quad \sigma = \eta\log(t); \quad \phi = \xi\log(t) + \phi_0. \tag{3.4}$$

For the latter's convenience, we will also introduce the following new parameters:

$$u = V_0\exp[\lambda\phi_0], \tag{3.5}$$

$$v = p_A^2 h_0^{-2}\exp[-2\rho\phi_0], \tag{3.6}$$

$$\kappa = f_0\exp[\tau\phi_0]. \tag{3.7}$$

It is clear that u, v are positive constants. As a result, γ can be shown to be

$$\gamma = \frac{1}{\sqrt{1 - \xi^2\kappa t^{\tau\xi-2}}}. \tag{3.8}$$

Note that γ becomes a constant if $\tau\xi = 2$. We will set this constant as γ_0 such that

$$\gamma \equiv \gamma_0 = \frac{|\tau|}{\sqrt{\tau^2 - 4\kappa}}. \tag{3.9}$$

Apparently, $\tau^2 > 4\kappa$ is required for the existence of such solutions. This inequality implies that $\tau^2 > 4f_0\exp[\tau\phi_0]$ which could be realized if $f_0 \ll 1$. Hence we will focus on the solutions with $\tau\xi = 2$ from now on. Note that $\gamma > 1$, serving as the Lorentz factor, characterizes the motion of the brane. A constant γ implies a constant velocity of the D3-brane in the throat domain.

The field equations (2.8), (2.9), (2.10), and (2.11) can be reduced to the following equations,

$$-\xi + \frac{3\zeta\xi}{\gamma_0^2} + \frac{\lambda u}{\gamma_0^3} + \frac{(\gamma_0+2)(\gamma_0-1)\xi}{(\gamma_0+1)\gamma_0} - \frac{\rho v}{\gamma_0^3} = 0, \tag{3.10}$$

$$-\zeta^2 + \eta^2 + \frac{\gamma_0^2\xi^2}{3(\gamma_0+1)} + \frac{u}{3} + \frac{v}{6} = 0, \tag{3.11}$$

$$-\zeta + 3\zeta^2 - \frac{\gamma_0(\gamma_0-1)\xi^2}{2(\gamma_0+1)} - u - \frac{v}{6} = 0, \tag{3.12}$$

$$-\eta + 3\zeta\eta - \frac{v}{3} = 0, \tag{3.13}$$

along with the following constraint equations that make all terms in the field equations have the same power in time:

$$\rho\xi + 2\zeta + 2\eta = 1, \tag{3.14}$$

$$\lambda\xi = -2. \tag{3.15}$$

Equation (3.15) and $\tau\xi = 2$ imply that $\tau = -\lambda < 0$.

Recall that $\gamma > 1$, serving as the Lorentz factor, characterizes the motion of the brane. A constant γ implies a constant velocity of the D3-brane in the throat domain. We have shown that the condition $\gamma =$ constant leads naturally to the relation $\tau = -\lambda$. This has to do with the fact that the power-law solution of the form shown above, requiring each term in the field equations to have the same power in time, exists only when this condition is met. We will also show again, in Sec. IV, that this set of power-law solutions is also the fixed point solution of this system under the choice $\gamma =$ constant, or equivalently $\tau = -\lambda$. Therefore this model appears to be the only interesting model with possible fixed point solutions from a dynamical view. Hence we will focus on the model with $\tau = -\lambda$ only [65].

In addition, the choice $\tau = -\lambda$, or equivalently $f(\phi)V(\phi) = f_0 V_0 =$ constant, gives a DBI term of the following form:

$$\frac{1}{f(\phi)}(\sqrt{1 + f(\phi)\partial_\mu \phi \partial^\mu \phi} - 1) = \frac{4}{\lambda^2 \chi^2}[(1 + \partial_\mu \chi \partial^\mu \chi)^{1/2} - 1] \quad (3.16)$$

with $\chi \equiv 2\sqrt{f_0}\exp[-\lambda\phi/2]/\lambda$. Therefore, the parameter $\tau = -\lambda$ represents the strength of coupling shown above. Indeed, if we rewrite the potential $V(\phi)$ as $V(\phi) = 4 f_0 V_0/(\lambda\chi)^2$, as a result, the scalar part of the Lagrangian becomes

$$\frac{1}{f(\phi)}(\sqrt{1 + f(\phi)\partial_\mu \phi \partial^\mu \phi} - 1) + V(\phi) = \frac{4}{\lambda^2 \chi^2}[(1 + \partial_\mu \chi \partial^\mu \chi)^{1/2} - 1 + f_0 V_0]. \quad (3.17)$$

Therefore, the contribution of the effective scalar action becomes negligible when $\lambda \to \infty$. This is consistent with the result of looking at the behavior of $V(\phi)$ in the limit $\lambda \to \infty$. Indeed, V will be very large when $\phi > 0$ in this limit. Note that ϕ tends to decrease under the influence of the exponential potential with positive λ. As a result, once ϕ crosses over the zero point, V will become extremely small and negligible. Therefore, the scalar field will appear to be a free field once $\phi < 0$.

Note that Eq. (3.14) can be written as

$$\eta = \frac{1}{2} - \zeta + \frac{\rho}{\lambda}. \quad (3.18)$$

Hence we can solve Eq. (3.13) as

$$v = -\frac{3(3\zeta - 1)[(2\zeta - 1)\lambda - 2\rho]}{2\lambda} \quad (3.19)$$

with the help of Eq. (3.18). In addition, the constant u can be solved as

$$u = \frac{2(3\zeta - 2)\gamma_0^2}{\lambda^2(\gamma_0 + 1)} - \frac{3[6\rho\lambda\zeta^2 - (6\rho^2 + 5\rho\lambda + 4)\zeta + \rho(\lambda + 2\rho)]\gamma_0}{2\lambda^2(\gamma_0 + 1)} - \frac{3\rho(3\zeta - 1)[(2\zeta - 1)\lambda - 2\rho]}{2\lambda^2(\gamma_0 + 1)}, \quad (3.20)$$

with the help of Eqs. (3.10), (3.18), and (3.19). As a result, ζ can be shown to obey the following equation:

$$6\lambda(\lambda + 2\rho)\zeta^2 - (\lambda^2 + 8\rho\lambda + 12\rho^2 + 8\gamma_0)\zeta = 0. \quad (3.21)$$

Other than a trivial solution $\zeta = 0$, there is another non-trivial solution

$$\zeta = \frac{\lambda^2 + 8\rho\lambda + 12\rho^2 + 8\gamma_0}{6\lambda(\lambda + 2\rho)}. \quad (3.22)$$

Finally, η, u, and v can now be written as functions of the parameters ρ, λ, and γ_0:

$$\eta = \frac{\lambda^2 + 2\rho\lambda - 4\gamma_0}{3\lambda(\lambda + 2\rho)}, \quad (3.23)$$

$$u = u_0 - \frac{2\gamma_0(\gamma_0 - 1)}{(\gamma_0 + 1)\lambda^2}, \quad (3.24)$$

$$v = \frac{(\lambda^2 + 2\rho\lambda - 4\gamma_0)(-\lambda^2 + 4\rho\lambda + 12\rho^2 + 8\gamma_0)}{2\lambda^2(\lambda + 2\rho)^2}, \quad (3.25)$$

with

$$u_0 = \frac{(\rho\lambda + 2\rho^2 + 2\gamma_0)(-\lambda^2 + 4\rho\lambda + 12\rho^2 + 8\gamma_0)}{2\lambda^2(\lambda + 2\rho)^2}. \quad (3.26)$$

If the solutions we are looking for represent expanding solutions, both $\zeta + \eta$ and $\zeta - 2\eta$ have to be positive. It is easy to see that $\zeta + \eta = 1/2 + \rho/\lambda > 0$ implies that

$$\lambda(\lambda + 2\rho) > 0. \quad (3.27)$$

Hence we will assume that both λ and ρ are positive constants. For $\zeta - 2\eta$ to be positive,

$$\zeta - 2\eta = \frac{-\lambda^2 + 4\rho^2 + 8\gamma_0}{2\lambda(\lambda + 2\rho)} > 0 \quad (3.28)$$

implies that

$$4\rho^2 + 8\gamma_0 > \lambda^2. \quad (3.29)$$

As a result, $-\lambda^2 + 4\rho\lambda + 12\rho^2 + 8\gamma_0 > 0$ is ensured. Therefore, the constraint $v > 0$ implies the following constraint:

$$\lambda^2 + 2\rho\lambda > 4\gamma_0. \quad (3.30)$$

By defining $E = \rho/\lambda$ and $F = \gamma_0/(\lambda^2 + 2\lambda\rho)$, we can write the constraints in a more compact form:

$$2E + 1 > 2E + 8F - 1 > 0. \quad (3.31)$$

Note that u_0 is always positive under the above constraints. The constraint $u > 0$ implies, however, the following constraint:

$$u_0\lambda^2 > \frac{2\gamma_0(\gamma_0 - 1)}{(\gamma_0 + 1)}. \quad (3.32)$$

In summary, the field parameters ρ, λ, γ_0 are required to obey three inequalities shown above in Eqs. (3.31) and (3.32). In addition, inflationary solutions will require that $\rho \gg \lambda$.

