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Validation of tolerance interval

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ABSTRACT

The tolerance interval receives very much attention in literature and is widely applied in industry. However, it is generally constructed through the criterion of minimum width by Eisenhart et al. (1947). Although effort for clarification of several prediction related intervals has been made recently by Huang et al. (2010), the appropriateness of the tolerance interval for its role in industry applications is insufficiently discussed. According to manufacturers' requests, a concept of admissibility of tolerance intervals is defined in this paper and we show that these types of tolerance intervals are not admissible due to short of confidence. We further prove that a $100(1-\alpha)\%$ confidence interval of a γ-coverage interval is admissible and is appropriate for use.

1. Motivation and introduction

The tolerance interval is popularly used for making the product quality assessments. In mass-production, the manufacturer is interested in an interval that contains a specified (usually large) percentage of the product and he knows that unless a fixed proportion (say γ) of the production is acceptable in the sense that the items' characteristics conform to the lower specification limit (*LSL*) and upper specification limits (*USL*), he will lose money in this production. With this concern, the manufacturer wants to know the following:

Do the production lots include at least a certain proportion of acceptable measurements with a stated confidence? (1)

Statisticians try to verify this problem of quality assessment through two steps. To begin, suppose that a random sample $\mathbf{X}=(X_1,\ldots,X_n)'$ is obtained from the same process of production which has distribution with probability density function $f_\theta(x)$. For the first step, the pioneer article by Wilks (1941) introduced a γ-content tolerance interval with confidence $1-\alpha$ defined as a random interval $(T_1,T_2)=(t_1(\mathbf{X}),t_2(\mathbf{X}))$ that satisfies

$$P_\theta\{P_\theta(X_0 \in (T_1,T_2)|\mathbf{X}) \geq \gamma\} \geq 1-\alpha \text{ for } \theta \in \Theta \quad (2)$$

where X_0 represents the future observation from the same production process. For the second step, let (t_1,t_2) be the observation of this tolerance interval. The general rule for verifying a manufacturer's problem using the tolerance interval is as follows:

If $(t_1, t_2) \subset (LSL, USL)$, the lot of product is acceptable because we have confidence $1-\alpha$ that at least $100\gamma\%$ of the population is conforming to specification limits (3)

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Statistical tolerance intervals are important in many industrial and pharmaceutical applications. Starting with Wilks (1941), much attention has been received for developing tolerance intervals. Guttman (1970) and Patel (1986) provide thorough overviews of the literature. A lot of different methods have been proposed to construct the different types of tolerance intervals under different problem structures, see, for example, Atwood (1984), Carroll and Ruppert (1991), Liao and Iyer (2004), and Cai and Wang (2009). In general, a common effort been made in the literature is to investigate the version with minimum width, for which Eisenhart et al. (1947) constructed an approximate minimum width tolerance interval for normal random variable. This normal tolerance interval is now popularly implemented in manufacturing industries and is presented in text books of engineering statistics. Since then, much effort has been done on developing this minimum width tolerance interval, parametrically or nonparametrically. For example, Jensen (2009) compares the approximations of tolerance intervals under normal distribution. Di Bucchianico et al. (2001) construct smallest nonparametric tolerance regions.

We consider here if the conclusion of (3) based on minimum width tolerance interval is actually appropriate. This is important since using an inappropriate tolerance interval may cause unexpected smaller confidence that provides the manufacturer an invalid conclusion regarding the confidence of having proportion γ of acceptable products.

This paper is organized as follows. In Section 2, we define an admissible tolerance interval that guarantees to solve the manufacturer's problem. Accordingly we also prove that the minimum width γ-tolerance interval is not admissible. In Section 3, we develop a necessary and sufficient condition for a tolerance interval to be admissible, which indicates that the confidence interval of a coverage interval extended from Owen (1964) is admissible. In Section 4, two simulations and one example are given. In the first simulation, we evaluate the approximate minimum width tolerance interval of Eisenhart et al. (1947) dealing with normal distribution through simulation studies showing that there is a risk leading to an invalid conclusion. Another simulation demonstrates the improvement on the confidence with the necessary and sufficient condition. An example is also given in the end of Section 4 to illustrate the concept of admissibility of tolerance intervals.