We can also define the average slow-roll parameter given by [59]

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = \frac{6\lambda(\lambda + 2\rho)}{\lambda^2 + 8\rho\lambda + 12\rho^2 + 8\gamma_0}, \tag{3.33}$$

and the anisotropy defined as

$$\frac{\Sigma}{H} \equiv \frac{\dot{\sigma}}{\dot{\alpha}} = \frac{2(\lambda^2 + 2\rho\lambda - 4\gamma_0)}{\lambda^2 + 8\rho\lambda + 12\rho^2 + 8\gamma_0}. \tag{3.34}$$

The relation between the average slow-roll parameter and the anisotropy is hence given by

$$\frac{\Sigma}{H} = \frac{1}{3} I\varepsilon, \tag{3.35}$$

with

$$I = 3\eta = 1 - \frac{4\gamma_0}{\lambda(\lambda + 2\rho)}. \tag{3.36}$$

Note that the inequality (3.30) leads us to the constraint $0 < I < 1$. Therefore, it is clear that the anisotropy will have to be small for all inflationary solutions.

Moreover, the power-law solutions we found will drive the gauge field to grow according to $\dot{A}_x \propto t^{-(\zeta+4\eta+2\rho\xi)} \propto t^{3\zeta-2}$. As a result, the gauge field grows very fast during the inflationary phase. Therefore the $F^2 = F_{\mu\nu}F^{\mu\nu}$ term in the field equation (2.6) grows as $t^{-4(\zeta+\eta-1)}$. In fact, this is exactly the special feature of the anisotropically expanding power-law solution [59]. It keeps the evolution of anisotropy, characterized by $\dot{\sigma}$ in Eq. (2.11), from decaying to nothing. Indeed, the power-law solutions we found require each single term in Eq. (2.11) to grow as $1/t^2$. The last term in Eq. (2.11) is derived from the h^2F^2 coupling in Eq. (2.6). Therefore, the F^2 term has to explode when $h^2(\phi)$ tends to zero even faster. As a result, the F^2 term can successfully compensate the decreasing rate of the contribution from the h^2 term. Nonetheless, we can only observe the effect of the gauge field through the complete coupling h^2F^2. Indeed, we can only observe $\exp[\lambda\phi]F_{\mu\nu}$ instead of $F_{\mu\nu}$ directly.

In fact, Eq. (2.10) can be interpreted as the total energy shared by the anisotropy (σ), the scalar field (ϕ), and the scalar-vector coupling (h^2F^2), which together drive the expansion of the universe characterized by $\dot{\alpha}^2$. According to Eq. (2.11), the growth of the anisotropy ($\ddot{\sigma}$) can be interpreted as being driven by the nonvanishing contribution from the h^2F^2 term. Indeed, it is easy to see that the trivial solution $\dot{\sigma} = 0$ does not exist in the presence of the nonvanishing coupling h^2F^2. In short, the coupled scalar potential $h(\phi)$ plays a very unique role in the evolution of the anisotropy that attracts our attention.

IV. STABILITY ANALYSIS OF THE EXPANDING SOLUTIONS

A. Dynamical equations

The field equations (2.8), (2.10), and (2.11) can be recombined and cast as a set of a dynamical system with autonomous equations of the following form:

$$\frac{dX}{d\alpha} = \frac{Z^2}{3}(X+1) + X\left[3(X^2-1) + \frac{Y^2}{2\hat{\gamma}}\right], \tag{4.1}$$

$$\frac{dY}{d\alpha} = \hat{\gamma}^3\lambda\left[3(X^2-1) + \frac{Y^2}{\hat{\gamma}(\hat{\gamma}+1)}\right] + 3(X^2-\hat{\gamma}^2)Y + \left[\tau\frac{(2\hat{\gamma}+1)(\hat{\gamma}-1)}{\hat{\gamma}+1} + \frac{Y}{\hat{\gamma}}\right]\frac{Y^2}{2} + \left[\frac{Y}{3} + \hat{\gamma}^3\left(\frac{\lambda}{2} + \rho\right)\right]Z^2, \tag{4.2}$$

$$\frac{dZ}{d\alpha} = Z\left[3(X^2-1) + \frac{Y^2}{2\hat{\gamma}} + \frac{Z^2}{3} - 2X - \rho Y + 1\right], \tag{4.3}$$

$$\frac{d\hat{\gamma}}{d\alpha} = -\frac{(1-\hat{\gamma}^2)Y}{2\hat{\gamma}}\left\{\tau + \frac{2}{Y^2}\frac{dY}{d\alpha} - \frac{2}{Y}\left[3(X^2-1) + \frac{Y^2}{2\hat{\gamma}} + \frac{Z^2}{3} + 3\right]\right\}, \tag{4.4}$$

with an e-folding number as the new time coordinate $d\alpha = \dot{\alpha}dt$ [59,65]. Here we have defined the following dimensionless variables:

$$X = \frac{\dot{\sigma}}{\dot{\alpha}}; \qquad Y = \frac{\dot{\phi}}{\dot{\alpha}}; \qquad Z = h(\phi)\exp[-\alpha + 2\sigma]\frac{\dot{\pi}}{\dot{\alpha}}; \qquad \dot{\pi} = h^{-2}\exp[-\alpha - 4\sigma]p_A. \tag{4.5}$$

We have also defined $\hat{\gamma} = 1/\gamma$ for convenience. Note that Eqs. (4.3) and (4.4) are in fact constraint equations representing the dynamics of the potentials $h(\phi)$ and $f(\phi)$ [59,65]. We have totally three field variables: α, σ, and ϕ. There should be totally three independent field equations for the system. After we parametrize these field variables as dimensionless variables, X, Y, and Z, the second order field equations (2.8), (2.10), and (2.11) reduce to a set of first order differential equations. Moreover, $\hat{\gamma}$ can be regarded as an auxiliary field representing the constraint on γ. As a result, Eq. (4.4) can be regarded as a constraint equation. We will be looking for consistent fixed point solutions for the system. Recall that we have already shown in Sec. III that the choice $\tau = -\lambda$ is in fact equivalent to the choice $\gamma =$ constant. Therefore, we are naturally led to add (4.4) as an additional dynamical equation. The condition will then emerge naturally as the fixed point equation requiring $d\hat{\gamma}/d\alpha = 0$.