2. Validity for minimum-width tolerance interval

We expect that a tolerance interval contains not only a proportion γ of measurements but also one γ coverage interval which is an interval $(a(\theta),b(\theta))$ satisfying $\gamma = P_\theta\{a(\theta) \le X_0 \le b(\theta)\}$. We now state this requirement to formulate the admissible tolerance interval.

Definition 1. Let (T_1,T_2) be a γ-content tolerance interval with confidence $1-\alpha$. We say that it is admissible if the following:

$$P_\theta\{P_\theta[X_0 \in (T_1,T_2)|T_1,T_2] \ge \gamma,(a(\theta),b(\theta)) \subset (T_1,T_2)\} \ge 1-\alpha \quad \text{for } \theta \in \Theta, \tag{4}$$

holds for some γ coverage interval $(a(\theta),b(\theta))$.

An admissible tolerance interval guarantees that when the observed tolerance interval is contained within the specification limits, we then have confidence $1-\alpha$ that the proportion γ products are all acceptable.

Let the random sample $X_1,\ldots,X_n$ be drawn from the normal distribution $N(\mu,\sigma^2)$ where μ and σ are both unknown. With symmetric density, the general form of two sided normal tolerance interval is of the form

$$(\overline{X}-kS,\overline{X}+kS), \tag{5}$$

where the minimum width normal tolerance interval searches k^* such that $(\overline{X}-k^*S,\overline{X}+k^*S)$ solves the following minimization problem:

$$\underset{k>0}{\operatorname{argmin}}\, P_{\mu,\sigma}\{P_{\mu,\sigma}[X_0 \in (\overline{X}-kS,\overline{X}+kS)|\overline{X},S] \ge \gamma\} \ge 1-\alpha. \tag{6}$$

The in-admissibility of this minimum width tolerance interval is shown in the following theorem.

Theorem 1. *The minimum width γ-content tolerance interval of the form $(\overline{X}-k^*S,\overline{X}+k^*S)$ with confidence $1-\alpha$ is not admissible.*

For the proof of Theorem 1, we first show that among γ coverage intervals $(\mu+z_\delta\sigma,\mu+z_{\gamma+\delta}\sigma)$ for $0<\delta<1-\gamma$ the tolerance interval of (6) achieves the maximum confidence when the coverage interval is the symmetric one with $\delta=(1-\gamma)/2$.

Lemma 1. *Let $C_\delta(\gamma)=(\mu+z_\delta\sigma,\mu+z_{\gamma+\delta}\sigma)$. Then*

$$\frac{1-\gamma}{2} = \underset{0<\delta<1-\gamma}{\operatorname{argmax}}\, P_{\mu,\sigma}\{P_{\mu,\sigma}[X_0 \in (\overline{X}-kS,\overline{X}+kS)|\overline{X},S] \ge \gamma, C_\delta(\gamma) \subset (\overline{X}-kS,\overline{X}+kS)\},$$

for any given $k>0$.

Proof. We observe that

$$\begin{aligned}&P_{\mu,\sigma}\{P_{\mu,\sigma}[X_0 \in (\overline{X}-kS,\overline{X}+kS)|\overline{X},S)] \geq \gamma, C_\delta(\gamma) \subset (\overline{X}-kS,\overline{X}+kS)\}\\&\quad = E_{\mu,\sigma}\{P_{\mu,\sigma}[\Phi_{\mu,\sigma}(\overline{X}+kS)-\Phi_{\mu,\sigma}(\overline{X}-kS)] \geq \gamma, z_{\gamma+\delta}\sigma-kS<\overline{X}-\mu<z_\delta\sigma+kS|S]\}\\&\quad = E_{\mu,\sigma}\{P_{\mu,\sigma}\{z_{\gamma+\delta}\sigma-kS<\overline{X}-\mu<z_\delta\sigma+kS|S\}\}.\end{aligned}$$

The last equality holds because $\{z_{\gamma+\delta}\sigma-kS<\overline{X}-\mu<z_\delta\sigma+kS\} \subset \{\Phi_{\mu,\sigma}(\overline{X}+kS)-\Phi_{\mu,\sigma}(\overline{X}-kS) \geq \gamma\}$ with given S and k. Hence, it suffices to maximize $P_{\mu,\sigma}\{z_{\gamma+\delta}\sigma-kS<\overline{X}-\mu<z_\delta\sigma+kS|S\}$ with respect to δ.

According to the normal probability density function, if $0<\delta<(1-\gamma)/2$, then $z_\delta-z_{(1-\gamma)/2}<0, z_{\gamma+\delta}-z_{(1+\gamma)/2}<0$, and $|z_\delta-z_{(1-\gamma)/2}| > |z_{\gamma+\delta}-z_{(1+\gamma)/2}|$ and hence we obtain

$$\begin{aligned}&P_{\mu,\sigma}\{z_{\gamma+\delta}\sigma-kS<\overline{X}-\mu<z_\delta\sigma+kS|S\}\\&\quad = P_{\mu,\sigma}\{-kS+z_{(1+\gamma)/2}\sigma+z_{\gamma+\delta}\sigma-z_{(1+\gamma)/2}\sigma<\overline{X}-\mu<kS+z_{(1-\gamma)/2}\sigma+z_\delta\sigma-z_{(1-\gamma)/2}\sigma|S\}\\&\quad \leq P_{\mu,\sigma}\{-kS+z_{(1+\gamma)/2}\sigma<\overline{X}-\mu<kS+z_{(1-\gamma)/2}\sigma|S\}.\end{aligned} \tag{7}$$

For the other case, if $(1-\gamma)/2<\delta<1-\gamma$, then $z_\delta-z_{(1-\gamma)/2}>0, z_{\gamma+\delta}-z_{(1+\gamma)/2}>0$, and $|z_\delta-z_{(1-\gamma)/2}|<|z_{\gamma+\delta}-z_{(1+\gamma)/2}|$ and hence the inequality (7) also holds. Therefore, $P_{\mu,\sigma}\{z_{\gamma+\delta}\sigma-kS<\overline{X}-\mu<z_\delta\sigma+kS|S\}$ is maximized at $\delta=(1-\gamma)/2$ and then the result follows. □

The proof of Theorem 1 is then as follows.

Proof. Given $\gamma,\alpha \in (0,1)$, let $A(k)=\{\Phi_{\mu,\sigma}(\overline{X}+kS)-\Phi_{\mu,\sigma}(\overline{X}-kS)\geq\gamma\}$. For $k_1<k_2$, we have that $A(k_1)\subset A(k_2)$ and then $P_{\mu,\sigma}(A(k_1)) \leq P_{\mu,\sigma}(A(k_2))$. Defining a function g from R^+ to $[0,1]$ by

$$g(k)=P_{\mu,\sigma}(A(k)), \quad \text{for } k \in R^+.$$

By definition, the function g is increasing. Besides, it goes to 1 as $k\to\infty$ and goes to zero as $k\to 0$. Moreover, g inherits the continuity from that of function Φ and distribution of $\overline{X}$ and S. Hence

$$\begin{aligned}&\inf\{k : P_{\mu,\sigma}\{P_{\mu,\sigma}[X_0 \in (\overline{X}-kS,\overline{X}+kS)|\overline{X},S]\geq\gamma\}\geq 1-\alpha\}\\&\quad = \inf\{k : P_{\mu,\sigma}\{\Phi_{\mu,\sigma}(\overline{X}+kS)-\Phi_{\mu,\sigma}(\overline{X}-kS)\geq\gamma\}\geq 1-\alpha\}\\&\quad = \inf\{k : g(k)\geq 1-\alpha\}.\end{aligned}$$

It is seen that the value $1-\alpha$ is achievable at some positive number k^*.

By the definition of admissibility and Lemma 1, it then suffices to consider the following set:

$$A_{C(\gamma)}(k)=\{\Phi_{\mu,\sigma}(\overline{X}+kS)-\Phi_{\mu,\sigma}(\overline{X}-kS)\geq\gamma, C(\gamma)\subset(\overline{X}-kS,\overline{X}+kS)\}, \tag{8}$$

where $C(\gamma)$ is the set $C_{(1-\gamma)/2}(\gamma)$ defined in Lemma 1.