In addition to the above set of field equations, we also have the Hamiltonian constraint derived from Eq. (2.9):

$$3(X^2-1) + \frac{Y^2}{\hat{\gamma}(\hat{\gamma}+1)} + \frac{Z^2}{2} = -\frac{V(\phi)}{\dot{\alpha}^2}. \tag{4.6}$$

At the fixed points where $dX/d\alpha = dY/d\alpha = dZ/d\alpha = d\hat{\gamma}/d\alpha = 0$, Eqs. (4.1) and (4.3) can be rearranged as

$$Z^2 = 3X(1-2X) - 3\rho XY, \tag{4.7}$$

$$3(X^2-1)+\frac{Y^2}{2\hat{\gamma}}=2X+\rho Y-1-\frac{1}{3}Z^2. \tag{4.8}$$

As a result, the fixed point equation derived from Eq. (4.4) implies that

$$Y=\frac{4}{\tau-2\rho}(X+1). \tag{4.9}$$

Here we have ignored all trivial solutions such as $\hat{\gamma}=1$, $X=0$, $Y=0$, and $Z=0$ since we are looking for nontrivial solutions. We can eliminate Y and Z from Eq. (4.8) and find that the equation of X becomes a linear equation if $X+1\neq 0$. As a result, X can be solved directly to give

$$X=\frac{2[\hat{\gamma}\tau(\tau-2\rho)-4]}{\hat{\gamma}(\tau^2-8\tau\rho+12\rho^2)+8}. \tag{4.10}$$

Hence we can obtain the fixed point solutions to Y and Z:

$$Y=\frac{12\hat{\gamma}(\tau-2\rho)}{\hat{\gamma}(\tau^2-8\tau\rho+12\rho^2)+8}, \tag{4.11}$$

$$Z^2=\frac{18[\hat{\gamma}\tau(\tau-2\rho)-4][\hat{\gamma}(-\tau^2-4\tau\rho+12\rho^2)+8]}{[\hat{\gamma}(\tau^2-8\tau\rho+12\rho^2)+8]^2}. \tag{4.12}$$

The fixed point equation $dY/d\alpha=0$ derived from Eq. (4.2) therefore gives us the following algebraic constraint on the field parameters:

$$(\lambda+\tau)\hat{\gamma}^3\left[3(X^2-1)+\frac{Y^2}{\hat{\gamma}(\hat{\gamma}+1)}+\frac{Z^2}{2}\right]=0. \tag{4.13}$$

The λ-independent combination in the above equation is nothing but $V(\phi)/\dot{\alpha}^2$ from the Hamiltonian constraint (4.6) . $V(\phi)$ will not vanish for all ϕ. Hence $\tau=-\lambda$ is the only fixed point solution to the dynamical system (4.1), (4.2), (4.3), (4.4), (4.5), and (4.6).

This is a very unique property of the DBI theory with an exponential potential adopted in this paper and also in Ref. [65]. For these models of interest, the only way to admit a stable fixed point solution is to set $\tau=-\lambda$. Recall that we derive this condition because this is the only way that each single term in the field equations will have the same order in time for the power-law solutions. Alternatively, we can also derive this condition from solving the fixed point solution in the dynamical approach. In fact, the dynamical approach reveals enriched information concerning the dynamical details of the system.

As a result, we have a set of fixed point solutions for $\tau=-\lambda$:

$$X=\frac{2[\hat{\gamma}\lambda(\lambda+2\rho)-4]}{\hat{\gamma}(\lambda^2+8\lambda\rho+12\rho^2)+8}, \tag{4.14}$$

$$Y=-\frac{12\hat{\gamma}(\lambda+2\rho)}{\hat{\gamma}(\lambda^2+8\lambda\rho+12\rho^2)+8}, \tag{4.15}$$

$$Z^2=\frac{18[\hat{\gamma}\lambda(\lambda+2\rho)-4][\hat{\gamma}(-\lambda^2+4\lambda\rho+12\rho^2)+8]}{[\hat{\gamma}(\lambda^2+8\lambda\rho+12\rho^2)+8]^2}. \tag{4.16}$$

Indeed, this set of fixed point solutions is exactly the set of power-law solutions we have presented in Sec. III.

In addition, we can also show that this set of fixed point solutions is an attractor solution to the dynamical equation given above. Indeed, the stability of the dynamical system can also be solved analytically in the inflationary phase where $\rho/\lambda \gg 1$ or equivalently $\zeta-2\eta\gg 1$ and $\zeta+\eta\gg 1$. Consequently, it can be shown that $X\simeq\lambda/(3\rho)\ll 1$, $Y\simeq -2/\rho\ll 1$, and $Z^2\simeq 3\lambda/\rho\ll 1$ during the inflationary phase. As a result, we can perturb the dynamical equations around the anisotropic fixed points (X, Y, Z) and obtain the following perturbation equations:

$$\frac{d\delta X}{d\alpha}\simeq -3\delta X, \tag{4.17}$$

$$\frac{d\delta Y}{d\alpha}\simeq -3\hat{\gamma}^2\delta Y+2\hat{\gamma}^3\sqrt{3\lambda\rho}\delta Z, \tag{4.18}$$

$$\frac{d\delta Z}{d\alpha}\simeq -\sqrt{3\lambda\rho}\delta Y-2\delta Z. \tag{4.19}$$

Setting the perturbation as [59,65]

$$\delta X\propto\exp[\hat{\omega}\alpha],\qquad \delta Y\propto\exp[\hat{\omega}\alpha],$$
$$\delta Z\propto\exp[\hat{\omega}\alpha],\qquad \delta\hat{\gamma}\propto\exp[\hat{\omega}\alpha], \tag{4.20}$$

the following eigenvalues $\hat{\omega}$ can be easily solved: $\hat{\omega}_1=0$, $\hat{\omega}_2=-3$, and

$$\hat{\omega}_{3,4}=-\frac{2+3\hat{\gamma}^2\pm\sqrt{(2+3\hat{\gamma}^2)^2-24\hat{\gamma}^2(1+\lambda\rho\hat{\gamma})}}{2}\leq 0. \tag{4.21}$$

Therefore, all modes are apparently stable modes. As a result, we can show that the fixed points we found are attractor solutions. In particular, the numerical result shown in Fig. 1 also shows clearly that the fixed point solution is indeed an attractor solution.

B. Power-law perturbations

We can show directly from the dynamical equation given above that the fixed point solutions are stable solutions following the method presented in Refs. [59,65]. The proof will not be easy once we introduce an additional phantom field later in Sec. V. Indeed, the perturbation method can only predict whether a solution is stable or not near the fixed point. It will not predict whether such a solution is an attractor solution or not. The perturbation method can, however, still be very helpful if we only want to know whether a fixed point is unstable. In such cases, the perturbation method turns out to be a very convenient method. In order to study the stability of this set of expanding solutions

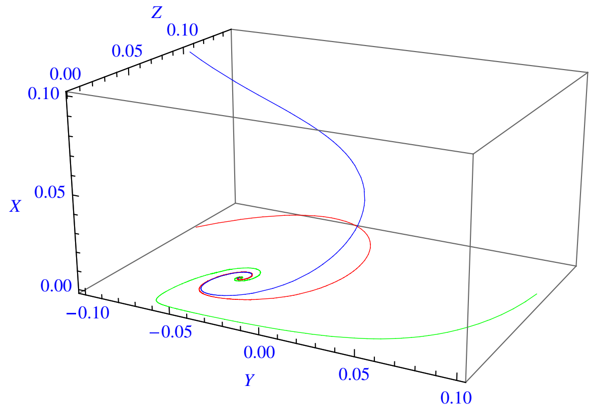

FIG. 1 (color online). The phase flow in the phase space span by X, Y, Z depicted for $\lambda = 0.1$, $\rho = 50$, and $\gamma = 1.5$ indicates that the trajectories converge to the anisotropic fixed point.

near the fixed point and compare it with a more complicated system involving additional phantom field coupling, we will consider the power-law perturbations of the following fields: $\delta\alpha = A_\alpha t^n$, $\delta\sigma = A_\sigma t^n$, $\delta\phi = A_\phi t^n$ [91]. In particular, $\delta\gamma$ can be shown to be

$$\delta\gamma = -\frac{2}{\lambda}\kappa\gamma_0^3(1+n)A_\phi t^n. \tag{4.22}$$

Note that the parameter κ can be written as a function of γ_0 following Eq. (3.9):

$$\kappa = \frac{(\gamma_0^2 - 1)\lambda^2}{4\gamma_0^2}. \tag{4.23}$$

Consequently, we can show that the perturbation of γ becomes

$$\delta\gamma = \left(\frac{1+n}{2}\right)\gamma_0(1-\gamma_0^2)\lambda A_\phi t^n. \tag{4.24}$$