Let $B(k^*)=\{\overline{X}-k^*S>\mu-z_{(1+\gamma)/2}\sigma, \Phi_{\mu,\sigma}(\overline{X}+k^*S)-\Phi_{\mu,\sigma}(\overline{X}-k^*S)\geq\gamma\}$. We then obtain that $B(k^*)\subset A(k^*), B(k^*)\cap A_{C(\gamma)}(k^*)=\emptyset$ and also $P_{\mu,\sigma}(B(k^*))>0$ due to normal distribution. Thus $P_{\mu,\sigma}(A_{C(\gamma)}(k^*))<1-\alpha$. Hence, the set $A_{C(\gamma)}(k^*)$ does not satisfy the inequality in the Definition 1. Thus the result follows. □

From the proof of Theorem 1, we obtain that the minimum width tolerance interval is usually too short to guarantee that the conclusion drawn in (3) is valid that may lead the manufacturer to make in-appropriate decision.

3. Coverage interval-based tolerance intervals

We have proved that not every tolerance interval of Wilks (1941) in (2) is admissible in Section 2. The next question will be the existence of a general technique for developing a tolerance interval that ensures the property of admissibility. Owen (1964) introduced a tolerance interval for normal distribution to control both tails. We extend this concept to a general setting.

Definition 2. We say that a random interval (T_1,T_2) is a $100(1-\alpha)\%$ confidence interval of a coverage interval $(a(\theta),b(\theta))$ if it satisfies

$$1-\alpha=P_\theta\{T_1\leq a(\theta)<b(\theta)\leq T_2)\} \text{ for } \theta\in\Theta. \tag{9}$$

The interest of this confidence interval is that if it serves as an admissible tolerance interval.

The following theorem states a necessary and sufficient condition for a tolerance interval to be admissible. This theorem also addresses the connection of admissible tolerance interval with the confidence interval of a coverage interval.

Theorem 2. *A random interval (T_1,T_2) is an admissible γ-content tolerance interval with confidence $1-\alpha$ if and only if it is a $100(1-\alpha)\%$ confidence interval of a γ coverage interval.*

Proof. Let $(T_1,T_2)=(t_1(\mathbf{X}),t_2(\mathbf{X}))$ be a random interval and $(a(\theta),b(\theta))$ be a γ coverage interval. We then obtain that

$$\{\mathbf{X} : (a(\theta),b(\theta))\subset(t_1(\mathbf{X}),t_2(\mathbf{X}))\}\subset\{\mathbf{X} : P_\theta(t_1(\mathbf{X})<X_0<t_2(\mathbf{X})|\mathbf{X})\geq\gamma\}$$

and thus,

$$\{\mathbf{X} : (a(\theta),b(\theta)) \subset (t_1(\mathbf{X}),t_2(\mathbf{X}))\} = \{\mathbf{X} : P_\theta(t_1(\mathbf{X}) < X_0 < t_2(\mathbf{X}) | \mathbf{X}) \geq \gamma, (a(\theta),b(\theta)) \subset (t_1(\mathbf{X}),t_2(\mathbf{X}))\}. \tag{10}$$

Hence, by definition of the admissibility of a γ-content interval with confidence $1-\alpha$ and the definition of a $(1-\alpha)\%$ confidence interval of a γ coverage interval $(a(\theta),b(\theta))$, the theorem follows. □

Practitioner can apply any version of tolerance interval based on rule of (3) to answer the manufacturer's question of (1). It is seen in this section that we are only sure that the conclusion is valid when the used version is the coverage interval-based tolerance interval.

4. Simulations and an example

In this section, we illustrate the results of the last two sections through simulated data. The first sub-section illustrates the fact that the approximate minimum width tolerance interval of Eisenhart et al. (1947) is not appropriate for solving the manufacturer's problem of (1). The second sub-section is the Monte–Carlo study on the necessary and sufficient condition for coverage interval-based tolerance interval proposed in Section 3. In the end, we give an example to illustrate the concept of admissibility to tolerance interval.

4.1. Approximate minimum width tolerance intervals

Our main concern is to see if the approximate minimum width tolerance interval of Eisenhart et al. (1947) is appropriate for solving the manufacturer's problem of (1). In this section, we will answer this question through two simulations. First, we evaluate its role as a Wilks' tolerance interval satisfying (2). Second, we perform another simulation to see if it is an admissible tolerance interval for some γ coverage interval. If a tolerance interval is not admissible but its observation (t_1,t_2) is contained in specification limits, there is a risk of drawing an invalid conclusion for the manufacturer.

For a given γ and $1-\alpha$, we select values k from the table developed by Eisenhart et al. (1947) to construct the normal tolerance interval of (5). This simulation is conducted with replication $m=100{,}000$. We then select a random sample of size n from normal distribution $N(\mu,\sigma^2)$ and let $\overline{x}_j$ and s_j^2 be the sample mean and sample variance for the sample of jth replication. The approximate confidence is defined as

$$\frac{1}{m}\sum_{j=1}^{m} I(\Phi_{\mu,\sigma}(\overline{x}_j+ks_j)-\Phi_{\mu,\sigma}(\overline{x}_j-ks_j) \geq \gamma), \tag{11}$$

where $\Phi_{\mu,\sigma}$ is the distribution function for $N(\mu,\sigma^2)$. As an approximate minimum width tolerance interval, we anticipate that its corresponding approximate confidence is around $1-\alpha$. For $\gamma = 0.9, 0.95, 0.99$, $1-\alpha = 0.9, 0.95, 0.99$ and $n = 10, 30, 50$, we display the corresponding simulated results in Table 1.

There are several comments we may draw from Table 1: (a) the simulated confidence levels for various situations of γ and n are all close to $1-\alpha$. This ensures that there is a proportion γ of products with characteristic values contained in the sample interval of (5) at an approximate confidence $1-\alpha$. (b) The sample space of any tolerance interval (T_1,T_2) may be partitioned into $\Omega_1=\{(t_1,t_2) : \int_{t_1}^{t_2} \phi_{\mu,\sigma}(x)\,dx \geq \gamma\}$ and $\Omega_2=\{(t_1,t_2) : \int_{t_1}^{t_2} \phi_{\mu,\sigma}(x)\,dx < \gamma\}$ where $\phi_{\mu,\sigma}$ is the probability density function for normal distribution $N(\mu,\sigma^2)$. From these simulation results, we have $P_{\mu,\sigma}\{(T_1,T_2) \subset \Omega_1\} \approx 1-\alpha$. However, there is a non-negligible subset $\Omega_{11} \subset \Omega_1$ with $\Omega_{11} = \{(t_1,t_2) : t_1 = -\infty \text{ or } t_2 = \infty\}$ such that $P_{\mu,\sigma}\{(T_1,T_2) \subset \Omega_{11}\} > 0$. Hence, its role in providing a valid conclusion for (3) is questionable.

Table 1
Confidence for approximate minimum width normal tolerance interval.

	$1-\alpha=0.9$	$1-\alpha=0.95$	$1-\alpha=0.99$
$\gamma=0.9$			
$n=10$	0.8962	0.9491	0.9887
$n=30$	0.8976	0.9482	0.9892
$n=50$	0.8976	0.9482	0.9900
$\gamma=0.95$			
$n=10$	0.8970	0.9491	0.9899
$n=30$	0.8978	0.9432	0.9895
$n=50$	0.8985	0.9488	0.9891
$\gamma=0.99$			
$n=10$	0.9003	0.9508	0.9896
$n=30$	0.8985	0.9499	0.9897
$n=50$	0.8995	0.9489	0.9903

Table 2
Confidence for normal tolerance intervals in covering interval $(\mu - z_{(1+\gamma)/2}\sigma, \mu + z_{(1+\gamma)/2}\sigma)$.