In addition, we will also define the following parameters for convenience:

$$Y_1 = -\frac{(\gamma_0 - 1)(2\gamma_0 + 1)}{\gamma_0(\gamma_0 + 1)}\lambda, \tag{4.25}$$

$$Y_2 = -\frac{\gamma_0(\gamma_0 - 1)(\gamma_0^2 + 2\gamma_0 - 1)}{2(\gamma_0 + 1)}\lambda, \tag{4.26}$$

$$Y_m = \frac{m\gamma_0^m(1-\gamma_0^2)}{2}\lambda, \quad \text{for } m = -2, -3, \tag{4.27}$$

$$M = \frac{(\gamma_0 + 2)(\gamma_0 - 1)}{\gamma_0(\gamma_0 + 1)}, \tag{4.28}$$

$$N = \frac{\gamma_0(\gamma_0 - 1)}{(\gamma_0 + 1)}. \tag{4.29}$$

With the help of these parameters we can show that the perturbation of Eqs. (2.8), (2.10), and (2.11) can be written as

$$A_{11}A_\alpha + A_{12}A_\sigma + A_{13}A_\phi = 0, \tag{4.30}$$

$$A_{21}A_\alpha + A_{22}A_\sigma + A_{23}A_\phi = 0, \tag{4.31}$$

$$A_{31}A_\alpha + A_{32}A_\sigma + A_{33}A_\phi = 0, \tag{4.32}$$

with

$$A_{11} = \left(\frac{6\gamma_0^{-2}n}{\lambda} - 4\gamma_0^{-3}\rho v\right); \qquad A_{12} = -4\gamma_0^{-3}\rho v;$$
$$A_{13} = \bigg\{-n^2 + \bigg[1 - 2M + \frac{2Y_1}{\lambda} - 3\gamma_0^{-2}\zeta + \frac{6}{\lambda}Y_{-2}\zeta + Y_{-3}(-\lambda u + \rho v)\bigg]n + \frac{2Y_1}{\lambda} + \frac{6}{\lambda}Y_{-2}\zeta - \gamma_0^{-3}(\lambda^2 u + 2\rho^2 v) + Y_{-3}(-\lambda u + \rho v)\bigg\}; \tag{4.33}$$

$$A_{21} = -\left[n^2 + (6\zeta - 1)n + \frac{2v}{3}\right]; \qquad A_{22} = -\frac{2v}{3};$$
$$A_{23} = -\left(\frac{2Nn}{\lambda} - \frac{2Y_2 n}{\lambda^2} - \lambda u + \frac{\rho v}{3} - \frac{2Y_2}{\lambda^2}\right); \tag{4.34}$$

$$A_{31} = -\left(3\eta n + \frac{4v}{3}\right); \quad A_{32} = -\left[n^2 + (3\zeta - 1)n + \frac{4v}{3}\right];$$
$$A_{33} = -\frac{2\rho v}{3}. \tag{4.35}$$

These equations can hence be written as a matrix equation,

$$\mathcal{D}\begin{pmatrix} A_\alpha \\ A_\sigma \\ A_\phi \end{pmatrix} \equiv \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\begin{pmatrix} A_\alpha \\ A_\sigma \\ A_\phi \end{pmatrix} = 0, \tag{4.36}$$

As a result, a nontrivial solution exists only when

$$\det\mathcal{D} = 0. \tag{4.37}$$

It is straightforward to show that this determinant equation gives a degree 6 polynomial equation of n of the following form:

$$n(-n^5 + \ldots + a_1) = 0. \tag{4.38}$$

In addition, it is also straightforward to show that the coefficient a_1 is

$$a_1 = -2v(\gamma_0^{-3}\lambda^2 u + Y_{-3}\lambda u - Y_{-3}\rho v - 6Y_{-2}\lambda^{-1}\zeta - 2Y_1\lambda^{-1})(5\zeta - \eta - 1) - 4\gamma_0^{-3}v(u + 2Y_2\lambda^{-3}) \times[(3\zeta - 3\eta - 1)\rho\lambda - 2\gamma_0]. \tag{4.39}$$

Therefore, the necessary condition for the existence of a nontrivial solution n is that n has to obey the following equation:

$$f(n) = n^5 - a_5 n^4 - a_4 n^3 - a_3 n^2 - a_2 n + b_1 = 0 \tag{4.40}$$

with $b_1 = -a_1$ and

$$\frac{\gamma_0^3\lambda^2}{v}b_1 = \left\{2\lambda^2 u_0 + 3(\gamma_0^2-1)\left[\lambda^2 u_0 - (\rho\lambda v + 4\gamma_0\zeta) + \frac{2}{3}\gamma_0\right]\right\}[5\zeta - \eta - 1]\lambda^2 + 4[\lambda^2 u_0 - \gamma_0(\gamma_0^2-1)][(3\zeta - 3\eta - 1)\rho\lambda - 2\gamma_0]. \tag{4.41}$$

Note again that a_1 reduces to the coefficient c_1 in the stability equation of the KSW model when we take the limit $\gamma_0 = 1$ [91]. It is known that the polynomial equation $f(n) = 0$ will have at least one positive root if $b_1 < 0$. On the contrary, if $b_1 \geq 0$, the property of the roots to the equation $f(n) = 0$ is comparably difficult to predict without the highly complex analysis of the coefficients a_2, a_3, a_4, and a_5.

Furthermore, if the expanding solutions represent inflationary solutions, it will require that $\zeta - 2\eta \gg 1$ and $\zeta + \eta \gg 1$. In particular, $\zeta + \eta \gg 1$ implies that

$$\rho \gg \lambda. \tag{4.42}$$

In addition, the parameters ζ, u_0, and v behave as

$$\zeta \simeq \frac{\rho}{\lambda}, \qquad \eta \simeq \frac{1}{3}, \qquad u_0 \simeq 3\zeta^2, \qquad v \simeq 3\zeta, \tag{4.43}$$

in the inflationary phase. Note that $\zeta \simeq \rho/\lambda + 1/6$ with $1/6$ much smaller as compared to $\rho/\lambda \gg 1$. Therefore the small factor $1/6$ is omitted in the above approximation. In addition, $\eta = 1/3$ really means that $\eta \simeq O(1)$. The DBI term is considered as the leading correction to the KSW scalar field model. Therefore, γ is expected to be close to the limit $\gamma = 1$. Hence we will assume that γ_0 is of the order of 1, $\gamma_0 \simeq O(1)$, as compared to the scaling factor $\zeta \simeq \rho/\lambda$. For simplicity, we will also assume λ is also of the order of 1, $\lambda \simeq O(1)$. As a result, we can examine the approximated behavior of the coefficient b_1 during the inflationary phase:

$$\frac{\gamma_0^3\lambda^2}{v}b_1 \simeq 6\lambda^4\zeta^3[6\zeta + 5]. \tag{4.44}$$

It is apparent that $b_1 > 0$ during the inflationary phase unless γ_0 is extremely large. However, γ_0 cannot be too large because of the constraint $v > 0$. This constraint implies that $\lambda^2 + 2\rho\lambda > 4\gamma_0$ following Eq. (3.30). In order to extract the contribution of the γ_0 terms, we can assume that $\gamma_0 \simeq \rho\lambda$. If γ_0/λ^2 is of the same order as the term ρ/γ_0, v will pick up another correction: $v \simeq 3\zeta - 6\gamma_0/\lambda^2$. Therefore, by counting powers of γ_0/λ^2 and ρ/λ, the leading order terms that are large in the γ_0 contribution can be shown to come from those terms that are proportional to $(\gamma_0^2 - 1)$. Therefore, the DBI effect can be extracted by collecting all the dominating terms that are proportional to $\gamma_0^2 - 1$ in b_1.

It is then quite straightforward to show that the dominating effect of the large γ_0 is in fact positive definite:

$$\frac{\gamma_0^3\lambda^2}{v}\tilde{b}_1 = 18\gamma_0^3\rho^2 > 0. \tag{4.45}$$

Therefore, the γ contribution will not affect the positivity of the coefficient b_1. As a result, the inflationary power-law solutions in the BI space appear to be stable for the DBI model. Hence, we will turn our attention to the effect of an additional scalar field on the stability property of similar power-law expanding solutions in Bianchi type I space. Fortunately, we can also find a new set of power-law solutions for the two-scalar-fields DBI model.