	$1-\alpha=0.9$	$1-\alpha=0.95$	$1-\alpha=0.99$
$\gamma=0.9$			
$n=10$	0.8197	0.8989	0.9765
$n=30$	0.7736	0.8648	0.9617
$n=50$	0.7595	0.8518	0.9561
$\gamma=0.95$			
$n=10$	0.8403	0.9129	0.9812
$n=30$	0.8032	0.8756	0.9700
$n=50$	0.7917	0.8773	0.9648
$\gamma=0.99$			
$n=10$	0.8627	0.9278	0.9849
$n=30$	0.8356	0.9097	0.9776
$n=50$	0.8268	0.9020	0.9762

To determine precisely if the approximate minimum width tolerance interval is appropriate for solving the manufacturer's problem, we further perform a simulation where the conditions are unchanged. The approximate admissibility confidence that it will have a role of confidence interval of the coverage interval $C(\gamma) = (\mu - z_{(1+\gamma)/2}\sigma, \mu + z_{(1+\gamma)/2}\sigma)$ is defined as

$$\frac{1}{m}\sum_{j=1}^{m} I(\Phi_{\mu,\sigma}(\overline{x}_j + ks_j) - \Phi_{\mu,\sigma}(\overline{x}_j - ks_j) \geq \gamma, C(\gamma) \subset (\overline{x}_j - ks_j, \overline{x}_j + ks_j)).$$

The simulated results are listed in Table 2.

It can be seen that the confidences fluctuate for cases of sample size n and coverage γ. We expect that a tolerance interval should have a chance to satisfy the manufacturer's expectation when $P_{\mu,\sigma}\{X_0 \in (LSL, USL)\} \geq \gamma$ is true. However, when $C(\gamma) = (LSL, USL)$, there is no chance for Eisenhart et al.'s minimum width normal tolerance interval to match the manufacturer's expectation. Hence, a valid conclusion cannot be drawn from (3) whenever $C(\gamma) = (LSL, USL)$. For the cases that $n=50$, $\gamma = 0.9$ and $1-\alpha = 0.9$, this tolerance interval only achieves confidences with value 0.7595, remarkably smaller than 0.9, the expected value.

4.2. Coverage interval-based tolerance intervals

Next we evaluate the necessary and sufficient property of the coverage interval-based tolerance in Section 3 through a discussion of the following example. Let $X_1, \ldots, X_n$ be a random sample from normal distribution $N(\mu, \sigma^2)$ where μ and σ are both unknown.

With this normal random sample

$$\left(\overline{X} - t_{1-(\alpha/2)}(n-1, \sqrt{n}z_{(1+\gamma)/2})\frac{S}{\sqrt{n}}, \overline{X} + t_{1-(\alpha/2)}(n-1, \sqrt{n}z_{(1+\gamma)/2})\frac{S}{\sqrt{n}}\right) \tag{12}$$

is a $100(1-\alpha)\%$ confidence interval for the coverage interval $C(\gamma) = (\mu - z_{(1+\gamma)/2}\sigma, \mu + z_{(1+\gamma)/2}\sigma)$ where $t_u(\nu, \eta)$ is the uth quantile of the noncentral t-distribution $t(\nu, \eta)$ with degree of freedom ν and noncentrality parameter η. A brief proof is as follows:

We know that $\sqrt{n}(\overline{X} - \mu + z_{(1+\gamma)/2}\sigma)/S \sim t(n-1, n^{1/2}z_{(1+\gamma)/2})$ and $\sqrt{n}(\overline{X} - \mu - z_{(1+\gamma)/2}\sigma)/S \sim t(n-1, -\sqrt{n}z_{(1+\gamma)/2})$. Put $a = n^{-1/2}t_{\alpha/2}(n-1, -\sqrt{n}z_{(1+\gamma)/2})$, $b = n^{-1/2}t_{1-\alpha/2}(n-1, \sqrt{n}z_{(1+\gamma)/2})$. Then

$$\begin{aligned}
1-\alpha &= P_{\mu,\sigma}(\overline{X} - \mu + z_{(1+\gamma)/2}\sigma < Sb) - P_{\mu,\sigma}(\overline{X} - \mu - z_{(1+\gamma)/2}\sigma < Sa) \\
&= P_{\mu,\sigma}(Sa + z_{(1+\gamma)/2}\sigma < \overline{X} - \mu < Sb - z_{(1+\gamma)/2}\sigma) \\
&= P_{\mu,\sigma}(Sa < \overline{X} - \mu - z_{(1+\gamma)/2}\sigma < \overline{X} - \mu + z_{(1+\gamma)/2}\sigma < Sb) \\
&= P_{\mu,\sigma}(\overline{X} - Sb < \mu - z_{(1+\gamma)/2}\sigma < \mu + z_{(1+\gamma)/2}\sigma < \overline{X} - Sa).
\end{aligned}$$

Hence, the statement for interval (12) follows by the property of noncentral t distribution, $t_u(\nu, -\eta) = -t_{1-u}(\nu, \eta)$. Then the random interval of (12) is an admissible γ content tolerance interval at confidence $1-\alpha$.

We simulate the confidences for this normal tolerance interval of (12) defined as

$$\frac{1}{m}\sum_{j=1}^{m} I\left(C(\gamma) \in \left(\overline{X} - t_{1-(\alpha/2)}(n-1, \sqrt{n}z_{(1+\gamma)/2})\frac{S}{\sqrt{n}}, \overline{X} + t_{1-(\alpha/2)}(n-1, \sqrt{n}z_{(1+\gamma)/2})\frac{S}{\sqrt{n}}\right)\right).$$

We list the simulation results in the following table.

We have two comments drawn from Table 3: (a) the admissible tolerance interval proposed by Huang et al. (2010) does attain admissibility confidence $1-\alpha$. This is not true for the minimum width tolerance intervals. (b) Using the confidence

Table 3
Simulated confidences for normal tolerance intervals of (12).

	$1-\alpha=0.9$	$1-\alpha=0.95$	$1-\alpha=0.99$
$\gamma=0.9$			
$n=10$	0.9127	0.9559	0.9911
$n=30$	0.9079	0.9531	0.9904
$n=50$	0.9076	0.9523	0.9905
$\gamma=0.95$			
$n=10$	0.9174	0.9581	0.9918
$n=30$	0.9099	0.9550	0.9908
$n=50$	0.9114	0.9545	0.9901
$\gamma=0.99$			
$n=10$	0.9213	0.9609	0.9924
$n=30$	0.9167	0.9575	0.9909
$n=50$	0.9168	0.9574	0.9914

interval of a coverage interval to construct a tolerance interval is proved to be satisfactory to solve manufacturer's question.

For any tolerance interval of Wilks (1941) to be admissible with respect to a coverage interval, it must be a confidence interval of the same coverage interval. This is a great help in developing an admissible tolerance interval. When a 100 $(1-\alpha)$% confidence interval of a γ coverage interval $C(\gamma)$ is contained in the interval bounded by specification limits, we are then sure that there is a proportion γ of acceptable measurements at confidence $1-\alpha$ since the tolerance interval is admissible with respect to $C(\gamma)$.

We then illustrate the concept of admissibility of tolerance intervals in the following example.

Example 1. Consider aircraft parts that have a required diameter of 0.425 cm and specification limits (0.38 cm, 0.47 cm). Assume that we have a random sample of size $n=20$ of aircraft parts drawn from a stable process that is well modeled by a normal distribution. The computed sample mean and sample standard deviation are, respectively, $\overline{x}=0.4232$ and $s=0.0177$. The shortest 90% content tolerance interval at confidence 95% is $(\overline{x}-2.31\,s,\overline{x}+2.31\,s)=(0.3823,0.4643)$. Following the rule of (3), we should confirm that there is at least 90% of the population is conforming to specification limits at confidence 0.95 if we apply the shortest tolerance interval. However, according to the coverage interval-based tolerance interval of (12), the 90% content tolerance interval at confidence 95% is $(\overline{x}-t_{0.975}(19,1.645\sqrt{20})(s/\sqrt{20}),\overline{x}+t_{0.975}(19,1.645\sqrt{20})(s/\sqrt{20}))=(0.3776,0.4688)$. We are not allowed to make the same conclusion when we apply this tolerance interval.

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