V. THE TWO-SCALAR-FIELDS DBI MODEL

A. The model

The action of the two-scalar-fields DBI model [59–67] mentioned earlier is given by the following action:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2} - \frac{1}{f(\phi)}\left(\sqrt{1 + f(\phi)\partial_\mu\phi\partial^\mu\phi} - 1\right) - \frac{\omega}{2}\partial_\mu\psi\partial^\mu\psi - V_1(\phi) - V_2(\psi) - \frac{1}{4}h^2(\phi,\psi)F_{\mu\nu}F^{\mu\nu}\right]. \tag{5.1}$$

Here ψ is either a canonical or phantom field depending on the sign of ω. Field equations of the action (5.1) can be shown to be

$$\partial_\mu[\sqrt{-g}(h^2F^{\mu\nu})] = 0, \tag{5.2}$$

$$\ddot{\phi} = -\frac{3H}{\gamma^2}\dot{\phi} - \frac{\partial_\phi V_1}{\gamma^3} - \frac{\partial_\phi f}{2f}\frac{(\gamma+2)(\gamma-1)}{(\gamma+1)\gamma}\dot{\phi}^2 - \frac{h\partial_\phi h}{2\gamma^3}F_{\mu\nu}F^{\mu\nu}, \tag{5.3}$$

$$\ddot{\psi} = -3H\dot{\psi} - \frac{\partial_\psi V_2}{\omega} - \frac{h\partial_\psi h}{2\omega}F_{\mu\nu}F^{\mu\nu}, \tag{5.4}$$

$$\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) - \gamma\partial_\mu\phi\partial_\nu\phi - \omega\partial_\mu\psi\partial_\nu\psi + \frac{\omega}{2}g_{\mu\nu}\partial^\sigma\psi\partial_\sigma\psi + g_{\mu\nu}\left[\frac{1}{f}\left(\frac{1}{\gamma} - 1\right) + V_1 + V_2 + \frac{h^2}{4}F^{\rho\sigma}F_{\rho\sigma}\right] - h^2F_{\mu\gamma}F_\nu^\gamma = 0. \tag{5.5}$$

Similarly, we will consider the exponential potentials,

$$V_1(\phi) = V_{01}\exp[\lambda_1\phi], \tag{5.6}$$

$$V_2(\psi) = V_{02}\exp[\lambda_2\psi], \tag{5.7}$$

and the exponential gauge kinetic functions,

$$h(\phi,\psi) = h_0\exp[\rho_1\phi]\exp[\rho_2\psi], \tag{5.8}$$

$$f(\phi) = f_0 \exp[\tau\phi], \tag{5.9}$$

with V_{0i}, h_0, f_0, ρ_i, λ_i, τ as constants specified by the corresponding boundary values. Note that Eq. (5.2) can still be solved to give

$$\dot{A}_x(t) = h^{-2} \exp[-\alpha - 4\sigma] p_A, \tag{5.10}$$

with p_A a constant of integration. We will be looking for power-law solutions of the following forms [59]:

$$\alpha = \zeta \log(t); \qquad \sigma = \eta \log(t);$$
$$\phi = \xi_1 \log(t) + \phi_0; \qquad \psi = \xi_2 \log(t) + \psi_0. \tag{5.11}$$

For convenience, we will also introduce the following new variables:

$$u_1 = V_{01} \exp[\lambda_1 \phi_0], \tag{5.12}$$

$$u_2 = V_{02} \exp[\lambda_2 \psi_0], \tag{5.13}$$

$$\tilde{v} = p_A^2 h_0^{-2} \exp[-2\rho_1\phi_0 - 2\rho_2\psi_0], \tag{5.14}$$

$$\kappa = f_0 \exp[\tau\phi_0]. \tag{5.15}$$

Note that u_i and $\tilde{v}$ are all positive parameters. We can show that (see the Appendix for details) γ is a constant when $\tau\xi_1 = 2$:

$$\gamma \equiv \gamma_0 = \frac{|\tau|}{\sqrt{\tau^2 - 4\kappa}}. \tag{5.16}$$

Similar to the discussion in the previous section, we will also focus on the special case that $\tau\xi_1 = 2$. With the ansatz given by Eq. (5.11), the whole set of the field equations (A1)–(A5) can be solved to give the following solutions given by the parameters

$$\zeta = \frac{4(\lambda_1\rho_2 + \lambda_2\rho_1)(2\lambda_1\lambda_2 + 3\lambda_1\rho_2 + 3\lambda_2\rho_1) + \lambda_1^2\lambda_2^2 + 8(\omega\lambda_1^2 + \gamma_0\lambda_2^2)}{6\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)}, \tag{5.17}$$

$$\eta = \frac{\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1) - 4(\omega\lambda_1^2 + \gamma_0\lambda_2^2)}{3\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)}, \tag{5.18}$$

$$u_1 = \tilde{u}_0 - \frac{2\gamma_0(\gamma_0 - 1)}{(\gamma_0 + 1)\lambda_1^2}, \tag{5.19}$$

$$u_2 = \frac{\Omega \times [\lambda_1^2(\lambda_2\rho_2 + 2\rho_2^2 + 2\omega) + 2\lambda_1\lambda_2\rho_1\rho_2 + 4(\omega\lambda_1\rho_1 - \gamma_0\lambda_2\rho_2)]}{2[\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)]^2}, \tag{5.20}$$

$$\tilde{v} = \frac{\Omega \times [\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1) - 4(\omega\lambda_1^2 + \gamma_0\lambda_2^2)]}{2[\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)]^2}, \tag{5.21}$$

with

$$\tilde{u}_0 = \frac{\Omega \times [\lambda_2^2(\lambda_1\rho_1 + 2\rho_1^2 + 2\gamma_0) + 2\lambda_1\lambda_2\rho_1\rho_2 - 4(\omega\lambda_1\rho_1 - \gamma_0\lambda_2\rho_2)]}{2[\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)]^2}, \tag{5.22}$$

$$\Omega = 4(\lambda_1\rho_2 + \lambda_2\rho_1)(\lambda_1\lambda_2 + 3\lambda_1\rho_2 + 3\lambda_2\rho_1) - \lambda_1^2\lambda_2^2 + 8(\omega\lambda_1^2 + \gamma_0\lambda_2^2). \tag{5.23}$$

t It can then be shown that $\zeta + \eta = 1/2 + (\lambda_1\rho_2 + \lambda_2\rho_1)/(\lambda_1\lambda_2)$ is always positive for positive λ_i and ρ_i. Therefore, we can also show that

$$\zeta - 2\eta = \frac{4(\lambda_1\rho_2 + \lambda_2\rho_1)^2 - \lambda_1^2\lambda_2^2 + 8(\omega\lambda_1^2 + \gamma_0\lambda_2^2)}{2\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)}. \tag{5.24}$$

Consequently, the constraint of the expanding solutions with $\zeta - 2\eta > 0$ implies that

$$4(\lambda_1\rho_2 + \lambda_2\rho_1)^2 - \lambda_1^2\lambda_2^2 + 8(\omega\lambda_1^2 + \gamma_0\lambda_2^2) > 0. \tag{5.25}$$

In addition, $\Omega > 0$ if the inequality (5.25) holds. Moreover, $u_1 > 0$ implies that

$$\tilde{u}_0 > \frac{2\gamma_0(\gamma_0 - 1)}{(\gamma_0 + 1)\lambda_1^2}. \tag{5.26}$$

Note that u_0 is always positive if $\omega = -1$. Hence, for convenience, we will focus on the case that $\omega = -1$; namely, ψ will be assumed to be a phantom field from now on. In addition, the positivity of the parameters u_2 and $\tilde{v}$ implies two more constraints:

$$\lambda_1^2(\lambda_2\rho_2 + 2\rho_2^2 + 2\omega) + 2\lambda_1\lambda_2\rho_1\rho_2 + 4(\omega\lambda_1\rho_1 - \gamma_0\lambda_2\rho_2) > 0, \tag{5.27}$$

$$\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1) - 4(\omega\lambda_1^2 + \gamma_0\lambda_2^2) > 0. \quad (5.28)$$

Similarly to earlier results, we can also write the inequalities (5.25) and (5.28) in a more comprehensive form as

$$2\tilde{E} + 1 > 2\tilde{E} - 8\tilde{F} - 1 > 0, \quad (5.29)$$

by defining the following parameters:

$$\tilde{E} = \frac{\lambda_1\rho_2 + \lambda_2\rho_1}{\lambda_1\lambda_2},$$
$$\tilde{F} = -\frac{\omega\lambda_1^2 + \gamma_0\lambda_2^2}{\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + \lambda_2\rho_1)} > -\frac{1}{4}. \quad (5.30)$$

In summary, we have totally four independent inequalities to be observed: (5.26) and (5.27), if the power-law solutions we found are to represent expanding solutions in the BI space.

B. Stability analysis of the expanding solutions

In order to study the stability of this set of expanding solutions, we will consider the power-law perturbations of the following fields: $\delta\alpha = B_\alpha t^n$, $\delta\sigma = B_\sigma t^n$, $\delta\phi = B_\phi t^n$, $\delta\psi = B_\psi t^n$ [91]. Similarly, $\delta\gamma$ can also be shown to be

$$\delta\gamma = \left(\frac{1+n}{2}\right)\gamma_0(1-\gamma_0^2)\lambda_1 B_\phi t^n. \quad (5.31)$$

Note that we are trying to show that this new set of solutions is composed of unstable solutions; therefore power-law perturbation will be shown to be good enough for this task. Moreover, the perturbation equations of Eqs. (A1), (A2), (A4), and (A5) also form a set of algebraic equations that can be written as a matrix equation:

$$\tilde{\mathcal{D}}\begin{pmatrix} B_\alpha \\ B_\sigma \\ B_\phi \\ B_\psi \end{pmatrix} \equiv \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix}\begin{pmatrix} B_\alpha \\ B_\sigma \\ B_\phi \\ B_\psi \end{pmatrix} = 0, \quad (5.32)$$

with

$$B_{11} = \left(\frac{6\gamma_0^{-2}n}{\lambda_1} - 4\gamma_0^{-3}\rho_1\tilde{v}\right); \qquad B_{12} = -4\gamma_0^{-3}\rho_1\tilde{v};$$
$$B_{13} = \left\{-n^2 + \left[1 - 2M + \frac{2Y_1}{\lambda_1} - 3\gamma_0^{-2}\zeta + \frac{6}{\lambda_1}Y_{-2}\zeta + Y_{-3}(-\lambda_1 u_1 + \rho_1\tilde{v})\right]n + \frac{2Y_1}{\lambda_1} + \frac{6}{\lambda_1}Y_{-2}\zeta - \gamma_0^{-3}(\lambda_1^2 u_1 + 2\rho_1^2\tilde{v}) + Y_{-3}(-\lambda_1 u_1 + \rho_1\tilde{v})\right\};$$
$$B_{14} = -2\gamma_0^{-3}\rho_1\rho_2\tilde{v}; \quad (5.33)$$

$$B_{21} = \left(\frac{6n}{\lambda_2} + 4\rho_2\tilde{v}\right); \quad B_{22} = 4\rho_2\tilde{v}; \quad B_{23} = -2\rho_1\rho_2\tilde{v};$$
$$B_{24} = -[n^2 + (3\zeta - 1)n - \lambda_2^2 u_2 - 2\rho_2^2\tilde{v}]; \quad (5.34)$$

$$B_{31} = -\left[n^2 + (6\zeta - 1)n + \frac{2\tilde{v}}{3}\right]; \qquad B_{32} = -\frac{2\tilde{v}}{3};$$
$$B_{33} = -\left(\frac{2Nn}{\lambda_1} - \frac{2Y_2 n}{\lambda_1^2} - \lambda_1 u_1 + \frac{\rho_1\tilde{v}}{3} - \frac{2Y_2}{\lambda_1^2}\right);$$
$$B_{34} = \lambda_2 u_2 - \frac{\rho_2\tilde{v}}{3}; \quad (5.35)$$

$$B_{41} = -\left(3\eta n + \frac{4\tilde{v}}{3}\right); \quad B_{42} = -\left[n^2 + (3\zeta - 1)n + \frac{4\tilde{v}}{3}\right];$$
$$B_{43} = -\frac{2\rho_1\tilde{v}}{3}; \quad B_{44} = -\frac{2\rho_2\tilde{v}}{3}. \quad (5.36)$$

Note that we have used the same notations for Y_m, M, and N as defined in Eqs. (4.25), (4.26), (4.27), (4.28), and (4.29) except that λ is replaced by λ_1 in this section for the two-scalar-fields model.

Since nontrivial solutions exist only when

$$\det\tilde{\mathcal{D}} = 0, \quad (5.37)$$

it is straightforward to show that this determinant equation gives a degree 8 polynomial equation of n in the following form:

$$n(n^7 + \ldots + c_1) = 0, \quad (5.38)$$

with

$$c_1 = K_1(5\zeta - \eta - 1) + K_2(3\zeta - 3\eta - 1) + K_3. \quad (5.39)$$

The parameters K_i are defined, respectively, as

$$K_1 = -2\lambda_2^2 u_2\tilde{v}\left(\frac{\lambda_1^2 u_1}{\gamma_0^3} + \lambda_1 Y_{-3}u_1 - \rho_1 Y_{-3}\tilde{v} - 6\frac{Y_{-2}\zeta}{\lambda_1} - 2\frac{Y_1}{\lambda_1}\right), \quad (5.40)$$

$$K_2 = -4\lambda_2 u_2\tilde{v}\left\{\frac{1}{\gamma_0^3}[\lambda_1(\lambda_1\rho_2 + \lambda_2\rho_1)u_1 + 4\rho_1^2\rho_2\tilde{v}] - \rho_2\left[6\frac{Y_{-2}\zeta}{\lambda_1} - Y_{-3}(\lambda_1 u_1 - \rho_1\tilde{v})\right] + \frac{2}{\lambda_1}\left(\frac{\lambda_2}{\lambda_1}\frac{\rho_1}{\gamma_0^3}Y_2 - \rho_2 Y_1\right)\right\}, \quad (5.41)$$

$$K_3 = 8u_2\tilde{v}\left\{-\frac{1}{\gamma_0^2}\left[(\lambda_1^2 - \gamma_0\lambda_2^2)\frac{u_1}{\gamma_0} - 4\frac{\lambda_2}{\lambda_1}\rho_1\rho_2\tilde{v}\right] - Y_{-3}(\lambda_1 u_1 - \rho_1\tilde{v}) + 6\frac{Y_{-2}}{\lambda_1}\zeta + \frac{2}{\lambda_1}\left(\frac{\lambda_2^2}{\gamma_0^2\lambda_1^2}Y_2 + Y_1\right)\right\}. \quad (5.42)$$

From the definitions of the parameters Y_m, M, and N defined by Eqs. (4.25), (4.26), (4.27), (4.28), and (4.29), we can rewrite K_1, K_2, and K_3 explicitly as functions of the parameters λ_i and ρ_i:

$$K_1 = -\frac{\lambda_2^2 u_2 \tilde{v}}{\gamma_0^3}\bigg\{(3\gamma_0^2 - 1)\lambda_1^2 u_1 - 3(\gamma_0^2 - 1)[\lambda_1\rho_1\tilde{v} + 4\gamma_0\zeta] + 4\frac{(\gamma_0 - 1)(2\gamma_0 + 1)\gamma_0^2}{\gamma_0 + 1}\bigg\}, \tag{5.43}$$

$$K_2 = -\frac{4\lambda_2 u_2\tilde{v}}{\gamma_0^3}\bigg\{(\lambda_1\rho_2 + \lambda_2\rho_1)\lambda_1 u_1 + 4\rho_1^2\rho_2\tilde{v} + \frac{3}{2}(\gamma_0^2 - 1)\rho_2[\lambda_1(\lambda_1 u_1 - \rho_1\tilde{v}) - 4\gamma_0\zeta] - \frac{(\gamma_0 - 1)\gamma_0^2}{\gamma_0 + 1}\bigg[\bigg(\frac{\gamma_0^2 + 2\gamma_0 - 1}{\gamma_0}\bigg)\frac{\lambda_2}{\lambda_1}\rho_1 - 2(2\gamma_0 + 1)\rho_2\bigg]\bigg\}, \tag{5.44}$$

$$K_3 = \frac{4u_2\tilde{v}}{\gamma_0^3}\bigg\{3(\gamma_0^2 - 1)[4\gamma_0\zeta - \lambda_1(\lambda_1 u_1 - \rho_1\tilde{v})] + 2\bigg[-\lambda_1^2 u_1 + \gamma_0\lambda_2\bigg(\lambda_2 u_1 + 4\frac{\rho_1}{\lambda_1}\rho_2\tilde{v}\bigg)\bigg] - \frac{2(\gamma_0 - 1)\gamma_0^2}{(\gamma_0 + 1)\lambda_1^2}[2(2\gamma_0 + 1)\lambda_1^2 + (\gamma_0^2 + 2\gamma_0 - 1)\lambda_2^2]\bigg\}. \tag{5.45}$$

Furthermore, if the expanding solutions represent inflationary solutions, they will require that $\zeta - 2\eta \gg 1$ and $\zeta + \eta \gg 1$. First of all, $\zeta + \eta \gg 1$ implies that

$$\lambda_1\rho_2 + \lambda_2\rho_1 \gg \lambda_1\lambda_2. \tag{5.46}$$

As a result, the parameters ζ, η, $\tilde{u}_0$, u_2, $\tilde{v}$ can be shown to behave as

$$\zeta \simeq \frac{\rho_1}{\lambda_1} + \frac{\rho_2}{\lambda_2}, \qquad \eta \simeq \frac{1}{3}, \qquad \lambda_1\tilde{u}_0 \simeq 3\rho_1\zeta, \qquad \lambda_2 u_2 \simeq 3\rho_2\zeta, \qquad \tilde{v} \simeq 3\zeta, \tag{5.47}$$

during the inflationary phase. In addition, when γ_0 is close to 1, we can extract the leading order contribution of K_1, K_2, and K_3 as

$$K_1 \simeq -2\lambda_2^2 u_2\tilde{v}\lambda_1^2 u_1 < 0,$$
$$K_2 \simeq -4\lambda_2 u_2\tilde{v}[(\lambda_1\rho_2 + \lambda_2\rho_1)\lambda_1 u_1 + 4\rho_1^2\rho_2\tilde{v}] < 0,$$
$$K_3 \simeq 8u_2\tilde{v}\bigg[(\gamma_0\lambda_2^2 - \lambda_1^2)u_1 + 4\gamma_0\frac{\lambda_2}{\lambda_1}\rho_1\rho_2\tilde{v}\bigg]. \tag{5.48}$$

During the inflationary phase, K_1 and K_2 terms will dictate the sign of the coefficient c_1. Therefore, it is clear that c_1 is negative during the inflationary phase under the condition $\lambda_1\rho_2 + \lambda_2\rho_1 \gg \lambda_1\lambda_2$. As a result, the power-law perturbation admits at least a positive mode with $n > 0$. Hence, the power-law solutions become unstable consistent with the no-hair conjecture [3–5]. Note again that the result shown in this section reduces to the result shown in Ref. [91] in the limit $\gamma_0 \to 1$.

Similar to the arguments of the large-γ_0 effect, we wish to know the impact of the γ_0 terms in c_1. γ_0 can not be too large because of the constraint $\tilde{v} > 0$. This constraint implies that $\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1) + 4\lambda_1^2 > 4\gamma_0\lambda_2^2$ following the constraint Eq. (A25). In order to extract the contribution of the γ_0 terms, we can assume that $\gamma_0/\lambda_1^2 \simeq \zeta \simeq \rho_1/\lambda_1 + \rho_2/\lambda_2$. For convenience, we will define $\tilde{\gamma} = \gamma_0/\lambda_1^2$, $\tilde{\rho}_1 = \rho_1/\lambda_1$, $\tilde{\rho}_2 = \rho_2/\lambda_2$, and $\tilde{\rho} = \rho_1/\lambda_1 + \rho_2/\lambda_2$. We will also assume that they are all of the same order, i.e., $\tilde{\gamma} \simeq \tilde{\rho} \simeq \tilde{\rho}_1 \simeq \tilde{\rho}_2$.

As a result, the parameters ζ, η, $\tilde{u}_0$, u_2, $\tilde{v}$ behave as

$$\zeta \simeq \tilde{\rho}, \qquad \eta \simeq \tfrac{1}{3}, \qquad \tilde{u}_0 \simeq 3\tilde{\rho}_1\tilde{\rho} + 6\tilde{\gamma}\tilde{\rho}_2, \qquad u_2 \simeq 3\tilde{\rho}_2\tilde{\rho} - 6\tilde{\gamma}\tilde{\rho}_2, \qquad \tilde{v} \simeq 3\tilde{\rho} - 6\tilde{\gamma} \tag{5.49}$$

during the inflationary phase with the large-γ_0 effect included. Therefore, by counting powers of $\tilde{\gamma}$ and $\tilde{\rho}$, it is apparent that the leading order terms dominating the γ_0 contribution come from those terms that are proportional to γ_0^2. Therefore, the DBI effect can be extracted by collecting all the dominating terms that are proportional to γ_0^2 in c_1.

It is then quite straightforward to show that the dominating effect of the large γ_0 is in fact negative definite:

$$\tilde{c}_1 = -36u_2\tilde{v}\tilde{\rho}^2\lambda_2\rho_2 < 0. \tag{5.50}$$

Therefore, the γ contribution will not affect the sign of the coefficient c_1. In fact, it only enhances the unstable trend of the power-law solutions we found.

VI. CONCLUSION

The inflationary universe is a nice resolution to a number of important phenomena associated with the cosmic microwave background radiation. Trying to understand the physical origin of the highly isotropic universe is therefore very important. The cosmic no-hair conjecture is apparently one of the most important predictions along this line. It is based on a belief that all classical hair should disappear once the vacuum energy dominates.

Among the many advances in the study of this approach, a new set of anisotropic inflationary solutions seems to act as a heuristic and important counterexample to the no-hair conjecture [56,57]. Indeed, it was shown that, when a vector field is coupled to the inflaton, there could be a small anisotropic expansion in the Bianchi type I space. This type of newly found anisotropic inflation is also known to be an attractor solution [59] in BI space. Shortly after the discovery of this result, analytic power-law solutions were also found explicitly in a model with an exponential scalar potential motivated by supergravity theory[59].

In this approach, the anisotropic hair seems to persist in the absence of a cosmological constant in contrast to the earlier known counterexamples. A modified version of scalar field theory, the DBI model, is then introduced in this paper to study the DBI effect on the power-law solutions. As a result, a new set of power-law solutions is shown to exist in this paper. To investigate the stability of the obtained anisotropic power-law inflation, we extend the method proposed by Ref. [65] for the stability analysis of the DBI field in the isotropic universe. Stability analysis indicates clearly that this new set of inflationary solutions is a set of attractor solutions and remains stable under field perturbations. The large-γ effect, constrained by the inequality $v > 0$, is also shown to favor the stability during the perturbations. In fact, the inclusion of the large-γ effect enhances the stability tendency of the power-law solutions.

Finally, we turn our attention to a two-scalar-fields DBI model with an additional phantom field coupled to the system. We have studied the effect of this new field on the stability of the BI space. A new set of power-law anisotropic expanding solutions is then shown to exist in the BI space. The exponential potential introduced in this paper with $\tau = -\lambda$ is shown to be a physical condition critical to the existence of fixed point solutions in this model. After a detailed stability analysis was introduced, the result shows that the phantom field does lead the new two-scalar-field DBI solutions to collapse as expected. In addition, the large-γ effect, constrained by the inequality $\tilde{v} > 0$, is also shown to enhance the collapse of the power-law expanding solutions.

The results shown in this paper indicate that the high energy effect derived from the KSW model, the DBI model, and the effect of the phantom field coupling are all very heuristic in the study of the physics origin underlying the no-hair conjecture. The DBI scalar field, the exponential potentials, and the coupled phantom field all deserve greater attention for their roles in the evolution of the early universe.

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APPENDIX: FIELD EQUATIONS AND THE SOLUTIONS OF THE TWO-SCALAR-FIELDS MODEL

Equations (5.3) and (5.4) can be shown to become

$$\ddot{\phi} = -\frac{3\dot{\alpha}}{\gamma^2}\dot{\phi} - \frac{\partial_\phi V_1}{\gamma^3} - \frac{\partial_\phi f}{2f}\frac{(\gamma+2)(\gamma-1)}{(\gamma+1)\gamma}\dot{\phi}^2 + \frac{h^{-3}\partial_\phi h}{\gamma^3}\exp[-4\alpha - 4\sigma]p_A^2, \tag{A1}$$

$$\ddot{\psi} = -3\dot{\alpha}\,\dot{\psi} - \frac{\partial_\psi V_2}{\omega} + \frac{h^{-3}\partial_\psi h}{\omega}\exp[-4\alpha - 4\sigma]p_A^2. \tag{A2}$$

Moreover, the metric equation (5.5) reduces to the following component equations:

$$\dot{\alpha}^2 = \dot{\sigma}^2 + \frac{1}{3}\left[\frac{\gamma^2}{\gamma+1}\dot{\phi}^2 + \frac{\omega\dot{\psi}^2}{2} + V_1 + V_2 + \frac{h^{-2}}{2}\exp[-4\alpha - 4\sigma]p_A^2\right], \tag{A3}$$

$$\ddot{\alpha} = -3\dot{\alpha}^2 + \left[\frac{\gamma(\gamma-1)}{2(\gamma+1)}\dot{\phi}^2 + V_1 + V_2\right] + \frac{h^{-2}}{6}\exp[-4\alpha - 4\sigma]p_A^2, \tag{A4}$$

$$\ddot{\sigma} = -3\dot{\alpha}\,\dot{\sigma} + \frac{h^{-2}}{3}\exp[-4\alpha - 4\sigma]p_A^2. \tag{A5}$$

It is straightforward to show that γ reduces to

$$\gamma = \frac{1}{\sqrt{1 - \xi_1^2\kappa t^{\tau\xi_1 - 2}}}. \tag{A6}$$

Hence γ is a constant when $\tau\xi_1 = 2$:

$$\gamma \equiv \gamma_0 = \frac{|\tau|}{\sqrt{\tau^2 - 4\kappa}}. \tag{A7}$$

Similar to the discussion in the previous section, we will also focus on the special case that $\tau\xi_1 = 2$. With the ansatz given by Eq. (5.11), the whole set of the field equations (A1)–(A5) can be reduced to a set of algebraic equations,

$$-\xi_1 + \frac{3\zeta\xi_1}{\gamma_0^2} + \frac{\lambda_1 u_1}{\gamma_0^3} + \frac{(\gamma_0+2)(\gamma_0-1)\xi_1}{(\gamma_0+1)\gamma_0} - \frac{\rho_1\tilde{v}}{\gamma_0^3} = 0, \tag{A8}$$

$$-\xi_2 + 3\zeta\xi_2 + \frac{\lambda_2 u_2}{\omega} - \frac{\rho_2\tilde{v}}{\omega} = 0, \tag{A9}$$

$$-\zeta^2 + \eta^2 + \frac{\gamma_0^2\xi_1^2}{3(\gamma_0+1)} + \frac{\omega\xi_2^2}{6} + \frac{u_1+u_2}{3} + \frac{\tilde{v}}{6} = 0, \tag{A10}$$

$$-\zeta + 3\zeta^2 - \frac{\gamma_0(\gamma_0-1)\xi_1^2}{2(\gamma_0+1)} - (u_1+u_2) - \frac{\tilde{v}}{6} = 0, \tag{A11}$$

$$-\eta + 3\zeta\eta - \frac{\tilde{v}}{3} = 0, \tag{A12}$$

along with the following constraints equations that make all terms have the same power in time:

$$\rho_1\xi_1 + \rho_2\xi_2 + 2\zeta + 2\eta = 1, \tag{A13}$$

$$\lambda_1\xi_1 = -2, \tag{A14}$$

$$\lambda_2 \xi_2 = -2. \tag{A15}$$

Note that Eq. (A14) and the constraint $\tau\xi_1 = 2$ imply that $\tau = -\lambda_1$. In addition, we have also assumed λ_i, ρ_i are all positive parameters. Hence τ has to be negative under these assumptions. First of all, Eqs. (A12) and (A13) can be written, respectively, as

$$\tilde{v} = 3\eta(3\zeta - 1), \tag{A16}$$

$$\eta = \frac{1}{2} - \zeta + \frac{\rho_1}{\lambda_1} + \frac{\rho_2}{\lambda_2}. \tag{A17}$$

Therefore Eqs. (A9) and (A10) can be solved to give

$$u_1 = \frac{3\rho_1\eta(3\zeta - 1)}{\lambda_1} + \frac{2\gamma_0[(3\zeta - 2)\gamma_0 + 3\zeta]}{(\gamma_0 + 1)\lambda_1^2}, \tag{A18}$$

$$u_2 = \frac{3\rho_2\eta(3\zeta - 1)}{\lambda_2} + \frac{2\omega(3\zeta - 1)}{\lambda_2^2}. \tag{A19}$$

As a result, ζ can be shown to obey the following equation:

$$6\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)\zeta^2 - 4[(\lambda_1\rho_2 + \lambda_2\rho_1)(2\lambda_1\lambda_2 + 3\lambda_1\rho_2 + 3\lambda_2\rho_1) + \lambda_1^2\lambda_2^2 + 8(\omega\lambda_1^2 + \gamma_0\lambda_2^2)]\zeta = 0. \tag{A20}$$

In addition to a trivial solution $\zeta = 0$, the ζ equation gives another nontrivial solution:

$$\zeta = \frac{4(\lambda_1\rho_2 + \lambda_2\rho_1)(2\lambda_1\lambda_2 + 3\lambda_1\rho_2 + 3\lambda_2\rho_1) + \lambda_1^2\lambda_2^2 + 8(\omega\lambda_1^2 + \gamma_0\lambda_2^2)}{6\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)}. \tag{A21}$$

With the ζ given above, η, u_1, u_2, $\tilde{v}$ can be solved explicitly as

$$\eta = \frac{\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1) - 4(\omega\lambda_1^2 + \gamma_0\lambda_2^2)}{3\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)}, \tag{A22}$$

$$u_1 = \tilde{u}_0 - \frac{2\gamma_0(\gamma_0 - 1)}{(\gamma_0 + 1)\lambda_1^2}, \tag{A23}$$

$$u_2 = \frac{\Omega \times [\lambda_1^2(\lambda_2\rho_2 + 2\rho_2^2 + 2\omega) + 2\lambda_1\lambda_2\rho_1\rho_2 + 4(\omega\lambda_1\rho_1 - \gamma_0\lambda_2\rho_2)]}{2[\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)]^2}, \tag{A24}$$

$$\tilde{v} = \frac{\Omega \times [\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1) - 4(\omega\lambda_1^2 + \gamma_0\lambda_2^2)]}{2[\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)]^2}, \tag{A25}$$

with

$$\tilde{u}_0 = \frac{\Omega \times [\lambda_2^2(\lambda_1\rho_1 + 2\rho_1^2 + 2\gamma_0) + 2\lambda_1\lambda_2\rho_1\rho_2 - 4(\omega\lambda_1\rho_1 - \gamma_0\lambda_2\rho_2)]}{2[\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)]^2}, \tag{A26}$$

$$\Omega = 4(\lambda_1\rho_2 + \lambda_2\rho_1)(\lambda_1\lambda_2 + 3\lambda_1\rho_2 + 3\lambda_2\rho_1) - \lambda_1^2\lambda_2^2 + 8(\omega\lambda_1^2 + \gamma_0\lambda_2^2). \tag{A27}$$

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